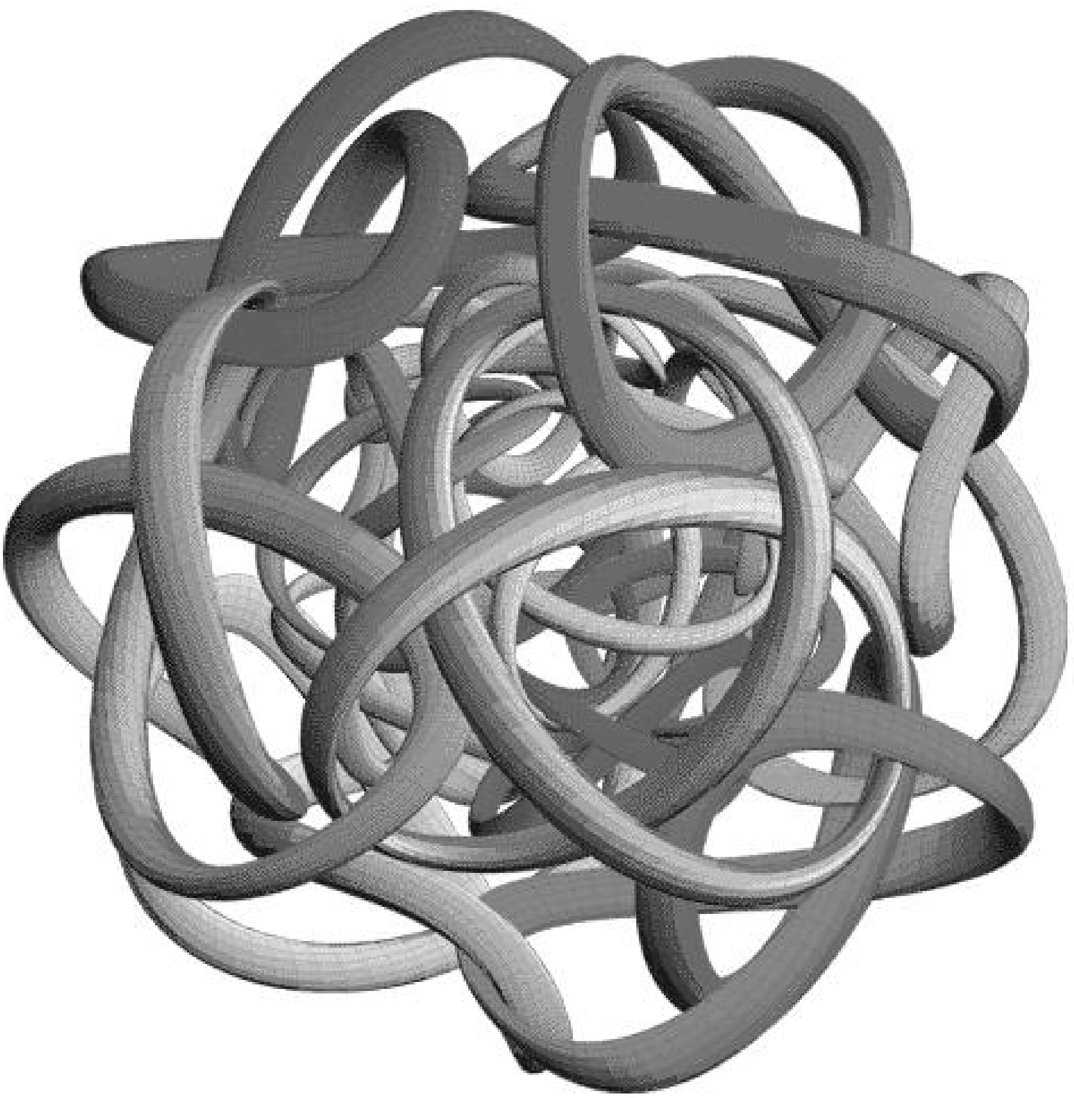
Issue 3, 2004

Paradox

THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY



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Paradox

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Words From the Editor

Welcome to the third and final edition of Paradox for 2004. This edition is, as ever, packed full of mathsy goodness and fun and games. New to Paradox is the mathematical cryptic crossword, which demands linguistic skill in addition to mathematical knowledge, and the exciting inauguration of the prestigious Barry Hughes Prize for Excellence in the Field of Appalling Maths Puns, hopefully the start of a great tradition! We have articles about that unreasonably fascinating number, 1.61803398874989484820458683436… otherwise known as the Golden Ratio. Captain Continuous is once again in action, and we have a useful summary, for the inexperienced reader, of Elementary Great Competition Theory, with an emphasis on its applications to the recent Melbourne University Puzzle Hunt (reading this article may help to understand some of the more obscure references in Captain Continuous). Bruce Craven has contributed a fascinating article about Pythagorean triples and n-tuples, and we probe the incredibly fertile mind of the Dutch graphic artist M.C. Escher. We also carry the saga of Ali Ibn Abi Taalib, the greatest solver of conundrums in all of Arabia, contributed by Usama.

Enjoy this offering, whether you are a committed Paradox reader of many years, or an internet reader, or even if you just picked this magazine up because it's something free to read on the train, and it's not MX! Whichever category you fit into, we encourage the readers of Paradox to take the next step and write an article. It is a most rewarding experience. There can be few better ways of really getting to know that subject you always thought might be interesting to learn a bit about, one day… Submissions of all varieties can be sent to the Paradox email account: `paradox@ms.unimelb.edu.au`. Happy perusing everyone, and make sure you get your entries in for the Barry Hughes prize!

— Nick Sheridan

> The Dean says to the physics department, "Why do I have to give you all this money for equipment and stuff. If only you were more like the mathematics department. All they need are pencils, paper, and wastebaskets. Or better still, the philosophy department. They only need the pencils and paper."

Words from the President

The end of the second semester is nigh; and while you may be preparing for the intense study ahead, MUMS is relaxing after one of its most intense periods ever. As promised, this semester has been abuzz with mathematically-inspired activities, involving everything from artwork to excavation. We have even been mentioned on radio and in UniNews.

The climax of the semester was Maths Week, which was kicked off by a seminar on Escher and his artwork, given by a light-sabre-weilding Dr. Burkard Polster. Like his previous seminars on juggling and origami, Dr. Polster's entertaining style and cute collection of toys provided for a fascinating lecture, which was made even more memorable by witnessing a theatre-full of people wearing 3D glasses and watching Escher's artwork jumping out at them.

The highlight of Maths Week was definitely our shiny new competition, the Puzzle Hunt. It spanned the whole week and attracted more than 200 team entries, including ones from faraway places like Sydney, Singapore, and even RMIT. The competitiors devoted a whole week of their productive lives to solving complicated conundrums devised by the Puzzle Hunt creators, while the creators themselves devoted their whole week to correcting typos, fixing bugs, and writing the accompanying storyline in real-time. Congratulations to all competitors and well done to "Team Room 106" who were the first to find the hidden treasure. The Puzzle Hunt was so successful that I am sure it will find itself a permanent fixture on the MUMS calendar. So, if you missed out this time, there's always next year!

The finale of Maths Week was our traditional University Maths Olympics. This year it was hosted by Professor Terry Speed. While being a newbie to the competition, it didn't take too long for him to perfect his lolly-throwing action. The two best schools from the Schools Maths Olympics, Scotch College and University High, managed to beat most of the uni teams and finished amongst the top five. The staff teams, after declining their traditional handicap, couldn't quite make the top five but gained respectable placings nonetheless. The stars of the day, however, were "The Quantum Mechanics: *no job too small*". Their combination of athleticism, stamina and preparation was, in the space of an hour, converted to a brilliantly huge winning score. (Don't worry about their name — they're not actually a physics team.)

Many hands, hours and brains were needed to make Maths Week happen. I would like to thank all of those who were involved in organising events, and all of those who assisted us in various ways — we couldn't have done without your support!

I hope you have all enjoyed our many activities. If there were any that you particularly liked, or disliked, please let us know so that we can improve next year.

