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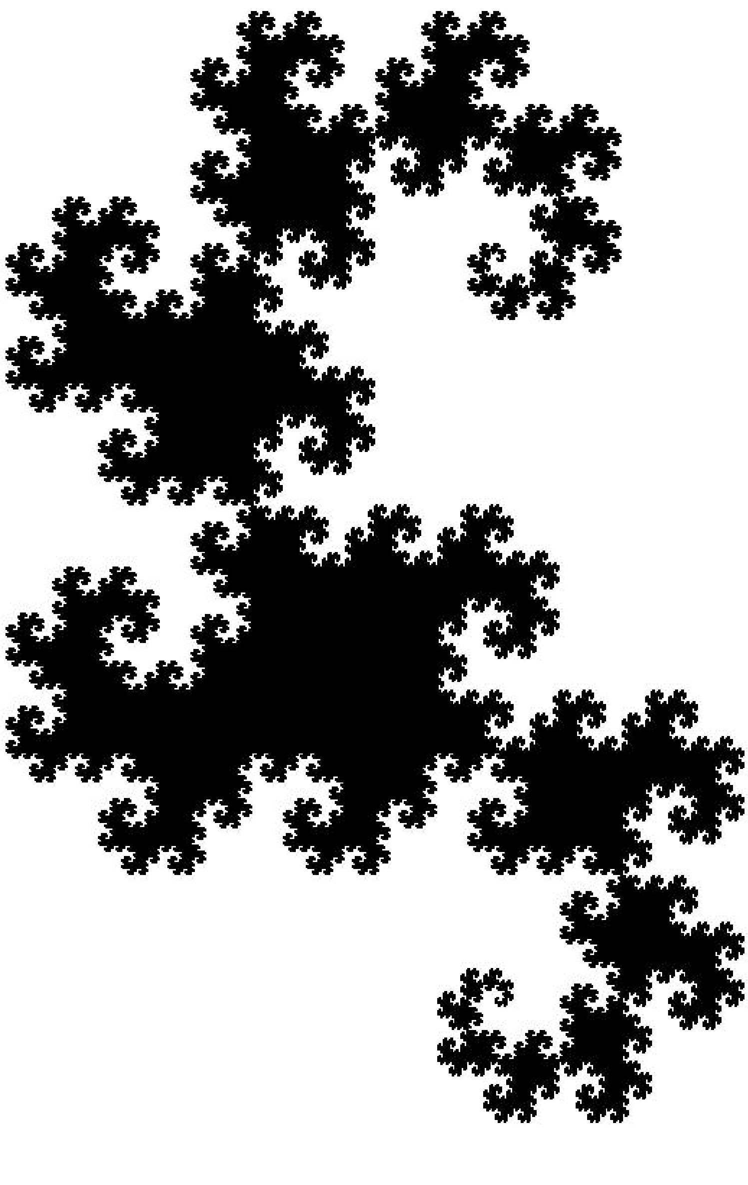
Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society



MUMS

President:	Damjan Vukcevic damjan@ms.unimelb.edu.au
Vice-President:	Julian Assange j.assange@ugrad.unimelb.edu.au
Treasurer:	Andrew Kwok a.kwok2@ugrad.unimelb.edu.au
Secretary:	Joanna Cheng j.cheng1@ugrad.unimelb.edu.au
Education Officer:	Maurice Chiodo m.chiodo@ugrad.unimelb.edu.au
Publicity Officer:	James Saunderson j.saunderson@ugrad.unimelb.edu.au
Editor of *Paradox*:	Nick Sheridan n.sheridan@ugrad.unimelb.edu.au
1st Year:	Bobbi Ramchen maple_leaf9@hotmail.com
2nd Year:	Michael de Graaf m.degraaf@ugrad.unimelb.edu.au
3rd Year:	Daniel Yeow d.yeow1@ugrad.unimelb.edu.au
Honours:	Chris Goddard c.goddard@ugrad.unimelb.edu.au
Postgraduate:	Norm Do norm@ms.unimelb.edu.au
Web Page:	http://www.ms.unimelb.edu.au/~mums
MUMS email:	mums@ms.unimelb.edu.au
Phone:	(03) 8344 3385

Paradox

Editor:	Nick Sheridan
Sub-editor:	Fendy Liauw
Layout:	Stephen Farrar
Web Page:	http://www.ms.unimelb.edu.au/~paradox
E-mail:	paradox@ms.unimelb.edu.au
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From the Editor

Students of mathematics and statistics, perusers of journals and papers, initiates to the wonders of Euclidean geometry, friends and allies of mathematics in all its forms, from the wise and learned to the young and moderately curious, welcome to the latest edition of Paradox! In this volume you will discover the secret to creating attractive fractals like the one on our front cover, hone your combinatorial skills, and come tantalisingly close to discovering the identity of the newest mathematical superhero of the department, Captain Continuous (our previous superhero, Knot Man, had to resign after being caught deriving while drunk – he claimed to have derived the Axiom of Choice!). You will find out what the hell Mersenne primes are and how they are linked to perfect numbers, learn how to make witty conversation about that perennial topic of dinner-party conversation, the chromatic number of the plane, and receive a timely reminder of the axioms of that sadly neglected practice of mathematical etiquette and political correctness.

And remember, you phantasms of mathematical spirit hidden deep within the heart of the fresh-faced first year student, the writing of Paradox articles is not only for Russian professors and people with large eyebrows, it is for you too! We rely on submissions from people all over the department, including students. Four of the articles in this edition were written by first-time Paradox authors. So if you have an idea for an article based on a cool problem that you have seen or invented, or something you have read somewhere, email me! We'd love to have you on board.

— Nick Sheridan, Paradox Editor

n.sheridan@ugrad.unimelb.edu.au

Suppose a mathematician parks his car, locks it with his key and walks away. After walking about 50 metres the mathematician realizes that he has dropped his key somewhere along the way. What does he do? If he is an applied mathematician he walks back to the car along the path he has previously travelled looking for his key. If he is a pure mathematician he walks to the other end of the parking lot where there is better light and looks for his key there.

Words from the President

The MUMS veterans among you will know that the second semester is traditionally much busier than the first, and this year that is still the case. We will be running our regular seminars as usual, as well as both of the Maths Olympics... but wait, there's more! We have two *completely new* events in the works!!

The first will be a social lunch that we like to call "Meet the Boss", where you will have the opportunity to meet Hyam Rubinstein (the Head of the Maths & Stats Department) and a few other professors, on social terms. This is a great opportunity to meet the people who make it all happen, without having to worry that they'll be asking you why your assignment is still a week late.

The other newbie on the calendar is "Maths Week" — a collection of events that celebrate the beauty and intrigue of mathematics, all packed into a mere five days. You will be able to witness mathematics in artwork, find out about research in the Department, and participate in the finale of the week: the University Maths Olympics! There have even been rumours of a maths-style Scav-Hunt-like competition...

