Issue 2, 2002

Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society

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Paradox

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Words from the Editor...

Welcome to the second issue of *Paradox* for 2002, the magazine of the Melbourne University Mathematics and Statistics Society (MUMS). *Paradox* has provided an entertaining and informative read for those of mathematical bent since the days when your maths lecturers were students themselves. Following from the success of its precursor, this issue of *Paradox* promises to be more bumper than ever, packed with hilarity and mathematical goodness.

This issue features an interview with current staff member, Tony Guttmann, along with articles about "Donutworld" and "Farey Sequences". Also included are maths jokes to impress your friends with. And, as always, there are some problems for you to try your hand at, with cash prizes up for grabs. For those of you who might remember the antics of *Paradox Kid*, we here at *Paradox* have endeavoured to provide you with a bigger, better and brighter mathematical comic strip hero. Appearing in a worldwide debut right here in the centrefold of this issue of *Paradox*, I am proud to present to you the one and only *Knot Man*!

We are always interested to hear from our readers, so if you have any comments or contributions, please email us at `paradox@ms.unimelb.edu.au`.

— Norman Do, *Paradox* Editor

... and some from the President

Each year on Discovery Day, MUMS runs the Schools Maths Olympics. Some of you may have competed in this in the past, or maybe even in the University Maths Olympics. Perhaps you've picked up this copy of *Paradox* on Discovery Day as you were going to the Schools Maths Olympics?

For the uninitiated, the Maths Olympics are best described as a maths relay. It is a fun and exciting competition which combines both physical ability and mathematical prowess. If you happen to be at Melbourne Uni on Discovery Day, I would encourage you to come along and spectate between 12 and 1pm in Theatre A in our beloved Richard Berry Building. If you're a university student and have never competed before, it may give you the chance to work out the winning strategy that will bring you glory in the upcoming University Maths Olympics, to be held at 1pm on Friday 13 September.

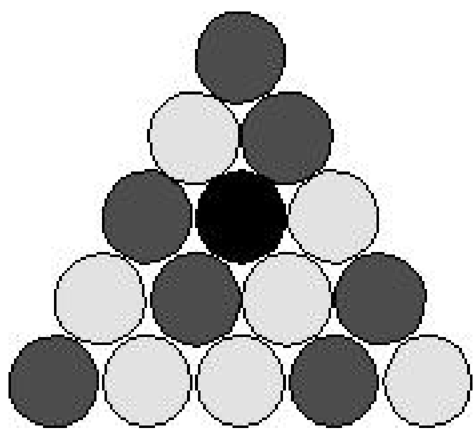
As always, I would encourage you to sign up to the MUMS events mailing list, by sending your name and email address to `mums@ms.unimelb.edu.au`. Apart from the Schools and University Maths Olympics, this semester we plan to have an information session about honours and further study in mathematics, a trivia night, and our weekly seminars on various interesting mathematical topics.

— Luke Mawbey, President of MUMS

The President's Pool Problem for Paradox

It all began a few weeks ago, as I was playing pool (eightball) with Norm, the fantastic editor of *Paradox* . After spending considerable time racking up the balls in a somewhat haphazard manner that clearly seemed to be sub-optimal, I started to wonder: "What is the largest number of moves I should have to make to rack up the balls correctly?" Let me explain...

I stick my $2 into the coin slot, and the balls roll out. I randomly place all of them within the triangle and then I begin to order them. I define a move as a direct swap of two balls, or a rotation (by 120°) of the triangle. I don't care much about symmetries, so if the setup is reflected through the middle, then that's fine. Also, if all of the balls of one type are swapped with all of the balls of the other type, then that is also fine.



I reckon a required configuration can be achieved in 5 moves from any starting position, though could it perhaps always be done in 4? Can you prove this? Can you devise an algorithm for the swapping of balls and rotating of the triangle? Send your answers in to `paradox@ms.unimelb.edu.au` to receive a special prize!

A word of warning though – people may look at you strangely if you spend all your time at the pool table racking up the balls and never play a game!

— Luke Mawbey, President of MUMS

Meanderings on Donuts

Donutworld

Have you ever wondered what it would be like to live on a donut? (Mmm...) No? Well it's an interesting thought for many reasons, including mathematical ones. Our world may be shaped kind of like a sphere, but it might be much more interesting if it were donut-shaped.

For one thing, you can walk around a donut-shaped world in many more different and interesting ways than on a boring old (almost) spherical planet. Walking around Earth, for example, you can walk all the way around the planet in a loop by circumnavigating the globe, or alternatively by walking in a circle around your house, but it's all pretty boring. Sure, you might see some interesting scenery, but in both cases you just walk in a circle, more or less.

Instead, imagine yourself living on a donut-shaped world. You can walk around it and come back to your starting point in many different ways. The simplest way (and the most boring!) is to walk in a little circle around your local neighbourhood – i.e. around your bedroom in your Donutworld house. Two slightly more interesting paths are to walk around a *meridian* of the donut (i.e. around a disc of dough in a real donut), or around a *longitude* (i.e. around the whole [hole?] of Donutworld, doing a lap of the donut). These two paths are completely different, but both correspond to circumnavigating Donutworld, in two different senses.

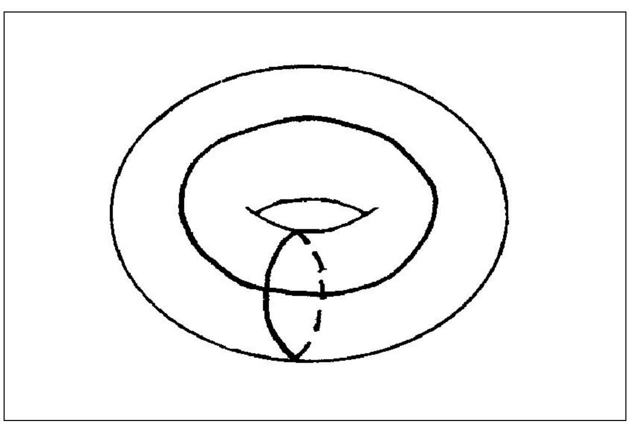


Figure 1: Diagram of a torus showing the meridian (going around the dough) and the longitude (going around the hole).

Now as you walk around, you are walking along a path in space. So if you ignore the actual surface of the donut for a minute, you can see that in walking around the meridian, you have simply walked around a circle in 3-dimensional space. Walking around the longitude produces a completely different circle, but it's still just a circle.

There are, of course, many possible walks around meridians and longitudes, at different sites on the donut, but all of them produce circles in 3-D space. The circles might be in different places, and you might walk around in a bit of a wobbly way, but speaking *topologically*, all your paths are the same. They are all circles in 3-D space!

"Topologically"?

But what does it mean to say that *topologically*, all paths are the same? Without getting too technical, topology is the study of the shapes and arrangements of objects, but not with particularities such as lengths, angles, areas and volumes. (These more familiar geometric notions are all a bit difficult for topologists, it seems!) So for instance, in topology a cube and a ball are the same, because if you think of them as made of rubber, then you can stretch and squeeze one into the other without tearing, ripping, puncturing, popping or otherwise, doing nasty things to your rubber. For this reason topology is sometimes popularly called "rubber sheet geometry".