Of course, the semester is not over yet and our usual regular events will continue right up to the very end!

— Damjan Vukcevic

The Adventures of Ali Ibn Abi Taalib – I

A person was about to die, and before dying he wrote his Will, which went as follows...
"I have 17 Camels, and I have three sons. Divide my Camels in such a way, that my eldest son gets half of them, the second one gets one third of the total and my youngest son gets one ninth of the total number of Camels."
After his death, when the relatives read his will, they got extremely perplexed and said to each other, "How can we divide 17 camels like this?"
So, after thinking long and hard, they decided that there was only one man in Arabia who could help them: Ali Ibn Abi Taalib.
So they all came to the door of Ali and put forward their problem.
Ali said, "OK. I will divide the camels according to the man's will."
Ali said, "I will lend one of my camels to the total which makes it 18. Now let's divide the camels as per his will."
The eldest gets half of 18, or 9 camels.
The second gets one third of 18, or 6 camels.
And the youngest gets one ninth of 18, or 2 camels.
Now the total number of camels we have distributed among the brothers is 17.
So Ali said, "Now I will take my camel back!"

The Inaugural Barry Hughes Prize for Excellence in the Field of Appalling Maths Puns

Paradox is pleased to announce the inauguration of a prize to recognize the excellent efforts of certain among the lecturers of the Mathematics and Statistics department to brighten the lives of countless students with countless incredibly poor mathematical puns. The prize is named after Barry Hughes, a lecturer in applied mathematics. He is one of the heavyweights in the area of mathematical puns, and it was unanimously felt in the Paradox editorial committee that his efforts, notably in the Mathematical Methods lectures of the 8th and the 15th of September, could not go unrecognized. On these occasions, Professor Hughes produced the classic pun "The Euler Homogeneous Equation – so named because Euler was a man (homo in Latin) and he was a genius," and the gem of an aside, made when discussing a case in which Fuchs' theorem did not work, that "In some sense, we are Fuchsed". His off-the-cuff remark, during a lecture that made reference to environmental impact models, that "We are dealing with Green functions" simply confirmed his status as the pre-eminent mathematical punster of the department.

Other powerhouses when it comes to mathematical punning have included the Notorious Derek Chan, who has entertained many a captive audience in recent times with the joke

$$\int \frac{dcabin}{cabin} = houseboat$$

(because, of course, this integral is equal to a log cabin plus sea). Paradox's own Knot Man, has always been well-equipped with snappy one-liners to lay on villains (as any half-decent superhero should be). The most famous of these was his reply to \$, the evil economic rationalist, when Knot Man answered \$'s request that the superhero release him from a trefoil knot trap with the droll remark "I'm a frayed **knot**!"

Some students have also got into the spirit, such as the anonymous second-year student who said, during a vector analysis tutorial, "The integral of rdr – I feel like a pirate." Vector analysis seems to have provoked a number of incidents of mathematical punnery, as another second-year student had ridicule heaped upon him when, indecisive about the message to write on a birthday card, he wrote

$$\frac{\int\int\int_{Messages}(message)dV}{Vol(Messages)}$$

The inaugural Barry Hughes prize is, of course, awarded to Barry Hughes. Nominations from students for the next winner of the Barry Hughes prize are called for. Send an email to paradox@ms.unimelb.edu.au with the name of the person being nominated, along with a sample of their finest mathematical puns. The MUMS committee will adjudicate, and the winner will be announced in the next edition of Paradox. Note that entries are not restricted to Mathematics and Statistics lecturers, or even staff members. Fellow students, or even lecturers from other faculties are all fair game.

The winner will be presented with a certificate recognizing their achievement, and the words "You Fuchsin' Legend!" inscribed on it.

— Nick Sheridan

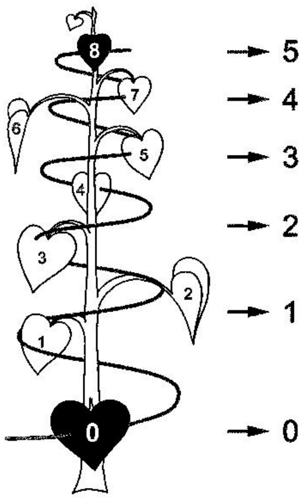
Q: What do you get when you cross an elephant with a banana?
A: A vector of magnitude $|\text{elephant}| \times |\text{banana}| \sin(\theta)$, where θ is the angle between elephant and banana, in a direction orthogonal to both elephant and banana according to the right-hand rule.

Q: What do you get when you cross a mosquito with a mountain climber?
A: You can't cross a vector with a scalar.

Q: How can you put fourteen sugar cubes in three cups of tea in such a way that there are an odd number of sugar cubes in each cup?
A: Put one in the first cup, one in the second cup, and twelve in the third cup.
Riposte: But twelve is not an odd number!
Reply: It's an odd number of sugar cubes to put in a cup of tea.

The Golden Ratio

I found my old textbook hidden away in the far corner of my house. As I searched though things that I have learnt and given back to my teachers, they reminded me of a tiny but significant memory of my education at secondary level. As I recall, it was in my Specialist class when I came across an interesting poster. On it, there were different types of plants and flowers. But what made this poster interesting was that, on the stem, there were numbers labelling how many leaves were on it before the last one lay *almost* on top of the first one. This is the computer generated version.



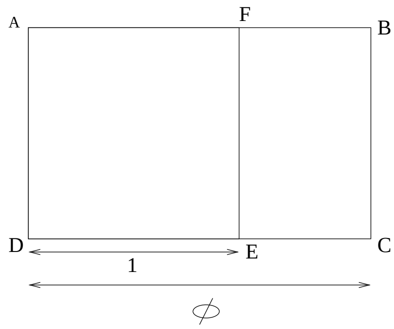
What is more interesting is that there is a set which these numbers are from:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

Don't they look familiar? They are the Fibonacci numbers. When counting how many turns it takes to satisfy this condition, believe it or not, it turns out to be a number from this group as well. This means that the angle between consecutive leaves is about $0.618\ldots \times 360^\circ = 222.5^\circ$. This corresponds to 1.618 leaves per turn.
The answer lies in this number, which is less well-known than π and e : **The Golden Section**. This article is quite technical in a way. If you do not want to stress over algebra, just skip over boxes and assume that the facts are right.

Definition of Golden Ratio

The Golden Section is defined in this way from geometry.



In this diagram, ABFE is a square and EFCD is similar to ABCD
Let ϕ be the longest length and 1 be the shortest length.
Then we have:

$$\phi - 1 = \frac{1}{\phi} \quad \Rightarrow \quad \phi = \frac{1+\sqrt{5}}{2}$$

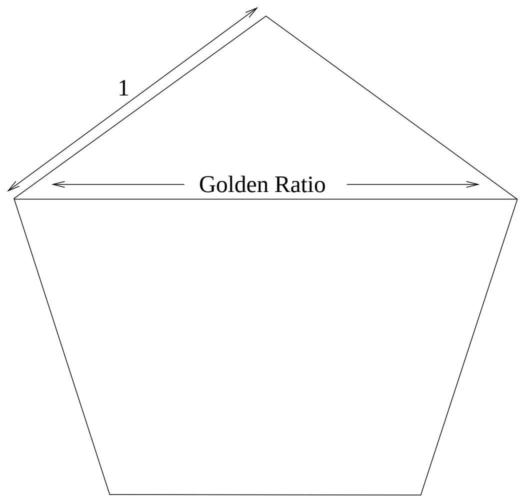
There are two golden sections, which are the reciprocals of one another. The other is $\frac{\sqrt{5}-1}{2}$. In this article, I will be focussed on $\phi = \frac{1+\sqrt{5}}{2}$.