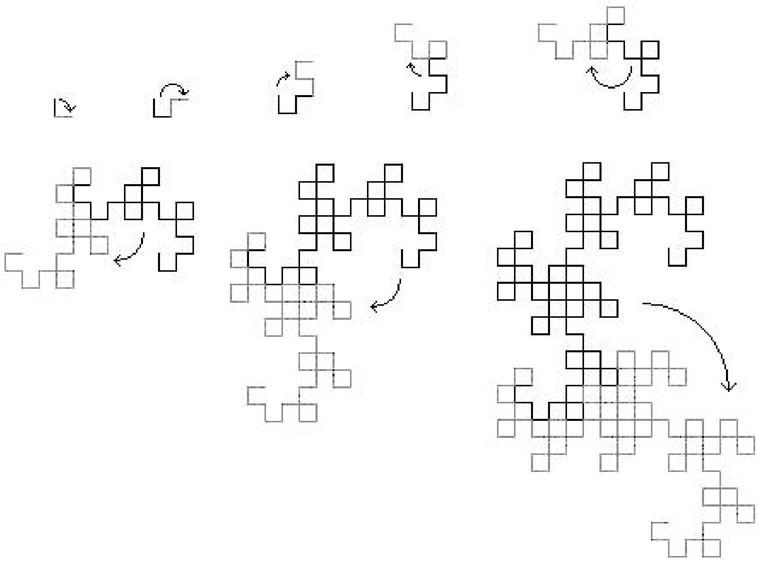
To add to the party atmosphere, we have decided to print some more of our very popular MUMS T-Shirts. In fact, they were so popular last time that they sold out. However, they'll soon be back... and not a moment too soon, with the Department about to bring out their own range of T-shirts. Pretty soon, we will have crazy maths fashion in every corridor!

— Damjan Vukcevic

Q: Why do Computer Scientists get Halloween and Christmas mixed up?
A: Because Oct. 31 = Dec. 25.

About the Front Cover

Have another look at the front cover. "Phwoaar!" you might say, or perhaps, "Humph! I have seen a fractal before." But this fractal[1] has been constructed in a very peculiar way. Let us suppose, for the sake of argument, that some bored group of people took it into their heads to take an enormous piece of paper, and to fold it exactly in half. Finding this not to have relieved their boredom, they fold it in half again, in the same direction as before, and again, and again, and again . . . Finally, when they have folded it an infinite number of times, they hand it to you. You start to unfold it, but you always unfold it so that, where originally there was a fold, there is now only a 90° fold. The first few steps are shown below, from side-on. The black bit represents the shape from the previous step, and the grey bit is the bit that has just been folded out.



The arrows are, in an air-stewardesque, obfuscatory sort of way, meant to show the direction the paper has just been folded. Already, the vague outline of the final picture is beginning to take shape. The final product can, of course, be seen on the front cover. The image on the front cover was produced in 'Paint' – I just used copy, paste and rotate 90° lots of times! An interesting fact about these shapes is that they tessellate – in fact, you can tessellate the whole plane with them!

— Nick Sheridan

[1] A fractal (vaguely) is a shape that repeats itself over and over again.

Proof by ... Combinatorics?

Sick of the standard recipe for induction? Confused by the logic used in proof by contradiction? Then you may want to consider using a combinatorial proof the next time you're solving a problem. While proof by contradiction and induction are certainly the two most common and widely applicable methods of proof, they can be quite fiddly and messy at times. In contrast, combinatorial proofs often provide more elegant solutions to the same problems.

"So what is a combinatorial proof?", you're probably asking at this stage. Well, the distinguishing feature in such a proof involves finding a physical interpretation to the result being proven. Usually you are given an algebraic expression and you want to interpret this as the number of different arrangements or combinations of some objects that satisfy particular conditions. Another common aspect of combinatorial proofs involves counting things in two different ways. As such, they're mostly used in proving mathematical identities.

Probably the best way to understand the power of combinatorial proofs is to go through a few examples. What follows is hopefully a selection of interesting results (each one named after a famous mathematician) which have constructive combinatorial proofs.

Pascal and Committees

Pascal's Formula is a well known result related to Pascal's triangle. While it can be proven algebraically, it is the classic example of using combinatorial proofs.

Pascal's Formula *If n and k are positive integers such that $n > k$, then*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. [1]

Proof: Suppose Paul is a member of the Maths and Stats Society, which has n members. The society is conducting its annual general meeting and needs

[1] $\binom{n}{k}$ is defined to be the number of ways of choosing k objects from n different objects (where the order of the k objects does not matter). It can be shown that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $n! = n \times (n-1) \times \ldots \times 2 \times 1$. In this article, it is not important to know this formula as long as you understand what $\binom{n}{k}$ stands for.

to elect a committee of k people from its n members. The question is how many different committees can be formed. This number can be counted in two different ways.

First way: Since k people need to be chosen from a group of n people, there are $\binom{n}{k}$ ways of forming the committee (this is the left hand side of the result to be proven).

Second way: Since Paul is a member, he may or may not be in the committee. If he is in the committee, then $k-1$ other people still need to be chosen from the remaining $n-1$ members. There are $\binom{n-1}{k-1}$ such committees. Otherwise Paul isn't in the committee, in which case the k committee people need to be chosen from the $n-1$ other members. There are $\binom{n-1}{k}$ such committees. So overall there are $\binom{n-1}{k-1} + \binom{n-1}{k}$ committees (the right hand side of the result).

Since both ways count the number of k-person committees chosen from n people, it follows that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, as required.

Fermat and Necklaces

Fermat's Little Theorem is a well-known result in number theory. The standard proofs of this result use induction or modular arithmetic and it is somewhat surprising that a combinatorial proof exists for this theorem.

Fermat's Little Theorem *If p is a prime number and n is any positive integer, then*

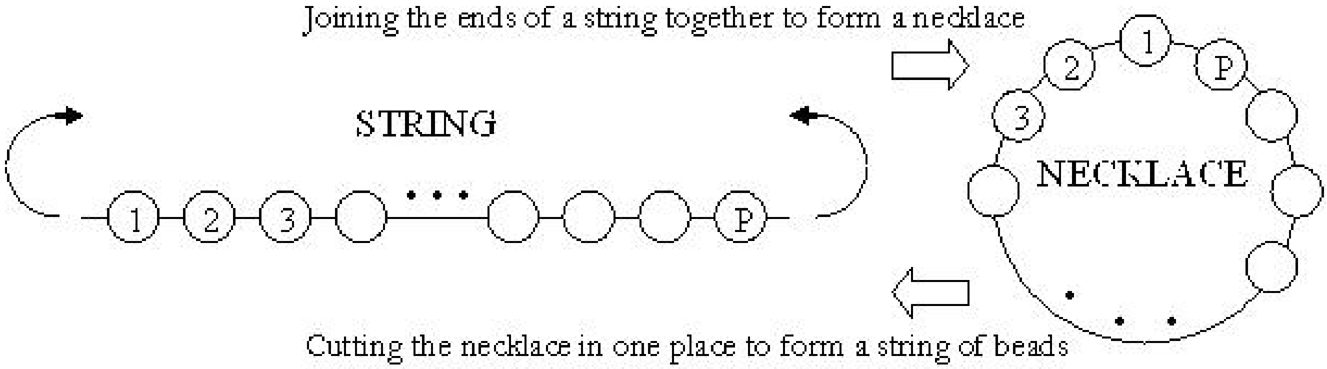
$$n^p - n \quad \textit{is divisible by } p$$

Proof: Suppose you want to make necklaces from exactly p beads. The beads come in n different colours and there are at least p beads of each colour.