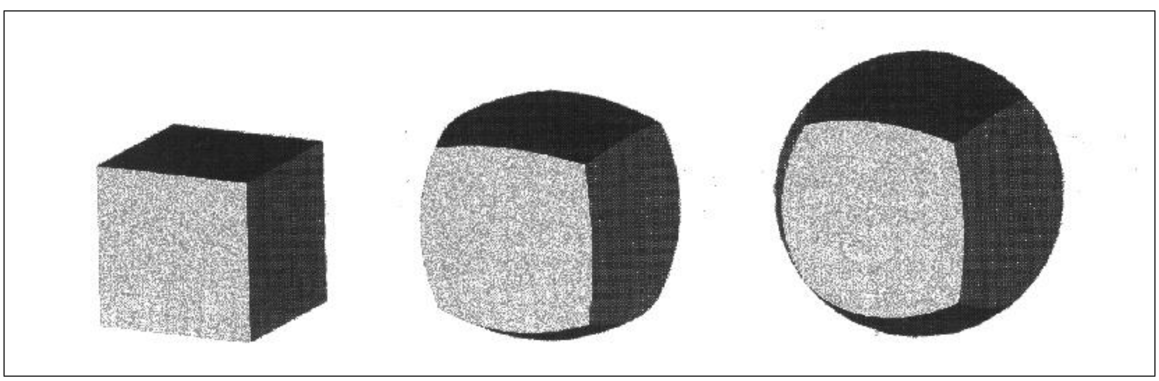


Figure 2: A cube is topologically the same as a sphere, since we can stretch and squeeze one of them to form the other.

But if everything can be made out of rubber, and deformed into everything else, then you might think that topology could be a very boring area of mathematics. Can't you just stretch and squeeze any shape to give any other shape? Well actually, you can't. While a soccer ball, football, hollow box, empty microwave and room of a house (with the door closed) might be the same to a topologist, a donut is different! So, although topology might treat lots of things the same that we would normally think of as different, it treats many classes of objects as different. You can't stretch and squeeze a ball into a donut without puncturing, cutting bits out or otherwise performing some horrible torture on your poor ball. That is why Earth and Donutworld are "topologically" different. Equally, it is why paths around meridians and longitudes of Donutworld (no matter how wobbly) are "topologically" all circles.

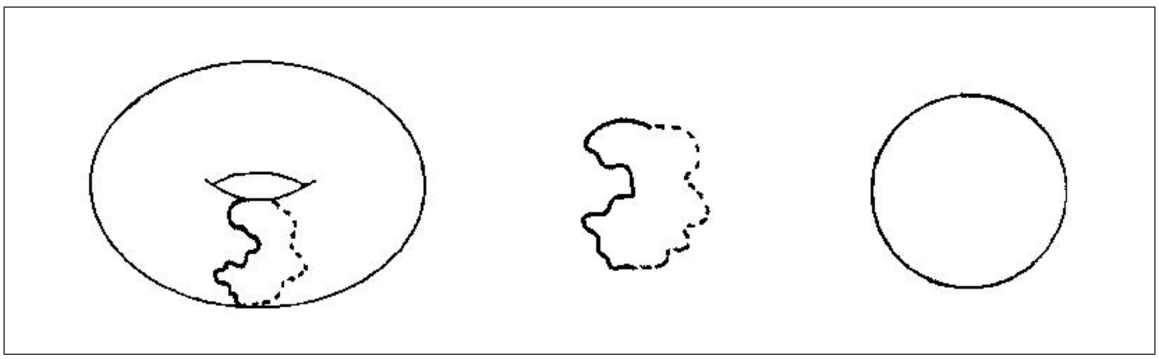


Figure 3: A wobbly path around the meridian of a torus is just a wobbly circle in space.

Mathematicians call a donut a *torus*, possibly because they don't want to get too hungry while engaged in their scholarly pursuits, and "torus" probably induces less salivation than the word "donut".

Let's return to our wanderings on Earth and Donutworld...

Torus Knots

We talked at first about walking around on Earth. We can now be a bit more precise. As you walk around on Earth in some sort of loop, without ever crossing over your path, and come back to your starting position, you trace out a path which is always (topologically) a circle, no matter how crazily you navigated.

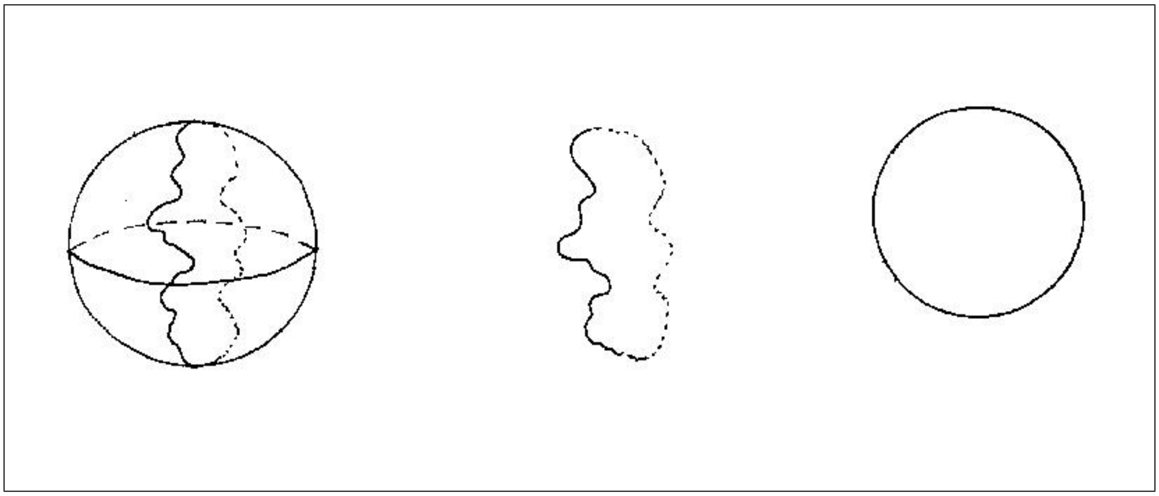


Figure 4: Every loop on a sphere is topologically just a circle in space.

This might seem obvious. You might think, isn't it possible to get only circles, since you're walking in a loop, whether on Earth, Donutworld or wherever? We already know that "boring" paths on Donutworld such as walking around the neighbourhood, and even meridians and longitudes, all merely produce circles. But the answer to the question may be surprising, because on Donutworld, very interesting paths are possible! Here is one.

Start at the top of the front of Donutworld, as shown below (1). Then walk to the right and forwards, so that you are walking around a meridian and longitude at the same time. But do it so that you go round the meridian $1\frac{1}{2}$ times as fast as you go around the longitude. So when you get to the back of the donut (i.e. half way around a longitude), you have actually travelled $\frac{3}{4}$ of the way around a meridian (2). You complete your first full meridian when you are $\frac{2}{3}$ of your way around a longitude (3). When you finish your first lap of the longitude, you have gone around $1\frac{1}{2}$ meridians, so you are at the bottom of the front of Donutworld (4). Keep going – when you have completed two full laps of the longitude, you will be back at the starting position (1).