Some Interesting Properties of Golden Section

The golden section can be written in many forms, and it appears in geometry a few times. These are a few interesting ways of writing the golden ration:

$$\begin{aligned}
\phi &= \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}} && && (1)\\
&= 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}} && \text{Continued fraction form} && (2)\\
&= 2\cos\frac{\pi}{5} && \text{Trigonometric Form} && (3)\\
&= \frac{13}{8}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)!}{(n+2)!n!4^{2n+3}} && && (4)\\
&= \lim_{n\to\infty}\frac{F_{n+1}}{F_n} && && (5)
\end{aligned}$$

where F_n is the *nth* Fibonacci number.
Equations (2) and (3) are closely related to the pentagon, decagon and dodecagon. For simplicity, I will use the pentagon as an example:



If the sidelength of the regular pentagon is 1, then the length of the diagonal of the pentagon is ϕ. You can prove this yourself using similar triangles. From this simple fact, you can directly prove equation (3). We can prove equation (2), which will become useful later, by first defining x so that:

$$x = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\dots}}}$$

Then

$$x = 1 + \frac{1}{x} \qquad \Rightarrow \qquad x = \frac{1+\sqrt{5}}{2} = \phi$$

Therefore the equation (2) is true.
For equation (5) we require a knowledge of equation (2):

Denote $[1,2,3,4] = 1 + \frac{1}{2+\frac{1}{3+\frac{1}{4}}}$ Then

$\frac{F_2}{F_1} = \frac{1}{1} = [1]$

Let's assume that $\frac{F_{k+1}}{F_k} = 1 + \frac{1}{\frac{F_k}{F_{k-1}}} = [\overbrace{1,1,\dots,1}^{k\ 1's}]$

Therefore

$\frac{F_{k+2}}{F_{k+1}} = \frac{F_{k+1}+F_k}{F_{k+1}} = 1 + \frac{F_k}{F_{k+1}} = 1 + \frac{1}{\frac{F_{k+1}}{F_k}} = [\overbrace{\ 1\ }^{added}, \overbrace{1,1,\dots,1}^{k\ 1's}] = [\overbrace{1,1,\dots,1}^{k+1\ 1's}]$

Therefore $\frac{F_{n+1}}{F_n} = [\overbrace{1,1,\dots,1}^{n\ 1's}]$ and $\phi = [1,1,\dots] = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$

This leads us back to our original problem. The plant wants to maximise exposure to the light. This means it does not want to to have any two leaves on top of each other. This means any number n of leaves per turn, will not work, when n is rational. This can be demonstrated using a little bit of algebra.

Let the number be $n = \frac{a}{b}$
Then every $(r + nb)th$ leaf, would be on top of the rth leaf. Thus, a rational number will not achieve the plant's objective.

Now we need to consider the possibility that n is an irrational number. Here we need to use the properties of continued fractions. Without going into great depth, continued fractions can be proven to be the best rational approximation to an irrational number on a given denominator.
For example, e can be expressed in this form(taken to 20 terms)

$$[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1]$$

Notice that big numbers (like 12) appear occasionally in the line. We can approximate the number quite well by 'cutting off' the continued fraction just before the chosen large number. That is because the term that we are neglecting will be equal to 1 divided by a large number, which will make little difference to the final answer. For e, if we cut it off before the number 12, we get:

$$[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1]$$

This is calculated to be $\frac{1084483}{398959}$ which is very close to $\frac{19}{7}$. If we express our number n as a continued fraction then, since we can always approximate an irrational number by a rational number by cutting it off before a large number in the continued fraction expansion, there will be some spots where two leaves are very close to being on top of each other. Therefore, nature finds its way to minimise the number of times at which large numbers occur. Thus, the ideal number would have to be:

$$[1, 1, \ldots]$$

since this is the continued fraction with the least number of large numbers in its continued fraction expansion. This corresponds to the **Golden Section**. This explains why nature has adopted this number.

Some other interesting appearances of the Golden Ratio:

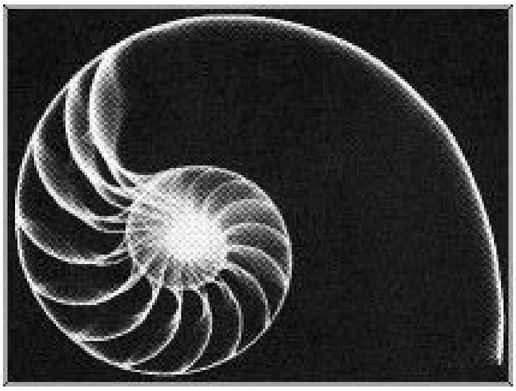
In Architecture: The golden section appears in many buildings in Greece. One clear example is the Parthenon. Its shape is designed to have the shape of the golden rectangle when viewed on different faces.

In Music: In Beethoven's Fifth, the famous opening is found to be repeated at calculated intervals. It appears in the first bar, the last bar(601) and exactly at 0.618 . . . of the piece(bar 372). Coincidence?

Many of Mozart's Sonatas are divided into two sections. This occurs at precisely 0.618 . . . of the piece.

In Art: Pacioli wrote a book in 1509, "*De Divina Proportione*" , which discussed the golden rectangle as being the most pleasing object. This later influenced Leonardo da Vinci in many of his drawings.

In Nature: The nautilus shell:



The ratio of radii from the centre to the furthest shell and to the second furthest shell is always the golden section.

— Tharatorn Supasiti

Pythagoras, Complex Numbers, Quaternions

A Pythagorean number triple is a set (p, q, r) of positive integers satisfying $p^2+q^2 = r^2$. It is well-known that such triples may be constructed from integers a, b by $p = a^2 - b^2, q = 2ab, r = a^2 + b^2$. In terms of the complex number $z = a+ib$, with integers a, b, we have $z^2 = p+iq$, so that $p^2 + q^2 = |z^2|^2$, where $|z^2|$ has an integer value. Thus the Pythagorean triples are generated from complex numbers with integer real and imaginary components. Morerover, we may take a as odd, say $a = 2m - 1$, and b as even, say $b = 2n$, so that the pairs of positive integers (m, n) also generate the triples. Some p, q may have negative signs, which can be ignored. A few examples are as follows:

m	n	$2m-1$	$2n$	p	q
1	1	1	2	−3	4
1	2	1	3	−8	6
1	3	1	6	−35	12
2	1	3	2	5	12
2	2	3	4	−7	24
3	2	5	4	9	40
4	1	7	2	45	28

Since complex numbers have a commutative and associative multiplication, there is a consequent multiplication also for the triples. However, complex conjugates give two different products, and so two different triples (since their signs don't matter) for each pair of complex numbers.

$(a+ib)(a'+ib')$	*product*	p	q	r	m	n
$(3+4i)(3+4i)$	$-7+24i$	7	24	25	2	2
$(3+4i)(5+12i)$	$-33+56i$	33	56	65	4	7
$(3+4i)(5-12i)$	$63-16i$	63	16	65	1	8
$(-7+24i)(-7+24i)$	$-527-336i$	527	336	625	7	28
$(4+7i)(4+7i)$	$-33+56i$	33	56	65	4	7

Of course, the identity:

$$(a_1^2 - a_2^2 - a_3^2 - \ldots - a_n^2)^2 + (2a_1a_2)^2 + \ldots (2a_1a_n)^2 = (a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2)^2$$