To form these necklaces, you start by making strings of beads. There are n choices of colour for the first bead, n choices for the second bead, ..., n choices for the p^{th} bead. So there are $\underbrace{n \times n \times \ldots \times n}_{p} = n^p$ different strings of beads.

Since the beads come in n different colours, exactly n of these strings will have beads all of the same colour. After removing these n strings, you have $n^p - n$ strings left, each containing beads of at least two different colours.

You then join the ends of these remaining strings together to form necklaces, as shown below. Consider two necklaces the same if, visually, you can rotate one of the necklaces clockwise or anticlockwise to obtain the other. Otherwise, consider the necklaces as different.



Since each necklace has p beads, there are p different ways to cut a necklace in exactly one place. So you can obtain p (possibly different) strings of beads by cutting one necklace in different places. In fact, each of these p strings must be different. Suppose otherwise. Then you would be able to rotate the original necklace clockwise (or anticlockwise) by some number of beads x and obtain the original necklace again. Choose x to be the smallest such number of beads needed to rotate the necklace before obtaining the same necklace again. By the same argument as above, you can rotate the necklace clockwise again by x beads and get the same necklace. Continuing this argument, it follows that x must divide p as there are p beads. Since p is prime, then $x = 1$ or $x = p$. If $x = 1$, then each bead of the necklace must be the same colour. This is not possible since each necklace has beads of at least two different colours. Otherwise $x = p$, but in this case, each of the p strings will be different, by the definition of x. It follows that each distinct necklace can be formed from p different strings.

So out of the $n^p - n$ necklaces, there will be p of each different type of necklace. Then there are exactly $\frac{n^p-n}{p}$ distinct necklaces. Because of this combinatorial interpretation, it follows that $\frac{n^p-n}{p}$ must be an integer. So $n^p - n$ is divisible by p, as required.

Fibonacci and Carriages

The sequence 1, 1, 2, 3, 5, 8... is known as the *Fibonacci sequence*, where each term (from the third onwards) is the sum of the two preceding terms. The following is an interesting formula for the Fibonacci numbers and a very nice combinatorial proof exists for this identity.

A Fibonacci Identity *If the Fibonacci sequence is defined by $F_0 = 1, F_1 = 2$ and $F_{n+2} = F_{n+1} + F_n$ for all positive integers n, then for all non negative integers n,*

$$F_n = \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \ldots = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Proof: A train has n carriages, each being a passenger carriage or a meal carriage (you get to choose). Each passenger carriage is identical to every other passenger carriage and the same applies for meal carriages. The question is how many ways can you couple the carriages in a row (behind the train engine) so that no two meal carriages are next to each other.

First way of counting: Let F_n be the number of ways of arranging the n carriages in a row so that no two meal carriages are next to each other. We prove $\{F_n\}$ is the Fibonacci sequence.

There are F_{n+2} ways of arranging $(n+2)$ carriages so that no two meal carriages are next to each other. Of the $(n + 2)$ carriages, the front carriage is either a passenger carriage or a meal carriage. If the front carriage is a passenger carriage: then the remaining $(n + 1)$ carriages are also arranged so that no two meal carriages are next to each other. There are F_{n+1} such ways of arranging these $(n + 1)$ carriages. Otherwise the front carriage is a meal carriage: since no two meal carriages may be next to each other, the second carriage must be a passenger carriage. The remaining n carriages are arranged once again so that no two meal carriages are next to each other. There are F_n such arrangements. It follows that $F_{n+2} = F_{n+1} + F_n$.

Clearly $F_0 = 1$ since there is only one way of arranging zero carriages. $F_1 = 2$ since you can either have one passenger carriage or one meal carriage. Since $F_{n+2} = F_{n+1} + F_n$, then it follows that $\{F_n\}$ is the Fibonacci sequence 1, 2, 3, 5, 8. . . .

Second way of counting: Consider the following simpler problem first. Suppose the train has exactly y passenger carriages and z meal carriages. The question is how many ways can the carriages be arranged so that no two meal carriages may be next to each other.

Representing each of the y passenger carriages by a **P**, we write down an **m** between each **P**.

$$\underbrace{\mathbf{m\ P\ m\ P\ m\ P\ m\ P\ m \ldots m\ P\ m\ P\ m\ P\ m\ P\ m}}_{y\ P's}$$

Since there are y passenger carriages P, there are $y+1$ positions marked by an **m**. The z meal carriages can be chosen as any of the $y+1$ positions marked by an **m**. Any such placement ensures that no two meal carriages are next to each other. So there are $\binom{y+1}{z}$ ways of coupling the carriages together so that no two meal carriages are next to each other.

Note that if $y+1<z$, then $\binom{y+1}{z}=0$, by definition. This makes sense since, in this case, the carriages cannot be arranged so that no two meal carriages are next to each other.

Now back to the main problem. If the train has exactly n carriages, there can be

	n	passenger carriages and 0 meal carriages
OR	$n-1$	passenger carriages and 1 meal carriage
OR	$n-2$	passenger carriages and 2 meal carriages
		$\vdots$

If there are n passenger carriages and 0 meal carriages, then there are $\binom{n+1}{0}$ ways of arranging the carriages so that no two meal carriages are next to each other (using the above result). If there are $n-1$ passenger carriages and 1 meal carriage, there are $\binom{n}{1}$ ways. Continuing this in a similar manner, in total, there are

$$\binom{n+1}{0}+\binom{n}{1}+\binom{n-1}{2}+\ldots=\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n+1-k}{k}$$

ways of arranging the carriages so that no two meal carriages are next to each other. The terms in the sum, $\binom{n+1-k}{k}$, eventually satisfy $n+1-k<k$, in which case $\binom{n+1-k}{k}=0$. $k=\lfloor\frac{n+1}{2}\rfloor$ is the largest value of k satisfying $n+1-k\geq k$, so that the last non-zero term in the sum Σ is $\binom{n+1-k}{k}$, where $k=\lfloor\frac{n+1}{2}\rfloor$.

Comparing both ways of counting the number of arrangements of n passenger and meal carriages in a row where no two meal carriages are next to each other, it follows that

$$F_n = \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \ldots = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} \qquad \text{as required.}$$

Problems to try for yourself

Hopefully the above proofs will have inspired you to attempt your own combinatorial proofs:

- For each positive integer n, prove that $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$
- If n, r and k are positive integers such that $n \geq r \geq k$, prove that $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$
- For each positive integer n, prove that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

- For each positive integer n, prove that

$$\sum_{k=0}^{n} \binom{n}{k} k^2 = n(n+1)2^{n-2}$$

This is an amazing identity which you can derive from a combinatorial proof:

- For each positive integer n, prove that

$$n! = n^n - \binom{n}{1}(n-1)^n + \binom{n}{2}(n-2)^n - \binom{n}{3}(n-3)^n + \ldots + (-1)^{n-1}\binom{n}{n-1}1^n$$

— Andrew Kwok

Politically Correct Maths

Considerable effort has been expended in recent times to reform our language so as to make it less offensive and rid it of deprecatory connotations. However, mathematics has been left untouched. Below we try to address this deficiency.