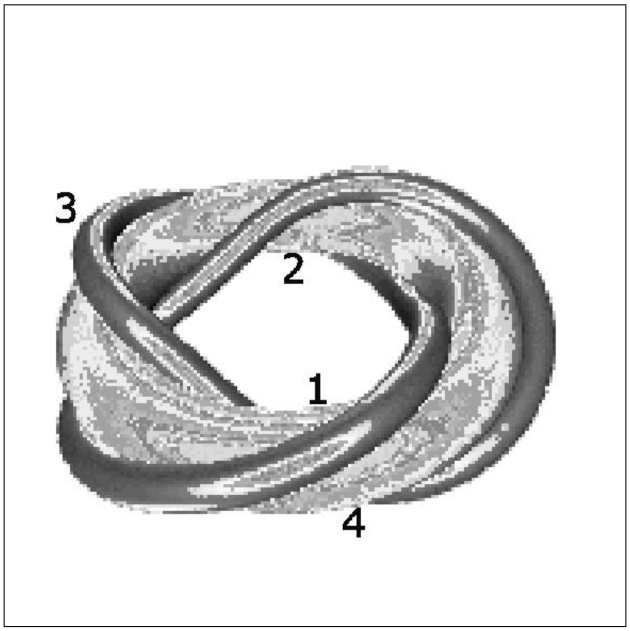


Figure 5: An interesting path on a torus!

What is this path you have just traced out? If you think about it carefully, you will see that you have actually not just walked along a boring circle, but your path is actually knotted up! The name of the path along which you have walked is called the *trefoil knot*. No matter how you bend, pull, shrink or twist the path, you cannot turn it into the usual circle – topologically, it is different from the unknotted circle. It is a mathematical *knot*. There is a whole branch of mathematics which deals with these sorts of objects. Not particularly surprisingly, it's known as *knot theory*.

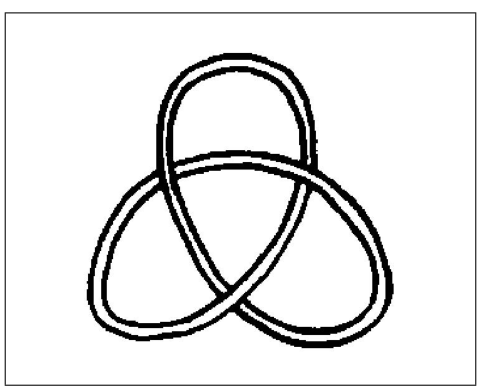


Figure 6: The interesting path on a torus gives us a knot called the trefoil knot.

Because, in our adventures, we traversed the meridian 3 times and the longitude twice, this knot is known as the $(3, 2)-$torus knot. But there is nothing special about 3 and 2. You can have a path that goes around the meridian 9 times and the longitude 5 times, for example, and you would get the $(9, 5)-$torus knot.

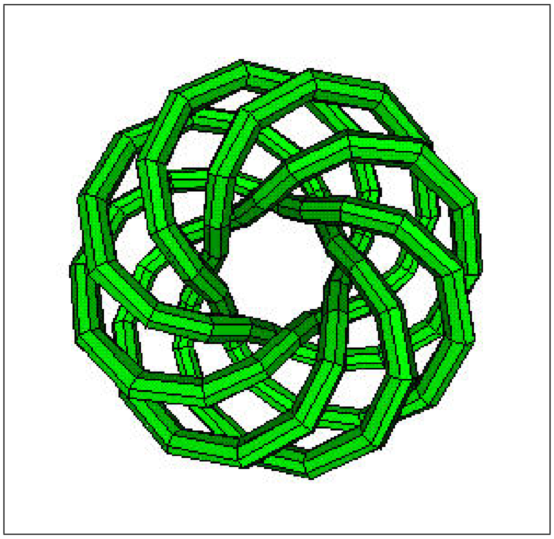


Figure 7: The $(8, 5)$-torus knot.

But you *could not* have, for example, a $(4, 2)$-torus knot. Why is this? Well, if you try to walk around Donutworld so that you steadily cover 4 meridians and 2 longitudes, you will find that after your first longitude lap, you will have covered 2 meridians and so you'll be back where you started! Remember that you're not allowed to cross your own path. So you will just trace out the $(2, 1)$-torus knot, twice. Even if you wobble your path slightly to avoid retracing the same path from the half-way mark, you'll find you have to cross your own path at some later point. The key thing here is that 4 and 2 have a common factor (of 2). But provided we have 2 numbers – call them p and q – which have no common factor other than 1 we can make a (p, q)-torus knot.

So, living on Donutworld, in fact, just by walking in loops around your planet, you can trace out an infinite variety of knots – which is surely much better than Earth's rather barren set of possibilities!

Do-It-Yourself Torus Knots

Here's the good news: you don't need to migrate to a different galaxy with donut-shaped planets to make torus knots! In fact, (and as seen in this issue's edition of *Knot Man*), you can make them in the comfort of your own home on your own pseudo-spherically shaped planet. It's very easy to construct some torus knots. I will show you how to make torus knots which go twice around the longitude – i.e. the torus knots of the form $(n, 2)$. (Note that since n and 2 can't have a factor in common, n must be odd).

Here is how to do it. Get a strip of paper from your local bookshelf, newsagent or photocopier. Preferably it should be narrow and long.

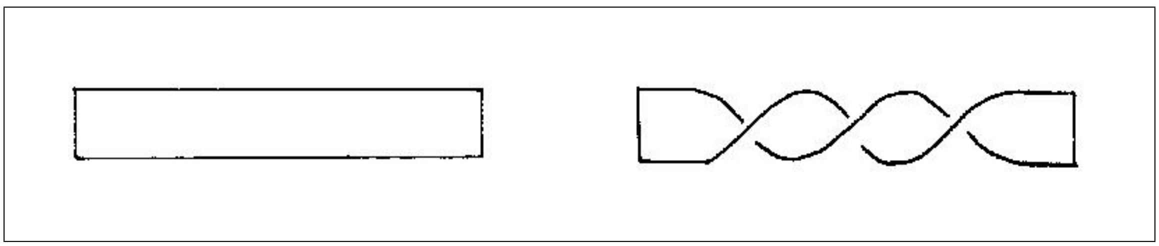


Figure 8: How to make a torus knot - part 1.

Now twist the strip of paper around. You could give the strip half a twist, or a full twist, or three half twists, two full twists, and so on. But stop after an *odd* number of half twists. Then glue the two ends of the strip together to create what I call (very technically, of course) a *twisty-strip*.

If you just did one half twist, then you now have a Möbius strip! Remember that the Möbius strip has only one side – you can walk around a Möbius strip and come back to the same point, upside down! If you did an odd number of half twists, then you should find that the twisty-strip you have just created also has only one side. (Technically speaking, all these "twisty-strips" are embeddings of the Möbius strip).

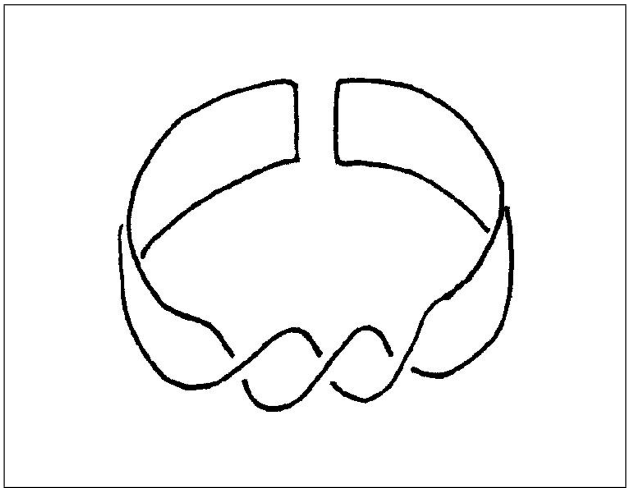


Figure 9: How to make a torus knot - part 2.