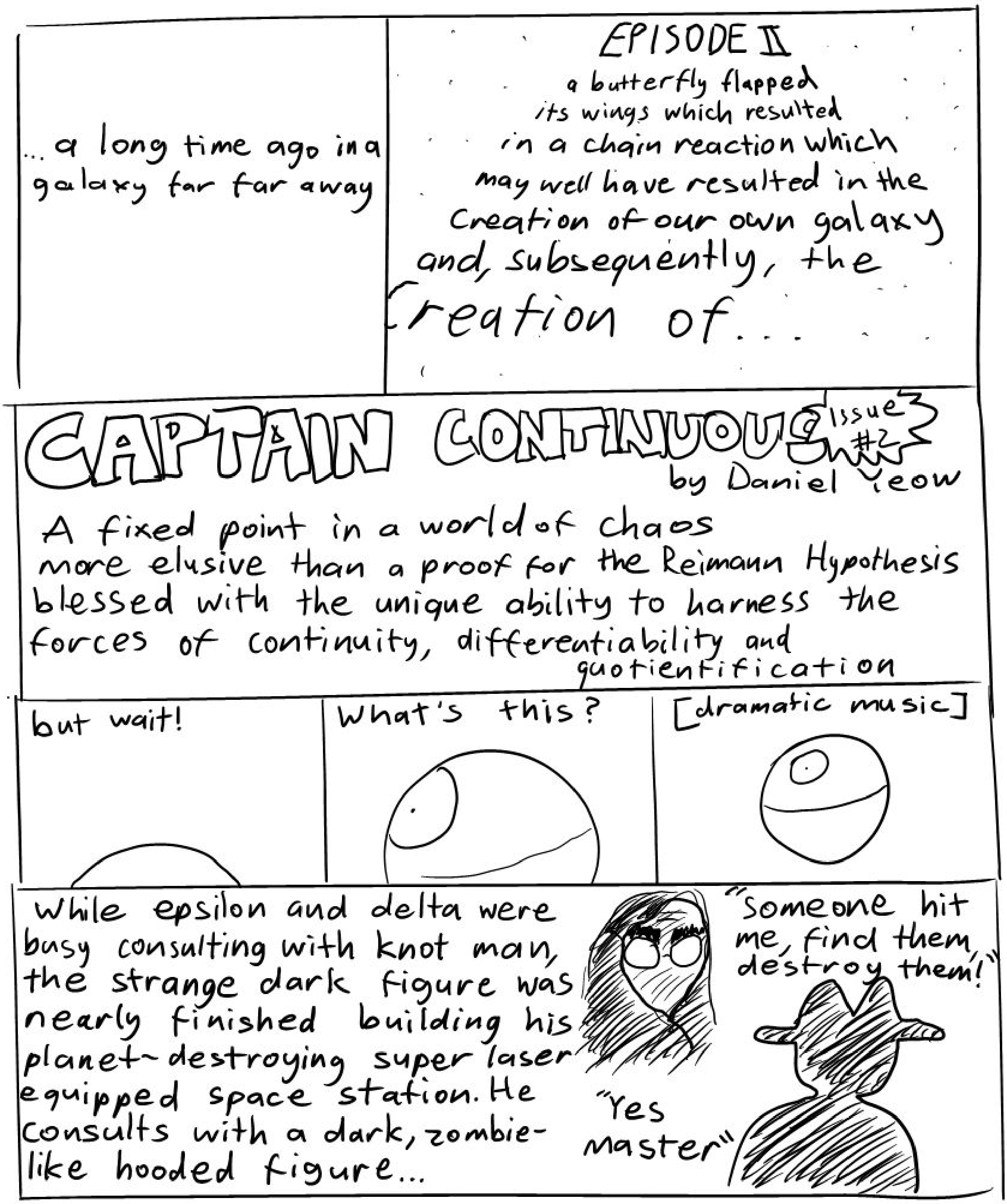
generates squares which are sums of n squares. But the algebraic structure only happens when $n = 2$ (complex numbers) and when $n = 4$ (quaternions).

Quaternions are "numbers" of the form $a + bi + cj + dk$, where a, b, c, d are real numbers, and the "units" i, j, k satisfy $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$; the analogue of squared absolute value of a complex number is $a^2 + b^2 + c^2 + d^2$.[1] The quaternions have a multiplication that is associative but not commutative. The division algebra of quaternions is equivalent to the algebra of complex 2×2 matrices of the form: $\begin{bmatrix} a - \mathbf{i}d & \mathbf{i}b + c \\ \mathbf{i}b - c & a + \mathbf{i}d \end{bmatrix}$, in which $\mathbf{i}$ is the imaginary unit. If $a_1 + a_2 i + a_3 j + a_4 k$ is a quaternion with integer components, then the above identity with $n = 4$ generates a sum of four squares which is also a square. For example, the quaternion $q = 2 - i + j + 2k$ produces $(-2)^2 + (-4)^2 + 4^2 + 8^2 = 10^2$, and thus a "quintuple" $(2, 4, 4, 8, 10)$ (or $(1, 2, 2, 4, 5)$, dividing by the common factor.) The quaternion $q' = 2 - 3i - j + 5k$ generates the "quintuple" $(31, 12, 4, 20)$, with $31^2 + 12^2 + 4^2 + 20^2 = 39^2$. The product $q\,q' = -8 - i - j + 18k$ generats the "quintuple" $(-262, -16, -16, -288, 390)$. But the associaive product of "quintuples" is very multivalued, since any combination of $\pm$ signs of the quaternion components will work.

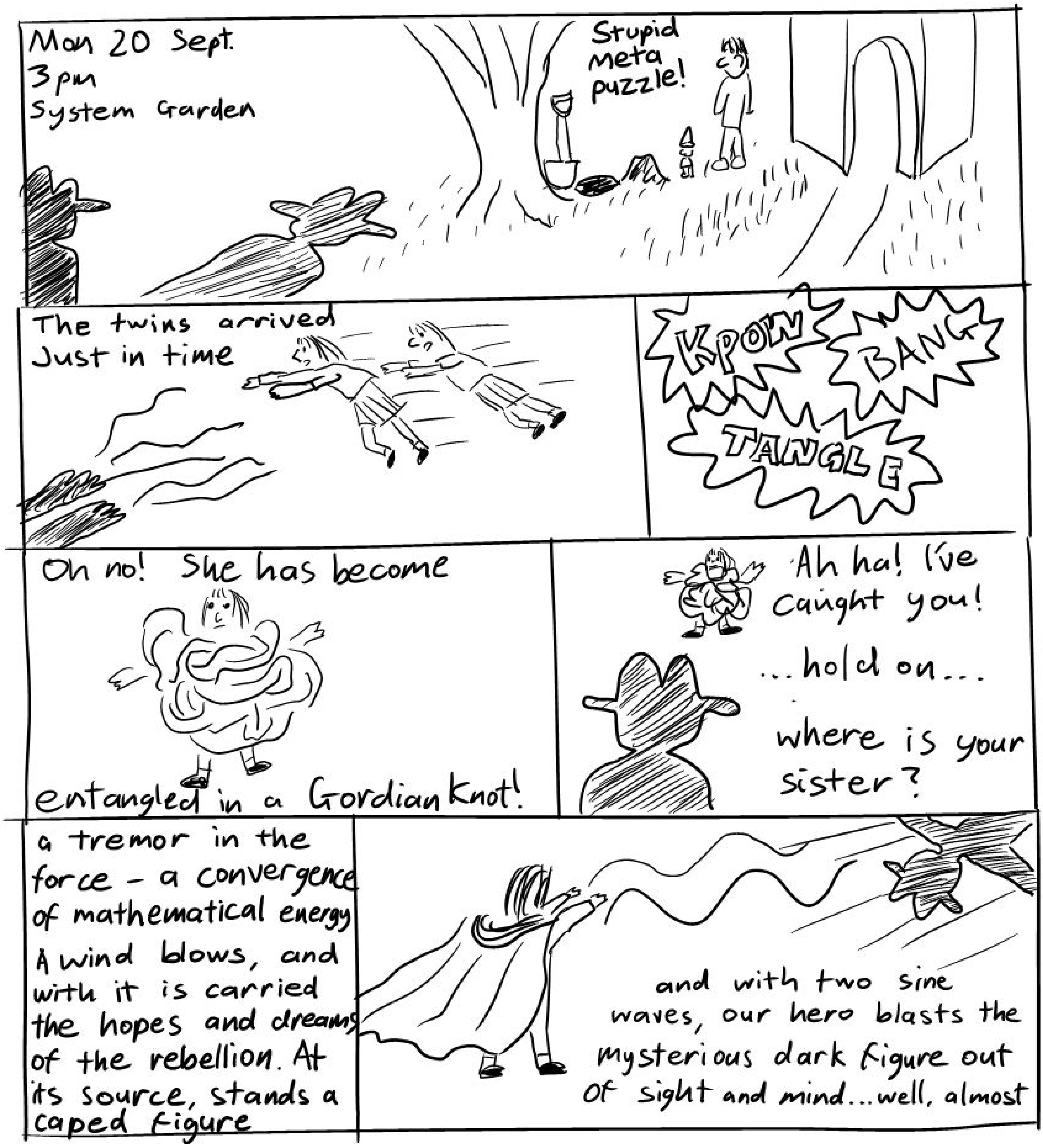
— Bruce Craven

In a foreign country, a mathematician, a physicist and an engineer are about to be guillotined. The mathematician has his head on the block, and the executioner pulls the rope, but the blade doesn't fall. "Aha! $P(E) = \frac{1}{2}$ so both events are equally likely, and all is well." He can't be executed twice for the same crime, so he walks free. The physicist is next up, but again the blade fails to fall. "Aha! $P.E. = \frac{1}{2}mv^2$ and $v = 0$, so all is well." He also walks free. Finally, the engineer has his head laid upon the block. He looks up at the release mechanism and says, "Hey! Wait a minute! I think I see your problem. . . "

[1] If you're new to quaternions, it's an interesting exercise to prove that the absolute value of a product of quaternions is the product of the absolute values.