1. Do not discriminate between x and y.

$$y \text{ or } x = m(x \text{ or } y) + b$$

$$f(x \text{ or } y) = \sin x \text{ or } y$$

2. Avoid terms with derogatory connotations.

INCORRECT	CORRECT
dummy variable	representative variable or generalized variable
negative value	nonpositive, nonzero value
discriminant	distinguisher
mean	expected value or average
irrational	not expressible as a terminating or recurring decimal

3. Avoid terms that indirectly deprecate something else.

   Do not use the normal distribution. It suggests that other distributions have something abnormal.

   Probability as a whole should be used with care. Assigning one event as less likely than another borders on discrimination.

   Avoid perfect squares for similar reasons. If they must be used, refer to them as squares of integers.

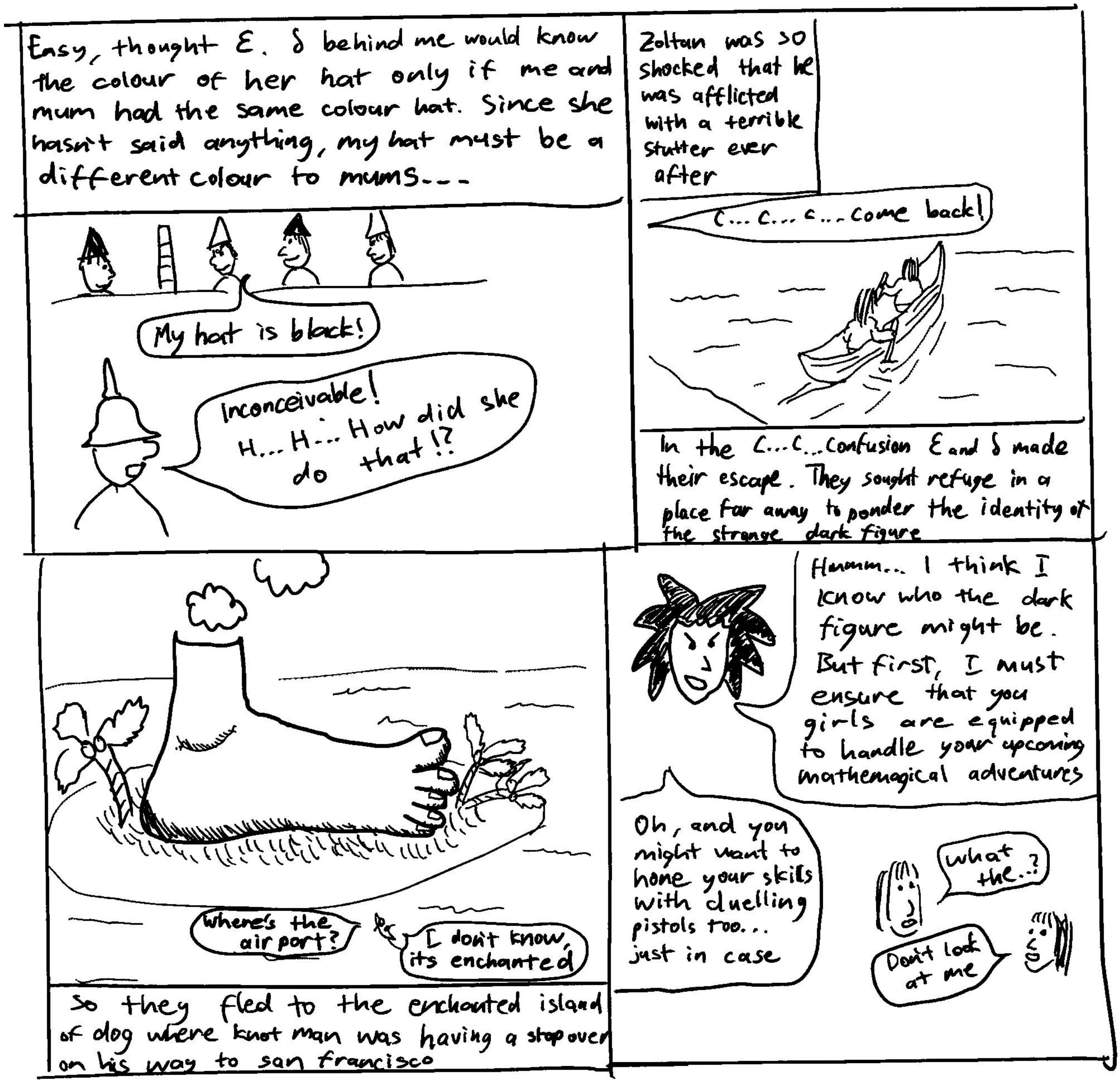
   Perfect numbers were carelessly named by idle mathematicians. The whole notion of such a number is discarded.