Now, find the middle of your twisty-strip, and get your scissors out! Cut along the middle line of the strip, and keep cutting until you get back to where you started. You might think that your twisty-strip should now fall into two pieces, but if it does, you've screwed up! If you have followed these instructions, your twisty-strip should not fall apart, and instead should now pull apart magically to become a torus knot! Specifically, if you gave n half twists at the start (and n is odd) then you have the $(n, 2)$ torus knot.

Doing this, you can discover that the $(1, 2)$ torus knot is the unknot. And you can easily construct the trefoil knot. What happens when you perform an even number of half-twists instead?

Why does this work? The following is a good way to think about it. An untwisted paper strip can be placed lengthwise inside a long, hollow cylinder.

Now when you twist your strip around, the cylinder twists around as well. But it is still exactly the same cylinder. And when you glue the ends together, you glue the ends of the cylinder together to get a donut. The edges of the twisty-strip now trace out a path on the donut, which covers 2 longitudes, and n meridians. But the twisty-strip connects the path all together. You need to separate the two edges of the twisty-strip from each other to get the knot, which is why you need to cut along the middle of it. And hence, the knot!

— Daniel Mathews

A Maths Joke

A group of mathematicians and a group of physicists met on the train, bound for a conference in Sydney. One of the physicists said, “I hope you absent-minded mathematicians have remembered to buy a ticket!”

To this, the mathematicians replied, “Actually, we only have ONE ticket between all of us.”

“How’re you going to get away with THAT?” cried the physicists, hoping that the mathematicians would be caught.

“You’ll see...” said the mathematicians coolly.

At that moment, the ticket inspector entered the carriage and the mathematicians all bolted to the other end and squeezed into one of the tiny toilet cubicles. The inspector checked all of the physicists’ tickets. Then, the physicists looked on with amazement as the ticket inspector knocked on the toilet door and cried out, “Tickets please!” Beneath the door slid the mathematicians’ only ticket which the inspector checked before moving on. After he had disappeared from the carriage, the mathematicians exited the toilet cubicle, grinning at the physicists’ astonishment.

It so happened that after the conference, the mathematicians and the physicists met once again on the train heading back to Melbourne. The physicists were looking very smug as they said, “We’ve only got ONE ticket between the lot of us this time!”

But the mathematicians replied, “Really? Well, we don’t even HAVE a ticket between the lot of us!”

“How’re you going to get away with THAT?” cried the physicists, sure that the mathematicians would be caught this time.

At that moment, the ticket inspector entered the carriage. The physicists all squeezed into one of the toilet cubicles and slammed the door shut while the mathematicians all squeezed into the toilet cubicle next to it. Then one of the mathematicians walked over to the other toilet, knocked on the door and cried out, “Tickets please!”

MUMS Seminars

The MUMS seminars, a series of informal talks on a wide range of topics in mathematics, form a regular fixture on the calendar of MUMS activities. Two seminars were held during the closing stages of last semester. The first of these was entitled “Mathematical Origami” and was given by Dr Burkard Polster of Monash University. Some of the topics included were folding cubes, shaping a parabola from a flat sheet of paper, unit origami, how to send gigantic telescope lenses on a thin rocket, and finally construction of dragon fractals by simple folds. Dr. Polster is a very entertaining speaker, so be on the look out for more talks from him this semester.

The second seminar was the screening of the video “N is a Number: A Portrait of Paul Erdös”. The video described the life of this enigmatic genius, his approach to mathematics and the way he influenced the people around him. Short and direct, the video gave vivid impressions about Erdös’ philosophy, his love for mathematics, and the legacy that is mathematics. The department has a collection of mathematics related videos (including this one) for loan to students. Please enquire at the general office if you would like to borrow them.

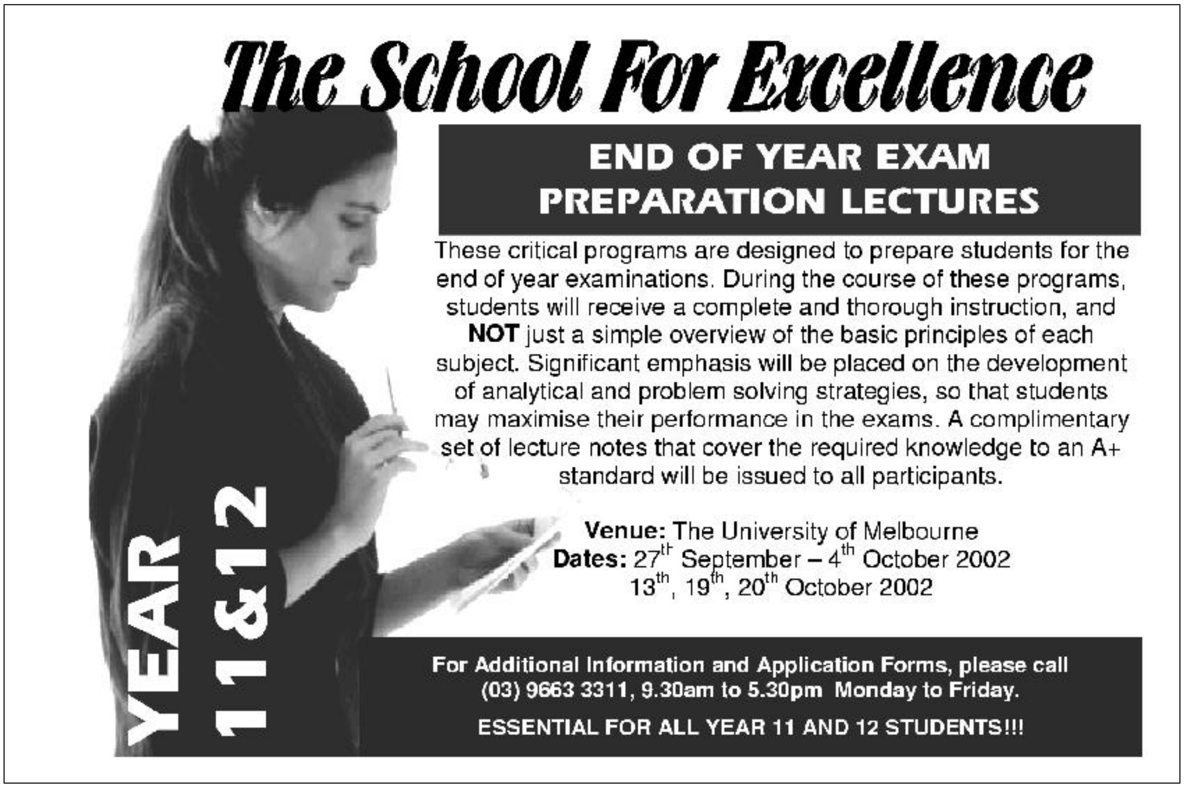
There will be many more seminars coming this semester. They are generally at 3:15pm on Fridays in the Thomas Cherry Room, followed by free refreshments. If you went to the last two seminars, I hope that you have enjoyed them. If you didn’t, do come to see one for yourself. You can subscribe to the MUMS events mailing list by sending your name and email address to `mums@ms.unimelb.edu.au`, to receive information on MUMS seminars and other MUMS related activities. Finally, if you are interested in giving a seminar, please email me at `g.zhang@ugrad.unimelb.edu.au`.