... a long time ago in a galaxy far far away
EPISODE II
a butterfly flapped its wings which resulted in a chain reaction which may well have resulted in the creation of our own galaxy and, subsequently, the creation of...
CAPTAIN CONTINUOUS
Issue #2
by Daniel Yeow
A fixed point in a world of chaos more elusive than a proof for the Reimann Hypothesis blessed with the unique ability to harness the forces of continuity, differentiability and quotientification
but wait!
What's this?
[dramatic music]
While epsilon and delta were busy consulting with knot man, the strange dark figure was nearly finished building his planet-destroying super laser equipped space station. He consults with a dark, zombie-like hooded figure...
"Someone hit me, find them destroy them!"
"Yes master"


Mon 20 Sept.
3pm
System Garden
Stupid meta puzzle!
The twins arrived Just in time
KPOW
BANG
TANGLE
Oh no! She has become
entangled in a Gordian Knot!
Ah ha! I've caught you!
...hold on...
where is your sister?
a tremor in the force - a convergence of mathematical energy
A wind blows, and with it is carried the hopes and dreams of the rebellion. At its source, stands a caped figure
and with two sine waves, our hero blasts the mysterious dark figure out of sight and mind...well, almost


Hey sis, looks like you're all tied up!
Yeah, you're funny! Help me out of this thing
Hmm... how to untie the Gordian knot?
I know! I'll use knot man's utility belt...
Let's see... String - no... paper strips umm... glue? mmm, donut, genus-1 mMm ah! Scissors!
What are you doing?
I'm cutting the Gordian knot...
but that's cheating
No more than guessing answers in the maths olympics...
UMO
Try 2004!
Wow! a zen moment! you did it sis!
Where did you get that belt?
Oh. I found it at a party. c'mon, let's go!
Where?
The death star is protected by a field generated in an underground particle accelerator under melbourne university.
really? where's the entrance?
BROWNLESS BIO MEDICAL LIBRARY
Duh, the brownless biomed library!

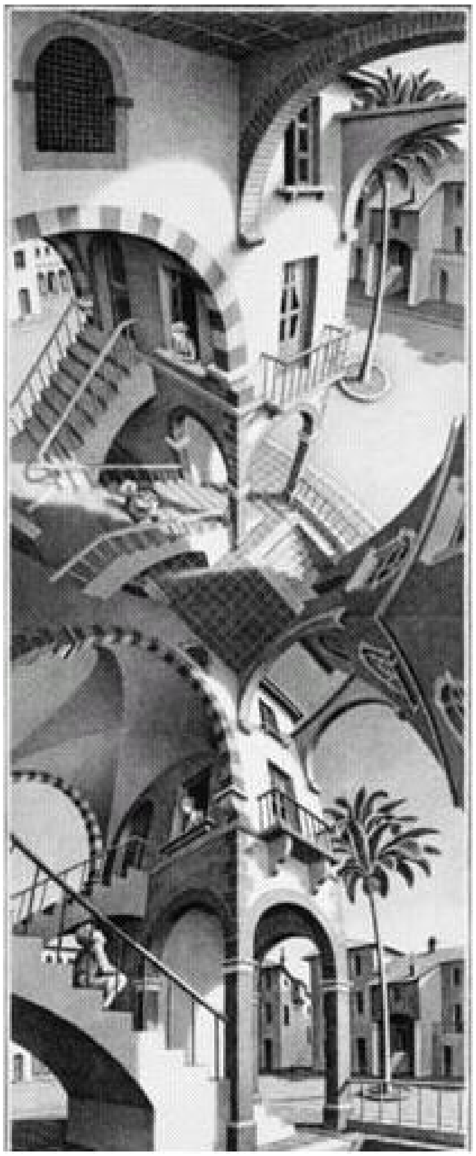

In the library, a librarian fumbles with his tie...
Stupid tie!
Mmm Cute librarian
Why don't you get changed sis?
Hey! I saw him first!
Let me help you with your tie...
I need a mirror
or maybe some mirror matter
here, let me help you with that...
mmm... you have really nice... eyes
phew, those leotards were not comfy
You are knot not man, but you know much about knots...
C'mon lets go
What! This particle accelerator is also a field generator for a space station?
Yes! A DEATH STAR! How do we get in to deactivate it?
The answer to that question is hidden in amongst all these slips of paper...
...they all yield one word...
Hey, maybe its this one
CSID
251, 1451, 2423
4937, 6389
ANSWERS TO dyeow@bigpond.com

M.C. Escher – The Soul of a Mathematician

Maurits Cornelis Escher has always been somewhat of a favourite among mathematicians. The Dutch artist's unique approach to art never sat comfortably with conventional art critics. His lithographs and woodcuts, however, have proven particularly fascinating for many mathematicians, who see in them striking parallels with areas of their own craft, including complex analysis, wallpaper groups and hyperbolic geometry. Escher, however, always professed to be ignorant of the mathematics behind his work – when two professors of maths tried to convince him that he had, in *Print Gallery*, drawn a Riemann surface, he said that

> I doubt if they are right, despite the fact that one of the characteristics of a surface of this kind seems to be that the centre remains empty. In any case, Riemann is completely beyond me and theoretical mathematics are even more so, not to mention non-Euclidean geometry.

The most incredible act of intuitive mathematics that Escher ever performed surely occurred in the making of *Print Gallery*. In it, the viewer's eye starts with the young man in the lower left-hand corner, and is drawn up into the print he is looking at, of a waterfront town. Following the buildings along the harbour, one comes across a print gallery, inside which the original man stands! Escher described how he conceived the idea behind this print: It was a "question of a cyclic expansion or bulge, without beginning or end." He conceded that *Print Gallery* caused him some "almighty headaches", and he made many attempts before he was able to produce the correct grid on which to draw the picture. He originally drew a grid so that the bars of the window and the houses in the harbour would be composed of perfectly straight lines, but scrapped this in favour of the curved version seen in the print, so that "the small squares (of his grid) could better retain their square appearance." In a recent article by B. de Smit and H.W. Lenstra Jr., the famous number theorist, the full mathematical complexity of the structure of this print was laid bare (see 'What's Going on in *Print Gallery*' for the details) Escher's intuitive decision to try to make the small squares retain their square appearance, of course, corresponded to trying to make his mapping conformal! Escher was also very interested in the theory of perspective. The classical theory of perspective, in which depth is depicted by drawing lines that are parallel in real life in such a way that



they converge at a vanishing point, was invented by Filippo Brunelleschi in the 15th century. This theory, however, is only an approximation. If, for instance, one were to look at two parallel lines on a flat surface from somewhere above the surface, they would appear curved. In fact, they would be shaped like sine curves. Escher made many prints studying exactly how such curved lines should be drawn, including *High and Low*, and measurement shows that his curves were very close to sine curves. It is also interesting how Escher played around with the vanishing points in this print, by making the centre of the picture the zenith of the lower half of the picture as well as the nadir of the upper half [2].