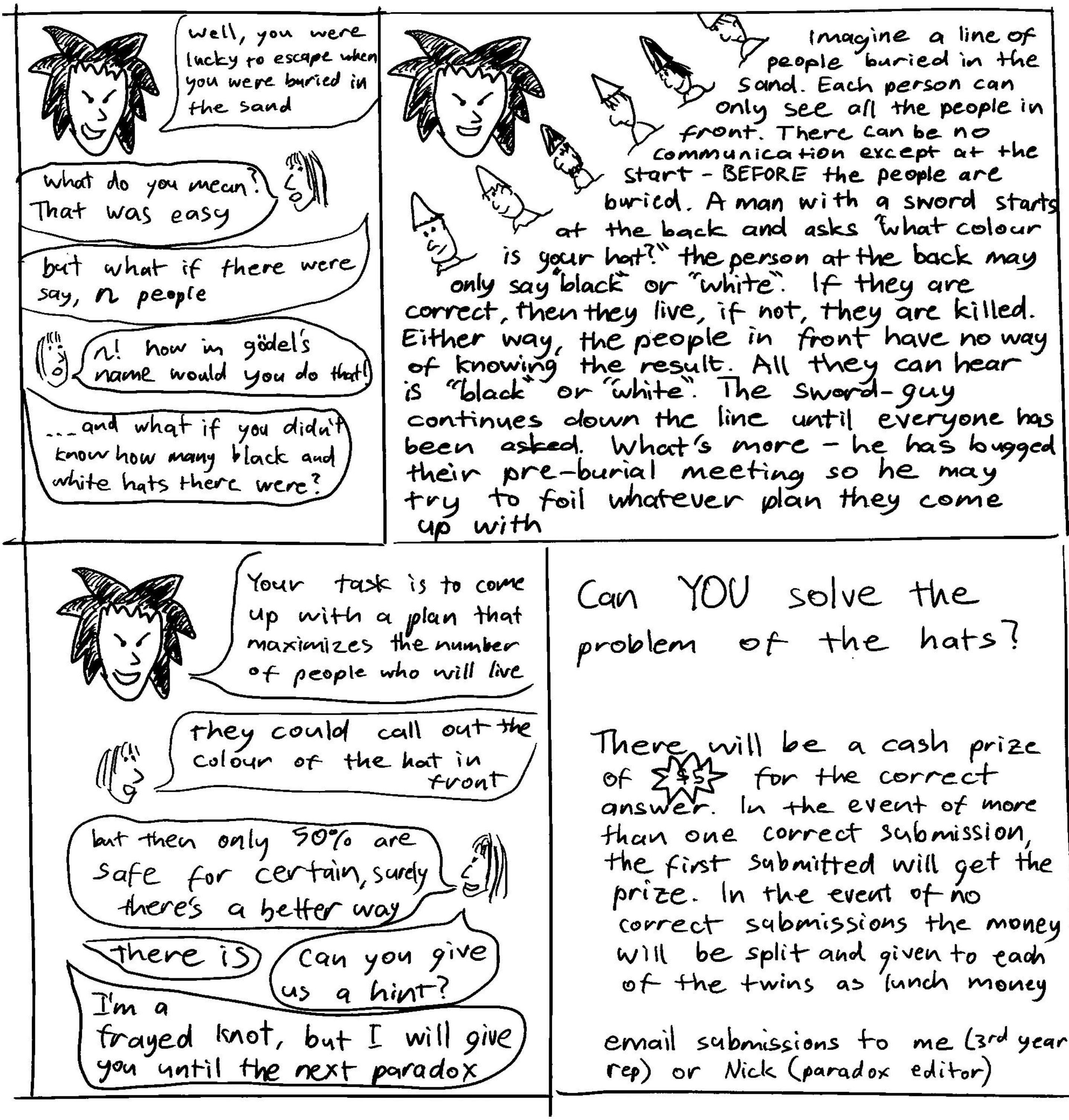
— George Politis

> "Mathematics is like checkers in being easy, suitable for the young, amusing, and without peril to the state."
>
> — Plato


The mathemagical adventures of
CAPTAIN CONTINUOUS
ISSUE #1
By Daniel Yeow
"The tradition of paradox cartoons continues... ...continuously" -Daniel Mathews
On a late afternoon, the warm flourecent glow of the MUMS room beckons those entering the Richard Berry building
but who is the mysterious dark figure working here during the semester break?
Epsilon (ε) and Delta (δ) possessed prodigous talent at mathematics
Can you construct a regular 17-gon for me too?
Not a regular pentagon? amazing!
wow! möbius-strip cookies!
You twins are great!
thanks for undoing the knots in my shoelaces!
but, of course, there were doubters
You Suck! what kind of a name is "Delta" anyway?

On a late afternoon playing chess by the sea the twins are set upon by a mysterious dark figure
Epsilon, Delta, tales of your mathematical abilities have reached far and wide. I am building a space station equipped with a planet-destroying super laser with the intention of bringing about world peace. If you join me I will share with you the spoils of world domin- oops, I mean peace.
Sensing that something wasn't quite right...
No! We will never turn to the dark side!
You misunderestimate the power of the dark side! Very well! Lieutenant Zoltan, take care of them!
Yes sir, right away sir!
Z
Dad
mum
ε
δ
Luckily Zoltan was an arts law student with a penchant for conundrums. "Given that Dad and Mum can't see anything, ε can see mum and δ can see mum and ε and that you all know that there are two black and two white hats between the four of you... one of you must correctly and with 100% certainty, identify the colour of your own hat. You have five minutes
MWUHAHAHA [diabolical laughter]


Easy, thought ε. δ behind me would know the colour of her hat only if me and mum had the same colour hat. Since she hasn't said anything, my hat must be a different colour to mums...
My hat is black!
Inconceivable! H... H... How did she do that!?
Zoltan was so shocked that he was afflicted with a terrible stutter ever after
C... C... C... Come back!
In the C...C...Confusion ε and δ made their escape. They sought refuge in a place far away to ponder the identity of the strange dark figure
Where's the airport?
I don't know, its enchanted
So they fled to the enchanted island of dog where knot man was having a stopover on his way to san francisco
Hmmm... I think I know who the dark figure might be. But first, I must ensure that you girls are equipped to handle your upcoming mathemagical adventures
Oh, and you might want to hone your skils with duelling pistols too... just in case
what the..?
Don't look at me


well, you were lucky to escape when you were buried in the sand
what do you mean? That was easy
but what if there were say, n people
n! how in gödel's name would you do that!
...and what if you didn't know how many black and white hats there were?
Imagine a line of people buried in the sand. Each person can only see all the people in front. There can be no communication except at the start - BEFORE the people are buried. A man with a sword starts at the back and asks "What colour is your hat?" the person at the back may only say "black" or "white". If they are correct, then they live, if not, they are killed. Either way, the people in front have no way of knowing the result. All they can hear is "black" or "white". The sword-guy continues down the line until everyone has been asked. What's more - he has bugged their pre-burial meeting so he may try to foil whatever plan they come up with
Your task is to come up with a plan that maximizes the number of people who will live
they could call out the colour of the hat in front
but then only 50% are safe for certain, surely there's a better way
there is
Can you give us a hint?
I'm a frayed knot, but I will give you until the next paradox
Can YOU solve the problem of the hats?
There will be a cash prize of $5 for the correct answer. In the event of more than one correct submission, the first submitted will get the prize. In the event of no correct submissions the money will be split and given to each of the Twins as lunch money
email submissions to me (3rd year rep) or Nick (paradox editor)

The Chromatic Number of the plane

Ever wondered what mathematicians do in their spare time? Play colouring games of course! So let me describe to you one particular 50-year-old colouring "game".

The Rules: Assign a colour to each point in the infinite plane such that any two points distance 1 apart are a different colour.

The Objective: Find the LEAST number of colours needed to do this.

We call this number the Chromatic Number of $\mathbb{R}^2$, and denote it $\chi(\mathbb{R}^2)$. We also say a colouring of $\mathbb{R}^2$ is a Minimal Colouring if, in addition to our rules above, the number of colours used is $\chi(\mathbb{R}^2)$ (the minimum number required)[1].

OK, so perhaps jumping into the $\mathbb{R}^2$ case is a bit too sudden. We'll start with $\mathbb{R}^0$, which is just a single point. Since $\mathbb{R}^0$ has only one point, and this point is distance 0 from itself, we only need 1 colour to colour in $\mathbb{R}^0$ according to our rules. So $\chi(\mathbb{R}^0) = 1$. YAY!

Now let's try and find $\chi(\mathbb{R}^1)$. $\mathbb{R}^1$ is just the real line, so we need to find a minimal colouring for the real line. We clearly can't do it using only 1 colour, since otherwise the points 0 and 1 would be the same colour, and they are distance 1 apart (which violates our rules). So $1 < \chi(\mathbb{R}^1)$. Now what if we coloured $\mathbb{R}^1$ using 2 colours, red and blue, in the following way:

Colour the semi-open interval $[0, 1)$ blue, $[1, 2)$ red, $[2, 3)$ blue, $[3, 4)$ red, and so on, alternating the colour of each semi-open interval as you go along the real line in each direction. Clearly, no 2 points in any semi-open interval are distance 1 apart, and intervals of the same colour are separated by an interval of length 1. So we have assigned a colour to each point of $\mathbb{R}^1$ according to our rules, and have done so using only 2 colours. So $1 < \chi(\mathbb{R}^1) \leq 2$, ie $\chi(\mathbb{R}^1) = 2$.