— Geordie Zhang, MUMS Education Officer


KNOT MAN
ISSUE #1
BY DANIEL MATHEWS & PRISCILLA BROWN
IT WAS A NORMAL DAY, LIKE ANY OTHER IN THE MATHS DEPARTMENT.
BLAH BLAH BLAH KNOT COMPLEMENT BLAH WHY CAN'T I GET THIS BACK BLACKBOARD TO WORK?
Theorem
THEODORE J KNOTT WAS BASKING IN THE WARM GLOW OF KNOT THEORY, SOCKS AND SANDALS AND MUNCHING ON HIS FAVOURITE SNACK IN CLASS...
MMM... GENUS-1 HANDLEBODIES!
MEANWHILE, OVER AT LATRASHKIN UNI, THE EVIL VICE-CHANCELLOR $ WAS PLOTTING WITH HIS ECONOMIC RATIONALIST HENCHMEN.
AND SO YOU SEE, THE DOWNTURN IN DEMAND ELASTICITY OF OUR STOCK PROFIT IS POSITIVELY CORRELATED TO...
...KNOT THEORY!
AH, YES KNOT THEORY HAS GOT TO GO!
JUST KNOT PROFITABLE ENOUGH HA HA HA.
PURE MATHS DEPARTMENT
...YES, SOON THEY WILL ALL BE DEPARTING!
TAKE HIM AWAY BOYS!!
$\mathbb{Z} \times \mathbb{Z}$
$ab = ba$
@$!%*#
Theorem
QED.
AND ALL THE PROFESSORS WERE TAKEN AWAY AND LOCKED UP IN THE COMMERCE DEPARTMENT.
...AND YOU WILL ALL STAY HERE UNTIL YOU EACH PRESENT ME WITH YOUR OWN BUSINESS PROPOSAL!
NEVER!!!
SAY, THESE ECONOMICS BOOKS ARE ALL TRIVIAL.
WHERE'S THE COFFEE?
THESE ASTROLOGY BOOKS SEEM TO MAKE MORE SENSE.


THEODORE J KNOTT AND HIS MATHSY FRIENDS WERE ENGROSSED IN CONVERSATION.
PARADOX HAS A NEW COMIC! COOL, HUH?
I WONDER IF PEOPLE IN THE COMIC WILL TALK ABOUT THE COMIC . . . OR ABOUT THEMSELVES TALKING ABOUT. . .
HEY GUYS! LISTEN TO THIS!
. . . AND THEY'RE ALL LOCKED UP IN THE COMMERCE DEPARTMENT!
EWWW!
WHAT CAN WE DO TO HELP?
HOW WILL THEY SURVIVE?!
HOW DO YOU PRONOUNCE THAT $ GUY'S NAME ANYWAY?
THIS SOUNDS LIKE A JOB FOR . . .
AND SO, WITH A MOUTHFUL OF HIGH ENERGY GENUS-1 DONUTS, AND DONNING HIS TOPOLOGICAL UTILITY BELT.. THEODORE J KNOTT BECOMES . . .
GLUE
KNOT MAN!
AND SO WITH TREFOIL T-SHIRT AND REPLETE WITH SOCKS AND SANDALS KNOTMAN SETS OFF TO LATRASHKIN UNIVERSITY..
. . . BY TRAIN OF COURSE LIKE EVERY OTHER UNI STUDENT.
HEY! NICE CAPE, MAN!
BRIGHTER THAN A MEDIUM-SIZED SUPERNOVA . . .
FASTER THAN A CRAY SV1
TOUGHER THAN THE RIEMANN HYPOTHESIS. . .
MORE ELEGANT THAN EUCLIDEAN GEOMETRY . . .
MORE DARING THAN A ROOM FULL OF PHYSICAL CHALLENGES...
UNBENDING AS THE THIRD YEAR ALGEBRA EXAM. . .
WELCOME TO LATRASH
PHYSICS
WOW! LOOK AT THAT RED SHIFT!
COMMERCE
COMMERCE
DEPRECIATION BLAH BLAH...
INTERNAL LABOUR MARKET... BLAH...
TAX DEDUCTION... BLAH... BLAH...
WHERE ARE THEY?


TRIVIAL... BLAH BLAH... NONSENSE BLAH COFFEE...
THIS KNOT THEORETICAL STRING COMES IN REALLY HANDY SOMETIMES!
EVENTUALLY KNOTMAN SCALES THE WALL AND ENTERS THE COMMERCE DEPARTMENT.
SMASH!
LET THEM GO $, YOU VILLAIN!
NEVER, KNOTMAN!! WHAT ARE YOU GOING TO DO, DIFFERENTIATE ME?!
THE VILLAINS ATTACK . . .
KNOTMAN! THE TREFOIL IS THE (3,2)-TORUS KNOT!
OF COURSE!
LET ME JUST GET MY SUPERSTRENGTH TRIPLE-HALF-TWIST MOBIUS STRIP!
WHAT PRACTICAL APPLICATION DOES THAT HAVE, YOU ACADEMIC?! ARE YOU TRYING TO DISORIENT ME?
?
SCISSORS ARE OF NO USE AGAINST MY POWER, TANGLE BOY!
TAKE THIS $-MEN!
(AND HOW DO YOU PRODUCE THAT IN TEX?)
NOOO!! TRAPPED IN A TREFOIL KNOT! LET US GO!
NO! I'M A FRAYED KNOT!
AND REMEMBER KIDS, YOU CAN TRY THIS AT HOME!
QED

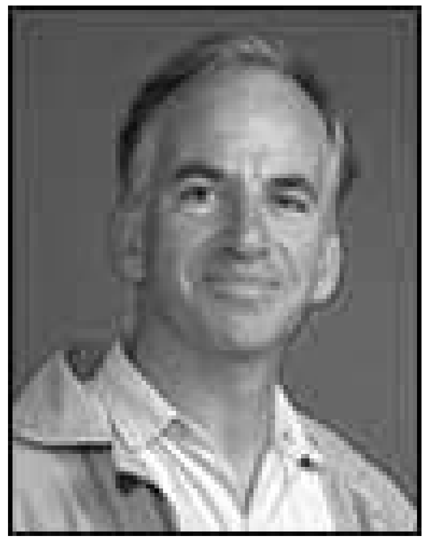


***Knot Man*'s Favourite Joke!**

Three strings went into a bar and sat down at a table. The first one got up, went over to the bar, and said to the bartender, "I'll have three Scotches." The bartender said, "We don't serve your kind here." "What kind?" said the string. "Strings, we don't serve strings here." So the string went back to the table and said to the other strings, "They won't serve us here." The second string said, "Oh yeah, we'll see about that." He got up, went over to the bar, pounded on the bartop, and said, "Hey bartender, I want three Scotches." The bartender said, "I told your friend, and now I'm telling you, we don't serve strings here. Now beat it." The string went back to the table and shrugged. The third string stood up. "Let me handle this", he said. He tied himself into a nasty tangle and pulled the strands out at his end, creating a wild mop of a hairdo. Then he walked over to the bar, leaned over close and said, "Bartender, I want three Scotches and I want them now." The bartender turned around and looked at him. He looked him up and down. Then he said, "You're not fooling me, you're one of those strings, aren't you?" The string looked him straight back in the eye and said, "Nope, I'm a frayed knot."