[2] The zenith is the vanishing point that is vertically up, while the nadir is the vanishing point that is vertically down

The mathematical complexity behind many of Escher's works is particularly interesting because he intuited the ideas behind most of his pictures, rather than starting from a mathematical understanding of the concept. Escher seems to have been unique in that he had a professional mathematician's highly-developed aesthetic sense of mathematical elegance without any real understanding of more than the rudiments of mathematics. He was inspired by the diagrams in a book given to him by H.S.M. Coxeter, the legendary geometer (co-author of *Geometry Revisited*, the most awesome book on Euclidean geometry ever written). A diagram of the Poincaré disk, in particular, "gave me quite a shock". Many of the works of his later years, when he was interested in ways of representing infinity, were inspired by pictures shown to him by Coxeter. These included diagrams from non-Euclidean geometry and complex analysis. Coxeter published a paper after Escher's death demonstrating that one of Escher's works based on the Poincaré disk (*Circular Limit III*) was accurate "to the millimetre" – an exact construction requires the use of hyperbolic trig functions and logarithms, but Escher used only his ruler, compass, and formidable intuition.

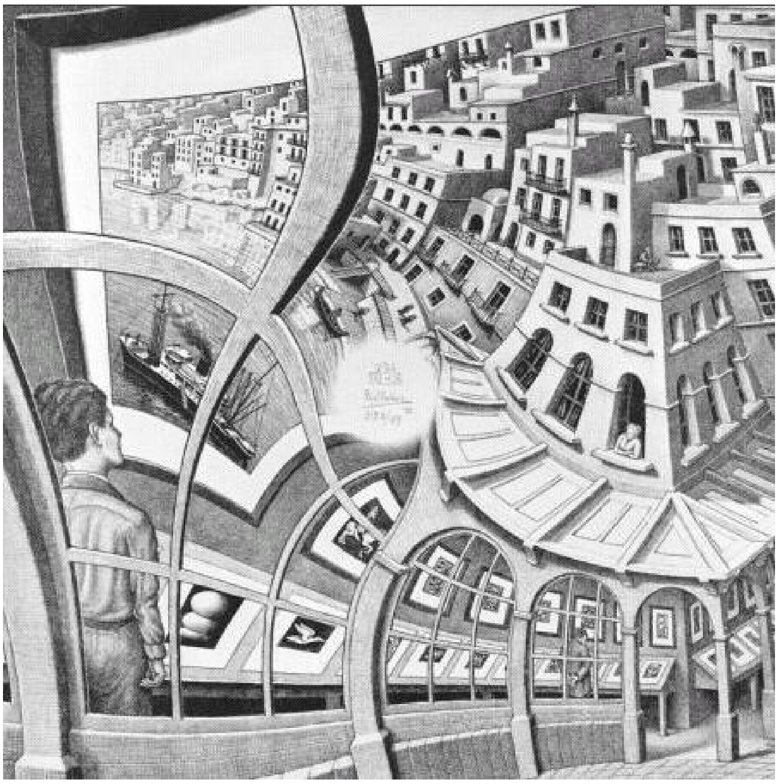
Maurits Escher was an unconventional artist. He never had any interest in expressing his emotions through his art, nor in the portrayal of the conventional ideas of beauty. Instead, he depicted the ideas or objects that fascinated him, and often, the ideas that took his fancy bore striking similarities to those that interest mathematicians. Escher himself said that

> I never feel quite at home among my artist colleagues; what they are striving for, first and foremost, is "beauty" – albeit the definition of that has changed a great deal since the seventeenth century! I guess the thing I mainly strive after is wonder, so I try to awaken wonder in the minds of my viewers.

This, certainly, seems more characteristic of a mathematician than an artist, and perhaps this is why Escher remarked that "I often seem to have more in common with mathematicians than with my fellow artists".

— Nick Sheridan

What's Going On in *Print Gallery*?

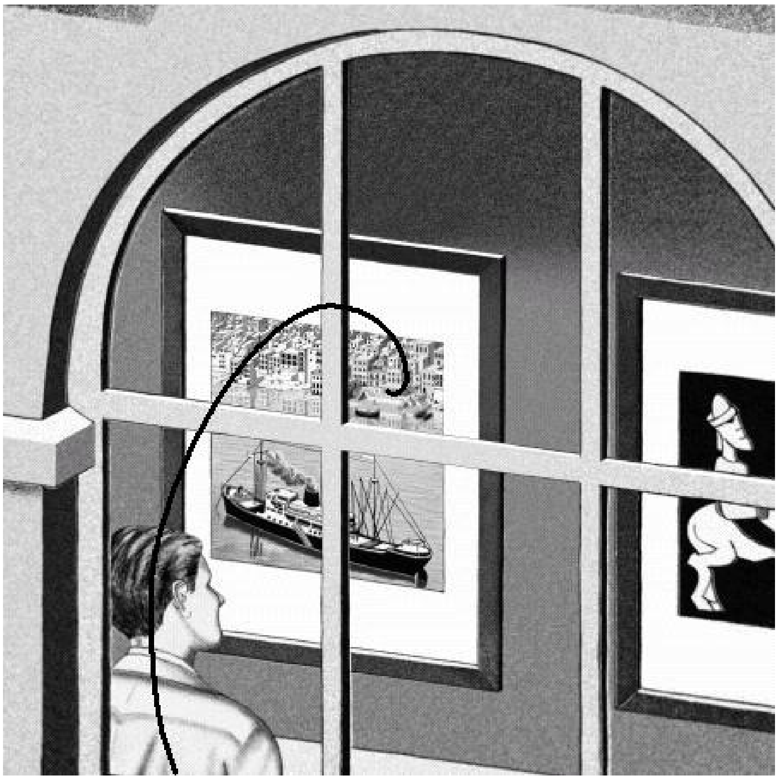


In order to explain the mathematics behind *Print Gallery*, Lenstra and de Smit began with an undistorted picture, showing the young man looking into a print which contained the town, and the print galley, and a smaller copy of the same man looking into a print, and so on ad infinitum (the smaller copy of the print gallery is at the centre of the spiral, and the smaller copy of the man is too small to be seen). The idea is to come up with a conformal mapping (conformal mappings look the nicest because they preserve angles) of the complex plane that has certain properties. We draw the exponential spiral parametrised by

$$z = e^{(\ln(256)+2\pi i)t}$$

on the undistorted picture.

This spiral passes through the young man's head then spirals around until it passes through the smaller copy of the young man's head, and keeps spiralling around and around the centre of the picture, passing through smaller and smaller heads. We would like to map this spiral onto the unit circle. To see why, look at the original *Print Gallery*. If we look at the unit circle



in this picture (which is centred on the blank spot in the middle, and passes through the young man's head), we see that it follows the same route through the distorted picture as the spiral follows through the undistorted picture. Now, it is not too difficult to find a conformal mapping that takes the spiral to the unit circle: $z \mapsto z^{\alpha}$, where $\alpha = (2\pi i)/(2\pi i + \ln(256))$ will do nicely, as the point $\exp((\ln(256) + 2\pi i)t)$ will clearly map to $\exp(2\pi it)$ under this mapping, which is on the unit circle. In fact, if we do perform this transformation on the undistorted picture, we obtain a picture very similar to the original *Print Gallery*[3]. This also enables us to fill in the blank centre of the original print. To see the results, and some excellent animations zooming in to the centre of the picture, go to `http://escherdroste.math.leidenuniv.nl/`.

References:

`http://escherdroste.math.leidenuniv.nl/`
`http://www.ams.org/notices/200304/fea-escher.pdf`
Ernst, B., *The Magic Mirror of M.C. Escher*, Tarquin Publications, 1985.

— Nick Sheridan

[3]There's a bit more to it than this – this function is not well-defined, but it turns out not to matter. See the original paper for details.