Now, what about $\chi(\mathbb{R}^2)$? How many colours are needed to colour $\mathbb{R}^2$ according to our rules? Well, clearly we can't do it with 1 colour. What about 2 colours? Try colouring the vertices of an equilateral triangle of edge length 1 using only 2 colours (see Figure 1).

There's no way to do it without violating our rules, so we need more than 2 colours for $\mathbb{R}^2$. Now can we colour $\mathbb{R}^2$ using 3 colours? Consider the 7

[1]For those who don't know, $\mathbb{R}^2$ is just the name mathematicians give to two-dimensional space. They call the real line $\mathbb{R}^1$, and three-dimensional space $\mathbb{R}^3$.

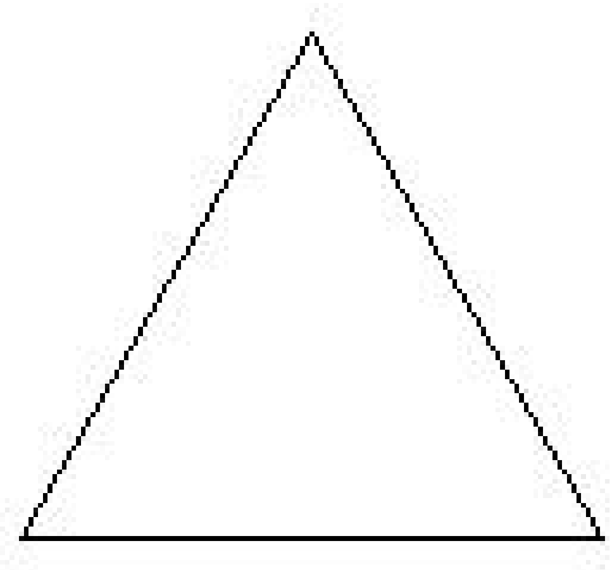


Figure 1: An equilateral triangle.

points in $\mathbb{R}^2$ arranged as shown in Figure 2. (The points form the vertices of 2 "diamonds", one of which is slightly tilted. Each edge length is unit distance. This figure is called the 'Moser Spindle'). It's easy to demonstrate that these 7 points cannot be coloured using only 3 colours. So we need more than 3 colours for $\mathbb{R}^2$.

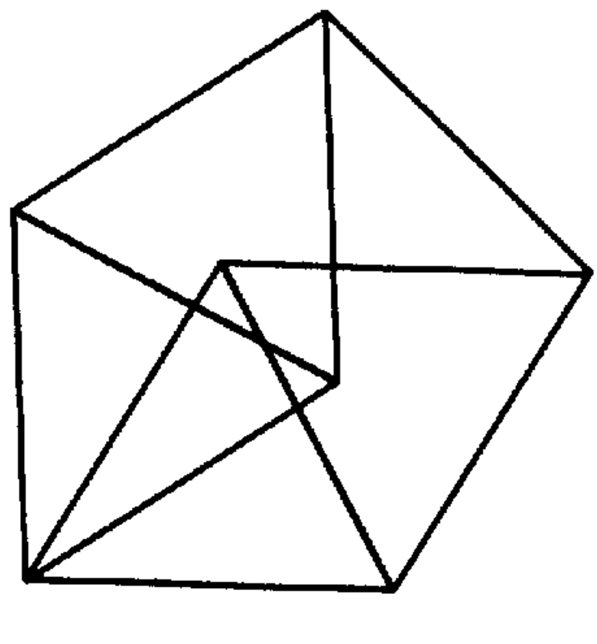


Figure 2: The Moser spindle.

From what I've said so far, you may be thinking that it is impossible to colour $\mathbb{R}^2$ according to our rules. But fortunately, it can be done! Lets see how: Suppose we divide $\mathbb{R}^2$ into squares of edge length 0.6. Now imagine grouping nine squares together to make a big 3×3 block. Now colour each of these little squares a different colour, thus using 9 colours in total. We number the colours from one to nine.

Now imagine we tessellated the infinite plane using these 3×3 blocks with the colouring as given above. Clearly no two points in any small square are a distance 1 apart, since the diagonal of a small square is $0.6 \times \sqrt{2}$ which is less than 1. Also, the smallest distance between any 2 different squares of the same colour is twice the width of a square, ie 1.2, which is more than 1. So we have found a way to colour every point in $\mathbb{R}^2$ according to our rules, using 9 colours. So $\chi(\mathbb{R}^2) \leq 9$.

1	2	3	1	2	3	...
4	5	6	4	5	6	...
7	8	9	7	8	9	...
1	2	3	1	2	3	...
4	5	6	4	5	6	...
7	8	9	7	8	9	...
⋮	⋮	⋮	⋮	⋮	⋮	⋱

Figure 3: A colouring of the plane with nine colours.

Can we do any better than 9? We can if we change our "base shape" from a square (as in the previous example) to a hexagon. Using a "flower" of 7 hexagons of diameter slightly less than 1 (see Figure 4), we can assign one of 7 colours to each of these hexagons, and tessellate $\mathbb{R}^2$ with them such that any 2 hexagons of the same colour are separated by a distance greater than 1 (I leave the geometric proof of this to the ever-vigilant reader). And since the diameter of any hexagon is less than 1, we have found a colouring of $\mathbb{R}^2$ according to our rules that uses only 7 colours. So $\chi(\mathbb{R}^2) \leq 7$.

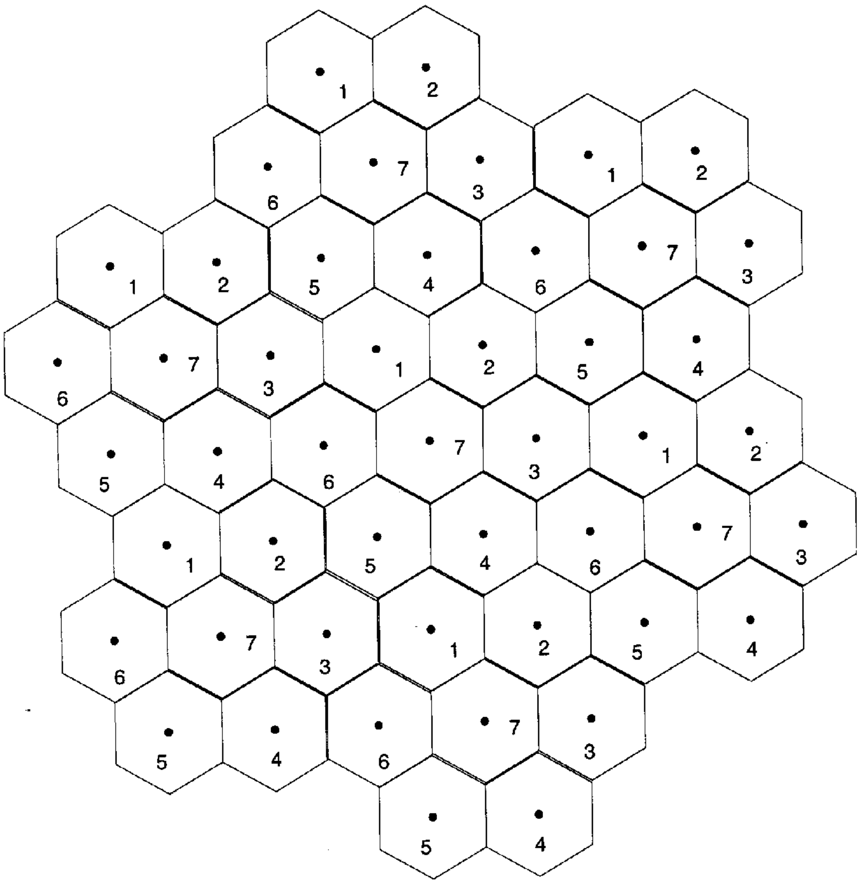


Figure 4: A colouring of the plane with seven colours.