Tony Guttmann

The Australian Mathematical Sciences Institute (AMSI) recently received funding to the tune of $1 million from the Victorian Government. One of the main lobbyists for the creation of such an institute, the first of its kind in the country, is Professor Tony Guttmann. Tony is currently the Acting Head of the Department of Mathematics and Statistics at The University of Melbourne, incoming President of the Australian Mathematical Society, and a competitive triathlete.



Tony Guttmann grew up in Melbourne, attending Wesley College, and from an early age harboured an interest in mathematics. He says, "When I was a child and we went to restaurants, I recall tallying the bill for entertainment!" This aptitude for mathematics led Tony to study Science at Melbourne University, continuing on to a Masters in physics, and finally a PhD in applied maths at the University of New South Wales. "Things were different for students back then," Tony explains. "Lecturers would address you as Mr or Miss and made it clear that they were to be addressed by their proper titles."

The next two years, Tony spent in London, working at the Wheatstone Physics Laboratory, University of London, before accepting a position in the maths department at The University of Newcastle. It was not until years later in 1986, after becoming a professor and dean of his faculty, that Tony resigned and decided to return to his home town of Melbourne. Since then, he has enjoyed a remarkable mathematical career with the Statistical Mechanics and Combinatorics research group in our very own department. Tony's current research involves devising algorithms and analytical techniques to count combinatorial objects. An example of such a problem which he is currently working on is the following:

Consider an $n \times n$ chessboard with a rook placed in the bottom left corner. It must move to the top right square but without ever passing through the same square twice. How many distinct paths are there for the rook to achieve its goal?

Amazingly enough, the interest in this problem arises from its application to telecommunication networks – determining the number of paths through a square array of nodes.

Apart from supervising six postgraduate students, continuing his research in mathematics and handling all of the administration involved in being head of the department, Tony still manages to find the time to pursue other matters. For example, he is the co-organiser of The University of Melbourne Schools Mathematics Competition, sits on the advisory board for the Board of Studies, was chair of the organising committee for the recent Formal Power Series and Algebraic Combinatorics conference and is the incoming President of the Australian Mathematical Society. In particular, as mentioned earlier, Tony has played an integral part in the Australian Mathematical Sciences Institute bid. Until now, Australia was the only developed country in the world that did not have such an institute. After trying for over ten years, initial funding of $1 million has recently been pledged by the Victorian Government, a sum which has been augmented by The University of Melbourne, various other universities from around the country, and CSIRO. Tony has been appointed the interim director of AMSI, having co-written the proposal to the government. He says that this national collaborative venture promises "to provide a focus for mathematical activity" and will help to "address some of the challenges facing the mathematical sciences in Australia".

But of course, life isn't all about maths for Professor Tony Guttmann. Apart from having a love of literature and being a listener to classical and modern music, many around the department will know that he is a seasoned, competitive triathlete. Among his sporting achievements, he has run eleven marathons, represented Australia in the Duathlon World Championships in France and is looking forward to competing in the upcoming World Masters Games to be staged right here in Melbourne. Tony modestly remarks that his success as a triathlete is "more a triumph of determination than sporting ability... most of the competitors in my age group are either injured, retired or dead!"

The International Mathematical Olympiad

As you all know, every so often Australia sends teams of talented athletes to compete in the Olympic Games. A lesser-known fact, though nonetheless interesting, is that for some time Australia has been sending teams of geeky[1], nerdy, high school students to compete in the International Mathematical Olympiad (IMO). Can maths nerds really be thought of as intellectual athletes, the champions in their chosen field? Apparently so. In July this year a team of six year twelve students represented Australia at the IMO in Glasgow, Scotland.

The IMO is an annual event and consists of two gruelling exams, each lasting $4\frac{1}{2}$ hours and each asking only three questions. It's pretty tough. In fact, this year 21 students travelled all the way to Glasgow and ended up scoring no marks at all, not even one. It's little wonder then that most of the contestants train very seriously. For instance it is rumoured that in Iran the IMO team members attend a maths camp where they learn only maths for 9 months! Whether or not this rumour is true, you must believe that some of the people attending the IMO are maths geeks of the highest calibre — perhaps the most geeky people in the whole world.

And so, you might ask, how do our Australian maths geeks compare to the rest of the world? This year the team won 1 gold medal, 2 silver medals, 1 bronze medal, and Australia was placed 26th out of 84 countries. This fine effort was achieved by the following six students:

NAME	SCHOOL	STATE
David Chan	Sydney Grammar School	NSW
Andrew Kwok	University High School	VIC
Nicholas Sheridan	Scotch College	VIC
Gareth White	Hurlstone Agricultural High School	NSW
Stewart Wilcox	North Sydney Boys High School	NSW
Yiying Zhao	Penleigh and Essendon Grammar School	VIC

For those people interested in seeing what an IMO exam is like, the questions from this year's competition appear below. The solutions, as well as other information, can be found at `http://www.kalva.demon.co.uk/imo.html`.

[1] When I use the words "geeky" and "nerdy" I am not being derogatory. One of my best friends was once voted geek of the month. I am also a geek I suppose (thought not to the extent of being voted geek of the month).

Problems from the 2002 IMO

1. Let n be a positive integer. Let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is coloured red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \leq x$ and $y' \leq y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y-set to be a set of n blue points having distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

2. Let BC be a diameter of the circle Γ with centre O. Let A be a point on Γ such that $0^\circ < \angle AOB < 120^\circ$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. Prove that J is the incentre of the triangle CEF.

3. Find all pairs of integers $m, n \geq 3$ such that there exist infinitely many positive integers a for which
$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$
is an integer.

4. Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$ where $1 = d_1 < d_2 < \cdots < d_k = n$.
Define $D = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$.
   (a) Prove that $D < n^2$.
   (b) Determine all n for which D is a divisor of n^2.

5. Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that
$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$
for all x, y, z, t in $\mathbb{R}$.

6. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centres by $O_1, O_2, \ldots, O_n$, respectively. Suppose that no line meets more than two of the circles. Prove that
$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

Farey Sequences

What is a Farey Sequence?

Suppose we start with our favourite positive integer, which just so happens to be 7. Now we write down all of the fractions from 0 to 1 in increasing order whose denominators are at most 7. Of course, there will be some overlap since there may be more than one representation for a given fraction – e.g. $\frac{2}{3} = \frac{4}{6}$. So we will only consider fractions $\frac{p}{q}$ which are in lowest terms; in other words, the greatest common divisor of p and q is 1. We now have a sequence which looks very much like the following:

$$\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}$$

This is known as the Farey sequence of order 7 and in a similar manner, we can write down a Farey sequence of order N, which we will denote by F_N. The first several Farey sequences are listed below and as we will soon see, they have some remarkable properties.

$$F_1 : \frac{0}{1}, \frac{1}{1}$$
$$F_2 : \frac{0}{1}, \frac{1}{2}, \frac{1}{1}$$
$$F_3 : \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$$
$$F_4 : \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$$
$$F_5 : \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$
$$F_6 : \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}$$

What is so special about them?