Elementary Great Competition Theory

Definition
A *competition* is any set of activities and events involving at least 2 parties, each aiming to demonstrate their superiority over all the others. A competition is said to be *great* if the following axioms hold:

1. The competition involves acts of extreme physical and mental endurance.
2. The competition has at least 500 competitors.
3. During the competition, there is at least one excavation expedition held between the hours of midnight and 3am.
4. During the competition, there is at least one controversial non-disqualification and at least one non-controversial disqualification.
5. During the competition, there is at least one terrorist threat.
6. The competition is worthy of the label 'great'.

Remark
This is a very natural definition. Yet a number of competitions are mistakenly labeled as 'great' in the literature. For example, the Olympic Games only satisfies axioms 2, 4, and 5, so is not *great*.

Theorem – (The Greatness Theorem)
There is exactly one *great competition*, the Melbourne University Puzzle Hunt (MUPH).
Proof
It is well known that the University Maths Olympics (UMO) and the MUPH are the only competitions worthy of the label great. Of the other greatness axioms, the UMO only satisfies axiom 1. Hence it is not *great*. Since the MUPH is the only competition that is worthy of the title great and satisfies all the other greatness axioms, the MUPH is the only *great competition*.

From this point on we will concern ourselves only with developing the theory of *great competitions*. Much of this classical field of study arose from the events of the 2004 MUPH.

Historical motivation for great competition theory

The 2004 MUPH saw competitors spend a week solving a selection of fiendishly difficult puzzles, some cryptographic, some musical, some linguistic, and some computational. Competitors entered in teams between 1 and 10 people. Between 3 and 4 puzzles were released daily for 5 days, the last of which used the answers to all the previous puzzles to indicate the location of a secret treasure. Competitors submitted answers electronically by entering them into an online submission form (impressive yet unreliable technology for its day). Hints to the puzzles were also periodically released so that all puzzles became solvable after a sufficiently long time. Some 627 people competed, coming from all faculties of the university and beyond.

Theorem – (The Shamelessness theorem)

Competitors lose all shame when faced with a significant mental challenge.

Proof

We will proceed by example (the general case is left as an exercise for the reader).

Answers submitted on the puzzle hunt answer-checking system in 2004:

- “monkeyspankingpoodle”
- “himumsroom…anyhintsforthemetta?…carn,justalittleone?”
- “zpsegwwippbctcdkia”

Corollary – (The Crossing The Line theorem)

When competitors are finding part of the competition particularly challenging, they may make jokes that are a little close to the line between humour and a quick call to the ‘Terrorist Hotline’.

Proof

Follows from the Shamelessness theorem.

Historical Remark

In the 2004 MUPH one team (who will remain nameless in case they are subsequently arrested by ASIO), rather dismayed at their inability to solve a puzzle, resorted to threatening the MUMS committee. Indeed they set an impressive ultimatum: either the organizers tell them the answer, or they bomb the MUMS room. Now the MUMS committee (as a matter of policy) does not negotiate with terrorists, so the threat was ignored. There have been no known bombings of the MUMS room to date.

Theorem – (The Jenkins-Jaschan theorem)
If someone is doing something unthinkably horrid then give them a job.
Proof
Beyond the scope of this book
Historical Remark
This startling, counter-intuitive result first arose when Mr Russell Jenkins, hacker-extraordinaire and key member of the Lambda Xi Fliers (an all star team of staff from the first year learning centre), decided to take a more 'creative' approach to solving puzzles. Putting his skills to the ultimate test he hacked into the answer submission system by a clever mix of local knowledge and good guesswork. Russell, being the honest man he is, admitted his dishonest ways to the event organizers, who were left with a conundrum. Should The Lambda Xi Fliers be disqualified for his dastardly act? As one member of the MUMS committee observed,

> "If we disqualify them, next we'll see Penny charging down the corridor, mace in hand, with Russell in her sights."

Just like the German Firewall provider Securepoint, who employed admitted hacker Sven Jaschan, the puzzle hunt organizers decided to give their loose cannon a job. And so that is how the answer-checker got fixed, and the puzzle hunt automated scoring system was born.

Lemma – (The Dethridge lemma)
If you can't beat them and you can solve 'disc', join them
Proof
Team Room 106 won the 2004 MUPH.
Comment
This theorem is often misquoted as:

> "If you can't beat them, join them"

This simplification makes the theorem non-sensical. One cannot join a team who are in a more powerful position without significant bargaining power.
Historical Note:
Team Room 106 (a collection of Statistics lecturers, honours students and others) were powering through the puzzles of the 2004 MUPH until they hit 'disc'. This puzzle involved the prime factorisation of:

37249588852282552285743302297396100822355179407163315972232664422849771058311

It's no wonder a number of teams called it 'evil'. But John Dethridge, who was going solo at that stage, had the goods and delivered them to Team Room 106 in return for a place on their illustrious team.

For completeness, we ought to quote a famous result from Extreme Competition Theory that originally arose in the context of Great Competition Theory.

The Assange-Chiodo Lemma
It is possible for a male to gain significant attention from a female he wishes to court by using only a necktie.
Proof
See http://www.ms.unimelb.edu.au/~mums/puzzlehunt/acts/act5.pdf

Theorem – (The Fundamental Theorem of Great Competitions)
If you miss out on the fun of the MUPH in any given year, don't worry, you can always compete the following year.

— James Saunderson

The Adventures of Ali Ibn Abi Taalib – II

One day a person came to Ali, thinking that, since Ali thinks he is so smart, I'll ask him such a tough question that he won't be able to answer it, and I'll have the chance to embarrass him in front of all the Arabs.
He asked "Ali, tell me a number, that if divided by any number between one and ten will always come in the form of a whole number and not as a fraction." Ali looked back at him and said, "Take the number of days in a year (in the lunar calendar) and multiply it by the number of days in a week, and you will have your answer."
The person was astonished but, as he was so arrogant, he still didn't believe Ali. He calculated the answer Ali gave him. To his amazement, he found that:
The Number of Days in a Year = 360 (in the Arabic Calendar)
The Number of Days in a Week = 7
The product of the two numbers = 2520
And 2520 is divisible by all of the numbers between 1 and 10.

Paradox Problems

The following are some maths problems for which prize money is offered. The person who submits the best (clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number. Solutions may be emailed to

`paradox@ms.unimelb.edu.au`

or you can drop a hard copy of your solution into the MUMS pigeonhole near the Maths and Stats Office in the Richard Berry Building. Congratulations to James Wan and Michael Dann, who submitted correct solutions to the problems from the last edition. Michael's solution to the problem of the hats is shown below. Shaun Gladman also submitted a correct solution to this problem (but Michael got there first).

The Problem of the Hats: Imagine a line of people buried in the sand. Each person can only see all the people in front. There can be no communication except at the start – BEFORE the people are buried. A man with a sword starts at the back and asks "What colour is your hat?" The person at the back may say "black" or "white". If they are right, then they live, if not, they are killed. Either way, the people in fron have no way of knowing the result. All they can hear is "black" or "white". The sword-guy continues down the line until everyone has been asked. What's more – he has bugged their pre-burial meeting so he may try to foil whatever plan they come up with.

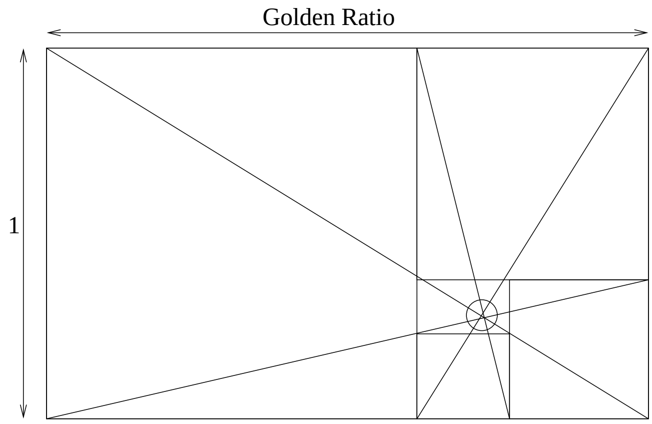
Michael Dann's solution: Firstly, suppose there are an even number of people in the sand. The person at the back of the line has an odd number of people in front of him. He must see either an even number of white hats and an odd number black hats, or vice-versa. He calls out the colour of the even number of hats. Suppose for argument's sake that he calls "white". He may or may not get his head chopped off. However, the next person is now safe: if he looks up and sees an odd number of white hats, he knows that he must be wearing white. If he sees an even number of white hats, he must be wearing black. So he calls the colour of his own hat and is safe. The rest of the group takes note of the call. If the call was "white", they know that there are an odd number of white hats remaining amongst them. If the call was "black", they know that there is an even number of white hats left. So the third guy in the line can deduce his own hat colour, and so on. Everyone is safe with this plan except for the one who starts.