To recap, we have shown that:

- $\chi(\mathbb{R}^0) = 1$
- $\chi(\mathbb{R}^1) = 2$
- $4 \leq \chi(\mathbb{R}^2) \leq 7$

Do we know the exact value of $\chi(\mathbb{R}^2)$? NO! In fact, the bounds on $\chi(\mathbb{R}^2)$ given above are the best known. Maybe you can find a colouring of $\mathbb{R}^2$ that uses less then 7 colours, or perhaps find a clever way to show that 4 colours isn't enough. I spent 8 weeks over summer trying to do just that (with no success).

And don't think that if you do find $\mathbb{R}^2$, that's the end of the problem. There's still $\chi(\mathbb{R}^3)$ (chromatic number of infinite 3-dimensional space), $\chi(\mathbb{R}^4)$, $\chi(\mathbb{R}^5)$ and so on to be found.

Happy Hunting. — Maurice Chiodo

Idiot's guide to Mersenne Primes

Phil: I read this article about the internet quest for the next Mersenne Prime. Do you know anything about it?

Sophie: Yes and no. Depends on what specifically you wanted to know.

Phil: Well, for starters, what IS a Mersenne Prime?

Sophie: There's nothing too mystical about them, they're just numbers which hold some certain properties. Well, if you are a die-hard mathematician, you might swear that knowing the key to Mersenne Primes holds the key to the universe[42],

then again... maybe not.

Phil: What properties? WAIT! Let me guess, they are PRIME numbers? But a "different" kind of prime numbers.

Sophie: Quite right. A Mersenne Prime is exactly what the name states. A Mersenne number is a number which is in the form of $M_n = 2^n - 1$. So there you have it, a Mersenne prime: a Mersenne number which is prime.

Phil: What's the big deal? Aren't "normal" primes "funky" enough?

Sophie: I'm not finished, here is where it starts to get freaky... it's not just ANY number, it has to follow a certain rule. You see, for a Mersenne Prime to BE prime, the index has to be prime AS WELL.

I can give you the basic structure of how to prove that. Now if n is composite, I can write $n = pq$ and thus, $M_n = 2^{pq} - 1$ and that is a binomial number which will always have a factor of either $(2^p - 1)$ or $(2^q - 1)$.

Phil: So, n HAD to be prime? That's the only way that the number $M_n = 2^n - 1$ is prime.

Sophie: Yupe.

Phil: I still don't get it, why bother making things more complicated? What is it with Mersenne Primes anyway?

Sophie: Mersenne Primes hold a very remarkable property. But before we deal

[42]The number 42 is the key to the Universe. Search Google to find out why!

with Mersenne Primes, I'd like to tell you a bit about PERFECT NUMBERS. Have you ever heard of them?

Phil: Perfect numbers? You mean integers? Aren't all integers perfect?

Integers are whole numbers, which might be described as "perfect", unlike rationals which are fractions, or even irrational numbers which are neither integers nor rationals. RIGHT?

Sophie: Perfect numbers ARE integers, but not all integers are perfect. "Perfect" here is a term that describes the "sum-of-factors" property. Quite simply, if you add up the factors of a number (excluding itself) you will get a number, and if that number is the same as the original number, then it's "perfect".

For example 6 is a perfect number, because 1, 2 and 3 are all factors of 6 and $1 + 2 + 3 = 6$. The next perfect number is 28, since $1 + 2 + 4 + 7 + 14 = 28$, and the next is 496.

The study of perfect numbers has been around since the time of the early Egyptians, maybe even as early as the invention of numbers. I suspect it's got something to do with people not having anything better to do. Or as we call it, "satisfying their innate curiosity"... whatever that is... (or even writing this article in the first place, or even for the readers who read them).

Phil: OK, I get that, what does it have to do with Mersenne Primes?

Sophie: Everything.

The sole reason people are actually interested in Mersenne Primes is the fact that each perfect number corresponds to a single Mersenne Prime.

Phil: I'm getting lost now. OK, how?

Sophie: It's actually not that difficult. Alright, like I said before, a perfect number is a number which is the sum of its factors excluding the number itself. Now let's include the number itself, so we will have twice the original number. So, we say a number is perfect if twice that number is the sum of all its factors.

6 is perfect because $1 + 2 + 3 + 6 = 12$ and $12 = 2 \times 6$.

28 is perfect because $1 + 2 + 4 + 7 + 14 + 28 = 56$ and $56 = 2 \times 28$.

In short, P is perfect if the sum of all factors of P is $2P$.

It's pretty tiring if I have to say "sum of all factors" over and over and over again. So to make life easier for both of us, we'll say P is perfect if $\delta(P) = 2P$ where $\delta(P)$ is the sum of all factors of P.

Phil: I'm digesting... So a number is perfect if all its factors sum up to twice the original number?

Let's say the number is 15. The factors are 1, 3, 5, 15. If you add them up, you will get $1 + 3 + 5 + 15 = 24$. 24 is NOT 2×15, thus 15 is NOT a perfect number. Am I right?

Sophie: Yes you are! Let's look at $\delta()$, what happens if we apply $\delta()$ to a prime ? Say $\delta(19) = ?$

Phil: The sum of all factors of 19 is $1 + 19 = 20$, are you saying that

$\delta(\text{prime}) = \text{prime} + 1$?

Sophie: Yes, EXACTLY! Now if we apply the function to numbers like 2^n, it will behave like the sum of geometrical series with a lower limit of 1, upper limit of 2^n and a ratio of 2.

So, $\delta(2^n) = 1 + 2 + \ldots + 2^n$

Phil: Let's see, the formula for sum of geometrical series is $S_n = \frac{x^{n+1}-1}{x-1}$

$\delta(2^n) = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1$

Sophie: That's right! Now before we move on, I'd like to state the one Fundamental Theorem which we will use. The Fundamental Theorem of Arithmetic.

Phil: DE JA VU! I've heard that before! I swear I have!

Sophie: The Fundamental Theorem of Arithmetic?

Phil: No, not that, that's YOU CHANGING THE TOPIC YET AGAIN!! "Before we move on, blah blah blah." Look at us! Now we're even further from our original topic. I get what a Mersenne Prime is, I just don't get what's so freakin' special about a number.