FAREY FACT 1: Now let's pick out two consecutive fractions in our sequence which we have denoted F_7 – for example, $\frac{4}{7}$ and $\frac{3}{5}$. If we multiply the numerators of each fraction by the denominators of the other and take the difference, we get the result:

$$4 \times 5 - 3 \times 7 = -1.$$

We can do the same for the next pair of consecutive fractions in the sequence – $\frac{3}{5}$ and $\frac{2}{3}$. This time, we get an answer of:

$$3 \times 3 - 5 \times 2 = -1.$$

And since we are in the mood for it, we might as well try the same calculation for the next pair of consecutive fractions in the sequence – $\frac{2}{3}$ and $\frac{3}{4}$:

$$2 \times 4 - 3 \times 3 = -1.$$

By now, you probably have the sneaking suspicion that it is not mere coincidence that we keep getting an answer of -1. This is confirmed by the amazing result which states that:

"If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive fractions in any Farey sequence, then $ad - bc = -1$."

(The astute reader may have already noticed that the expression $ad - bc$ is merely the determinant of the matrix with entries a, b, c, d.)

FAREY FACT 2: When given two fractions, for example $\frac{6}{11}$ and $\frac{3}{5}$, to sum together, a common mistake for the naive student is to add the numerators and add the denominators to give the result $\frac{6+3}{11+5} = \frac{9}{16}$. This is clearly fallacious and the veteran adder of fractions will recognise immediately that this is not the desired sum. In fact, given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, mathematicians refer to the expression $\frac{a+c}{b+d}$ (obtained by adding the numerators and denominators) as the *mediant* of the two fractions.

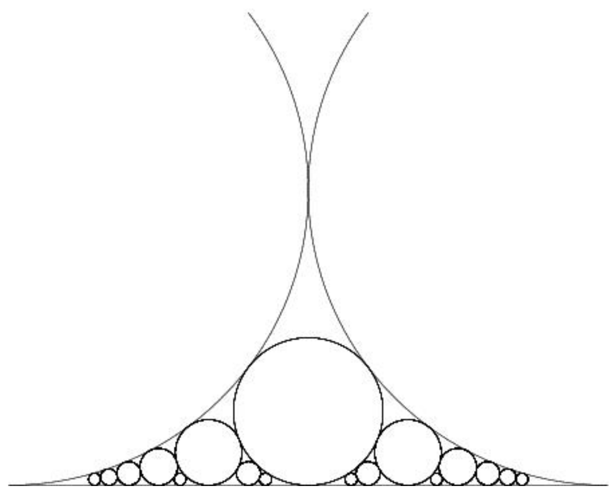
Armed with this new piece of knowledge, let us consider three consecutive fractions from the sequence F_7 – $\frac{1}{2}$, $\frac{4}{7}$ and $\frac{3}{5}$. Then we can make the observation that the mediant of $\frac{1}{2}$ and $\frac{3}{5}$ is $\frac{1+3}{2+5} = \frac{4}{7}$.

We could also have considered another group of three consecutive fractions from the sequence – $\frac{4}{7}$, $\frac{3}{5}$ and $\frac{2}{3}$. Once again, the mediant of the outer two fractions is $\frac{4+2}{7+3} = \frac{6}{10}$, which after cancellation leaves us with the middle term $\frac{3}{5}$. Once again, this is no mere coincidence, but a deep result about fractions which can be stated thus:

"If we consider three consecutive terms of any Farey sequence, then the middle term will be the mediant of the other two."

FAREY FACT 3: The *Ford circle* for the fraction $\frac{p}{q}$ is defined to be the circle which is tangent to the real number line at the point $\frac{p}{q}$ and which has diameter $\frac{1}{q^2}$. (There are actually two such circles, one above the line and one below – however, we will only consider the Ford circles which lie above the real number line.)

Now consider the terms of a Farey sequence F_N on the real number line. For each term of the Farey sequence, suppose we draw its corresponding Ford circle. For the case of $N = 7$, we obtain the pretty picture shown below.



The first thing to note is that no two circles intersect in more than one point. Secondly, we can determine exactly when two Ford circles will touch using the following result:

"The Ford circles for $\frac{a}{b}$ and $\frac{c}{d}$ will be tangent if and only if $ad - bc = -1$."

So this tells us that since $2\times7-5\times3 = -1$, the Ford circles which touch the real number line at $\frac{2}{5}$ and $\frac{3}{7}$ (and have diameters of $\frac{1}{25}$ and $\frac{1}{49}$ respectively) will be tangent to each other.

Of course, all of these results hold not only for the sequence F_7 but for all Farey sequences. So if your favourite positive integer had not been 7, but 32, 69 or perhaps even 666, then the three Farey facts stated above would still hold.

Farey, Farey, Quite Contrary...

Many mathematical results are named after the person who supposedly discovered them, such as Pythagoras' Theorem, Euler's Formula and Steiner's Porism. However, it is a well-known fact among mathematical historians that a large proportion of these results have been attributed to the wrong people. In fact, we can record this as Norm's Theorem:

"If Theorem X bears the name of Y, then it was probably first stated by and/or proved by Z."

Farey Sequences are a striking example of the aforementioned Norm's Theorem, since Farey never proved anything about Farey Sequences whatsoever. In fact, John Farey (1766-1826) was actually a geologist who is described by mathematicians Hardy and Wright as being "at best an indifferent mathematician". Despite his amateur status in mathematics, Farey stumbled upon the fact that each fraction in a Farey Sequence is the mediant of its two neighbours (which we called *Farey Fact 2* above). This prompted Farey in 1816 to write a letter to the *Philosophical Magazine* stating the result, along with the following addendum:

"I am not acquainted, whether this curious property of vulgar fractions has been before pointed out?; or whether it may admit of some easy or general demonstration?; which are points on which I should be glad to learn the sentiments of some of your mathematical readers..."

The challenge of finding a proof to this "curious property" was met by one of the most prolific mathematicians of the age, Augustin Cauchy. For some unknown reason, Cauchy decided to attribute the result to Farey himself, and it is for this reason that the English geologist's name has been immortalised in the world of mathematics.

However, in addition to this, subsequent mathematicians realised that not only did Farey not have a proof of his "curious property", but he was not even the first person to notice it. That honour goes to a certain Mr C Haros who anticipated Farey by fourteen years in a paper where he made use of the result. So it seems that once again, a mathematical result has been misnamed; in this case, after a somewhat versatile man of no mathematical repute whatsoever. Nonetheless, credit must go to Farey for making such a simple and elegant observation, one which eluded such great mathematicians as Fermat and Euler. Anyway, who knows - it may be that some little-known mathematician anticipated both Farey and Haros by a thousand or so years.

Paradox Problems

The following are some maths problems for which prize money is offered. The person who submits the best (i.e. clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number. Solutions may be emailed to `paradox@ms.unimelb.edu.au` or you can drop a hard copy of your solution into the MUMS pigeonhole near the Maths and Stats Office in the Richard Berry Building.

1. ($5) In a traditional LCD display some numbers, when viewed upside down, are images of other numbers. For example, 1995 becomes 5661. The first few numbers which can be read upside down are $1, 2, 5, 6, 8, 9, \ldots$. What is the millionth number that will appear in this sequence?