Now suppose there are an odd number of people in the line. The idea is to just begin the original plan with the second person from the back. But this leaves the first and second people in unsafe positions. However, the first person can help secure the second guy: The first guy knows what the second guy can see. He therefore knows what the second guy's call will be in the original plan. So he knows whether or not the second guy is going to get killed. The team agrees that the first guy's call will be interpreted as follows: "White" means "Don't worry second guy, you won't be killed." "Black" means "Change your call, second guy!" If the call is "white", the second guy is safe and the plan proceeds as usual. If the call is "black", the second guy changes his call so that he's safe. The group knows a change has occurred and just imagines that the second guy called the opposite colour. As before, everyone left is safe. So no matter how many people are in the line, only one is unsafe with this plan.

NB: It is possible to avoid this slightly complicated method for the odd case, by using a similar method to the even case. This solution is pretty ingenious though!

Problems for this edition:

1. ($) You are going for a drive on a circular path in the desert. There are a number of petrol stations on your route. The total amount of petrol in the stations around the route is just enough to travel the whole way round the track. You don't have any petrol in your car to start off with, so you need to choose a petrol station that will allow you to make it the whole way round the track without running out of petrol. Prove that you can always do this (what if there are an infinite number of petrol stations? Or if there is a continuous distribution of the petrol along the road?).
2. Prove that the lines that intersect in the small circle intersect at 45°:



3. Suppose n people, all having distinct heights, are standing in a single-file line. Call a person "visible" if he or she is taller than anyone in front of him or her (and so is visible to a person looking at the line from the front). Assuming a random distribution of the people into the lines, how large must n be in order that the expected number of visible people is 10?

A Mathematical Cryptic Crossword

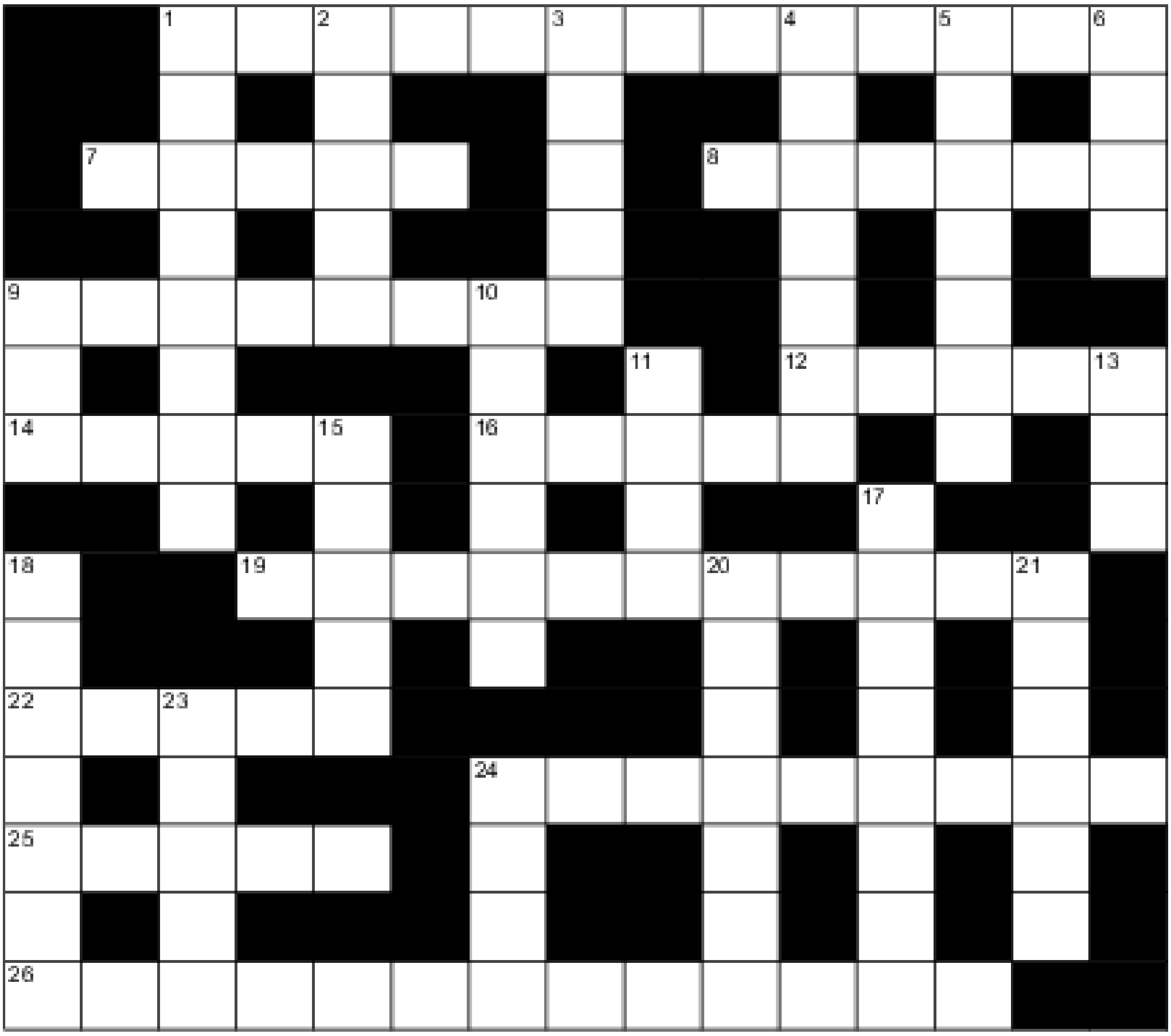
ACROSS

1. One plus twelve, written another way (3, 4, 6)
7. Trickery solved a theorem of Fermat (5)
8. Cardinal is feeling less than usual (6)
9. Write your name on this shape (8)
12. They seem to point to transcendental functions (5)
14. Mathematician sounds like he lubricates machinery (5)
16. A kind of lattice parasite (5)
19. Citing of RSA broken by doing this quickly? (11)
22. Without agreements and recesses (5)
24. Use this to get the answer to 24-across (9)
25. Bigger dose of caffeine in a mathematician (5)
26. Mmm... big number! (5, 8)

DOWN

1. Shape comes from calculating integral (8)
2. Zero times one million gives a letter from Greece (5)
3. Merger with the opposite of a charged atom (5)
4. I should sort of be raucous(7)
5. Lively rave after six initial beers (7)
6. I am the size of a number (4)
9. Food produces constant sound (3)
10. Able to be shaped like a football(6)
11. Better take time out for a drink (4)
13. Place a collection of elements (3)
15. This field is true south (5)
17. Military unit operation (8)
18. Sunburnt man is touching, but not intersecting (7)
20. Produces current and logical proof for natural numbers (7)
21. Mathematician mixed gas and oil (6)
23. Operations research and differential equations give right arrangement (5)
24. Irrational outbreak (4)

— Norman Do



Thanks

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Melbourne University Mathematics and Statistics Society
Presents:

Upcoming Events

Trivia Night

Thursday 28th of October

Seminar:	**Date:**
The Lost Notebook	**Monday 25th October**

For more information:

- keep an eye out for posters in the Richard Berry Building
- check out our website and subscribe to our mailing list at:

  `http://www.ms.unimelb.edu.au/~mums/mlist`