Sophie: Beats me, don't shoot the messenger, don't blame me for explaining. So do you still want to hear about the Fundamental Theorem of Arithmetic?

Phil: I still don't understand where this leads to, but yeah, what the heck.

Sophie: Wow! Such enthusiasm!

So anyway, the Fundamental Theorem of Arithmetic just states that you can express every natural number $n > 1$ in a unique way: as a product of prime factors.

So $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} .... p_r^{a_r}$ where $p_1, p_2, ..., p_r$ are all prime.

I'm just stating that so you can understand the next REMARKABLE property of $\delta()$. Let's say it's applied to a composite number. If we express the composite number as a product of prime factors, then δ(composite) = product of all δ(prime factors of that composite). Or, to be more exact,

$\delta(p_1^{a1} p_2^{a2} p_3^{a3} .... p_r^{ar}) = \delta(p_1^{a1})\delta(p_2^{a2})\delta(p_3^{a3})...\delta(p_r^{ar})$,

Phil: I still don't get what the Fundamental Theorem of Arithmetic has to do with any of this.

You're just obscuring things and making things even more complicated than they already are. You can just say the prime factorization of a composite number, and that will do. You're such a pain Sophie, you make me sick! I don't even remember why I asked you this in the first place.

Sophie: Well, that's encouraging.

Anyway, remember the examples of perfect number that I gave you, 6, 28, 496, ... Those are all even numbers. Now let's forget about odd perfect numbers, for our purpose here they don't exist, or just consider them to be non-existent.

Say we can write a perfect number in this form $P = q \cdot 2^{p-1}$ where q is prime.

Let's look at $\delta(P)$

$$\begin{aligned}
\delta(P) &= \delta(q \cdot 2^{p-1}) \\
&= \delta(q)\delta(2^{p-1}) && \text{from the prime factorization property} \\
&= (q+1)(2^p - 1) && \text{from my previous explanations}
\end{aligned}$$

and we know if P is to be perfect, then $\delta(P) = 2P$ thus,

$\delta(P) = 2P = 2(q(2^{p-1})) = q(2^p)$

Then, we just got the result that

$$\begin{aligned}
(q+1)(2^p-1) &= q(2^p) \\
q(2^p-1)+(2^p-1) &= q(2^p) \\
(2^p-1) &= q(2^p)-q(2^p-1) \\
(2^p-1) &= q(2^p-2^p+1) \\
(2^p-1) &= q
\end{aligned}$$

so q had to be a Mersenne Prime in order for P to be perfect, so can you see how special a Mersenne Prime is?

Phil: ....

Sophie: Or let me put it in another way.
Say we have a Mersenne prime M_p, $M_p = 2^p - 1$.
From the result I've just shown you,
we can write perfect numbers using this formula:

$P = (2^{p-1})(2^p-1) = \frac{(M_p+1)M_p}{2}$

So not only did p have to be prime for $M_p = 2^p - 1$ to be prime, we also see that each Mersenne Prime corresponds to a perfect number. Or is it the other way around.

So what is it about the article you read about the internet quest about the search of Mersenne Prime?

Personally, I think it's a good idea for the use of "unused resources" for the advancement of mathematics in general.

Sure, it costs electricity and it causes CO_2 emmission, but don't blame mathematicians for it, blame whoever invented an inefficient use of fossil fuels as a source of energy. Besides, mathematicians aren't evironmentalists, they shouldn't be held responsible for a thing they didn't do.

What do you think, Phil?
..
Hey Phill... ?
Phillllll?

— Fendy Liauw

Paradox Problems

The following are some maths problems for which prize money is offered. The person who submits the best (clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number. Solutions may be emailed to

`paradox@ms.unimelb.edu.au`

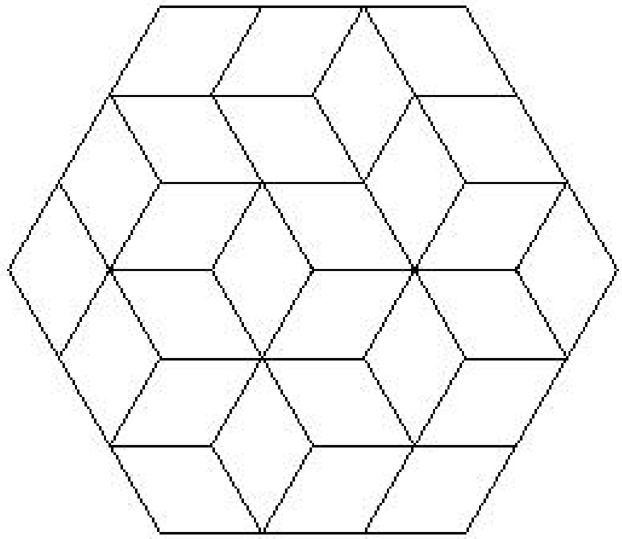
or you can drop a hard copy of your solution into the MUMS pigeonhole near the Maths and Stats Office in the Richard Berry Building. Congratulations to Jenny Theresia, Chris Cheung, David Gummersall and Jie Meng, who submitted correct solutions to the problems from the last edition.

1. ($5) A group of shepherds have 128 sheep among them. If one of them has at least half of the sheep, each other shepherd steals as many sheep from this shepherd as he already has. If two shepherds each have 64 sheep, one of these two shepherds steals all the sheep from the other. Suppose seven rounds of theft occur. Prove that one shepherd ends up with all of the sheep.

2. ($5) A positive integer is written on the board. We repeatedly erase its unit digit and add 5 times that digit to what remains. Starting with 7^{2004}, can we ever end up at 2004^7?

3. ($5) Duels in the town of Discretion are rarely fatal. When a duel is to be fought, each contestant arrives at a random moment between 5 a.m. and 6 a.m. on the appointed day and leaves exactly five minutes later, unless his opponent arrives within that time, in which case they duel. What fraction of duels end in violence?

4. ($5) A *calisson* is a French sweet, in the shape of two equilateral triangles joined edge to edge. *Calissons* come packed in hexagonal boxes. As you can see on the next page, they can have any of three orientations in the box. Show that, regardless of the size of the box or the arrangement of the *calissons*, there are the same number of *calissons* with each orientation.



5. ($10) A circuit board has 2004 contacts, any two of which are connected by a lead. The hooligans Vasya and Petya take turns cutting leads: Vasya (who goes first) always cuts one lead, while Petya cuts either one or three leads. The first person to cut the last lead from some contact loses. Who wins with correct play?

In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for re-election.

An absent-minded professor (alright, it was Norbert Weiner) was moving. His wife, knowing that Norbert would forget his address, took out a sheet of paper and wrote it down for him. Later that day, Norbert had a flash of insight, and, fumbling for a piece of paper, wrote down his new theorem on the paper his wife gave him. On further reflection, Norbert found a fallacy in his thinking and threw out the paper in disgust. When he came home that night, to the now-empty house he had moved from, he remembered he had moved, but had no idea where he had moved to. Just then, he spied a little girl on the street. "Little girl," he asked, "My name is Norbert Weiner, do you know where I live now?" "Yes Daddy," replied the girl, "Mummy thought you would forget."

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