2. ($5) A cylindrical hole six centimetres long has been drilled straight through the centre of a solid sphere. What is the volume remaining in the sphere?

3. ($5) Luke's calculator has a ten digit display but only the last four digits are working. He enters a four digit number into the calculator and squares it. To his surprise the display remains the same. What was the number?

4. ($10) The words of a mathematical problem are numbered in alphabetical order. Then the first word of the problem is written in the position denoted by 1, the second word in the position denoted by 2, etc. The result is:

   "FIVE RANDOM ORDER IS EIGHT THAT NUMBERS SIX ONE SQUARE FOUR ARE THE WHAT A WRITTEN DIGIT IS RESULTING NUMBER PROBABILITY AND THREE IN DOWN THE THE"

   Solve the (mathematical) problem!

5. ($10) After a mutual and irreconcilable dispute among Red, Black and Blue, the three parties have agreed to a three-way duel. Each man is provided a pistol and an unlimited supply of ammunition. Instead of simultaneous volleys, a firing order is to be established and followed until only one survivor remains. Blue is a 100 percent marksman, never

having missed a bull's-eye in his shooting career. Black is successful two out of three times on the average, and you, Red, are only a 1/3 marksman. Recognising the disparate degrees of marksmanship, everyone agrees that you will be the first, Black second, and Blue will come last in the firing order.

Your pistol is loaded and cocked. At whom do you fire?

Solutions to Last Issue's Paradox Problems

Here are solutions to some of the maths problems posted in the last issue of *Paradox*. Congratulations go to the following people for submitting correct solutions:

Problem 1: ($10) Geoffrey Kong
Problem 2: ($5) Ying Chee Lo
Problem 3: ($5) Daniel Lee Cameron

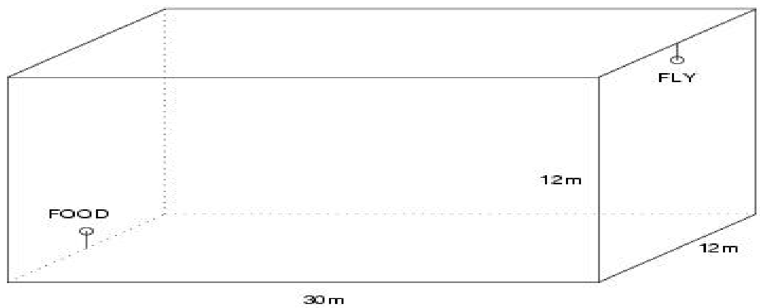
Please e-mail `paradox@ms.unimelb.edu.au` to pick up your prizes or drop by the MUMS room in the Richard Berry Building.

1. ($10) I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to the sum of the squares of the ages of the other three. In one year's time the sum of the squares of the ages of the oldest and the youngest will be equal to the sum of the squares of the other two children. Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

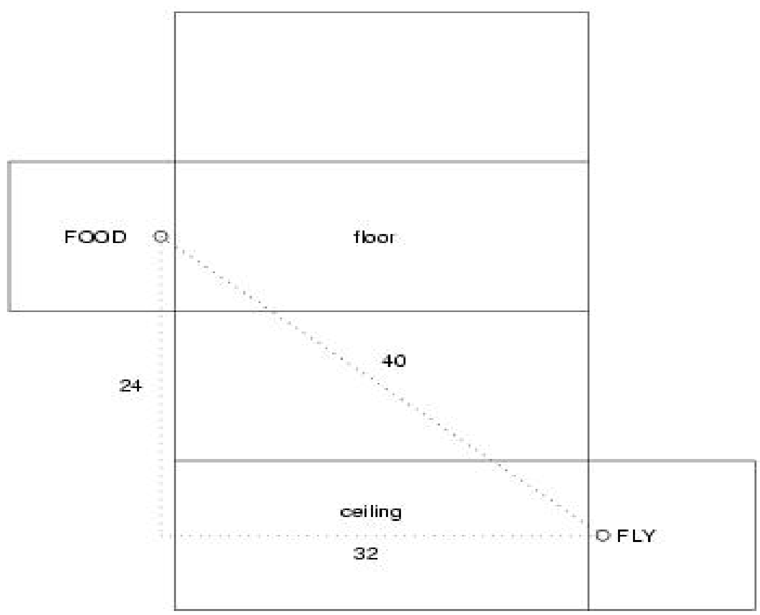
   Solution: There are actually two solutions for the ages of the children – $(3, 6, 15, 16)$ and $(3, 7, 10, 12)$.

2. ($5) Consider a room in the shape of a rectangular prism with a width of 12 metres, a length of 30 metres and a height of 12 metres. A fly stands in the middle of one of the walls, exactly 1 metre from the ceiling (as shown in the diagram below). On the middle of the opposite wall, exactly 1 metre from the floor, is a piece of food which the fly wants to eat. However, the fly is unfortunate enough to have broken one of

its wings, and can only reach the piece of food by walking along the walls of the room. What is the shortest distance that the fly can walk to get to the food?



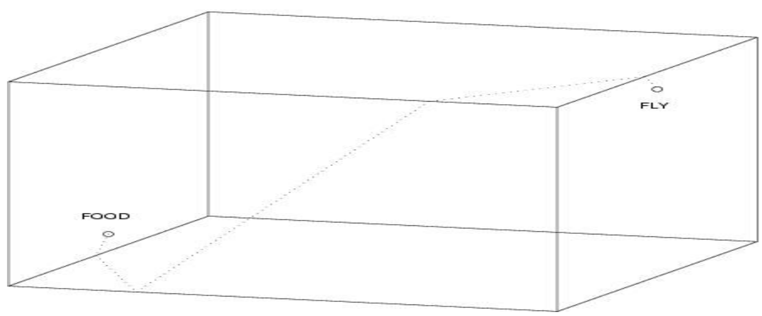
Solution: At first thought, it seems that the fly can reach the food after walking 42 metres in a somewhat obvious manner and that no shorter path exists. However, the following argument shows that amazingly enough, the fly can get to the food after walking along a shorter path.



Suppose we cut the room along its edges to create a flat "net" as indicated in the diagram above. The fly can then walk along the hypotenuse of the right-angled triangle indicated. And the length of this path can be calculated with the help of our good friend Pythagoras.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ \Rightarrow 24^2 + 32^2 &= c^2 \\ \Rightarrow c^2 = 576 + 1024 &= 1600 \end{aligned}$$

So using this path, the fly walks only 40 metres before reaching the food. The net can then be folded up again and the path can be traced along the walls of the room as shown in the diagram below.



3. ($5) Is it possible to place the digits from 1 to 9 (each exactly once) into the boxes to make this fraction sum correct?

$$\frac{\square}{\square\ \square} + \frac{\square}{\square\ \square} + \frac{\square}{\square\ \square} = 1$$

Solution: Yes, it most certainly is possible to do so. There is essentially only one solution (up to permutations of the fractions) as follows:

$$\frac{9}{12} + \frac{5}{34} + \frac{7}{68} = 1$$

Thanks

The *Paradox* team would like to thank the following people for their fantastic contributions to this year's issues: Leigh Angus, Priscilla Brown, Stephen Farrar, Jolene Koay and Daniel Mathews.