Paradox

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MUMS

PRESIDENT:	James Zhao j.zhao1@ugrad.unimelb.edu.au
VICE-PRESIDENT:	Maun Suang Boey m.boey@ugrad.unimelb.edu.au
TREASURER:	Han Liang Gan h.gan5@ugrad.unimelb.edu.au
SECRETARY:	Michael de Graaf m.degraaf@ugrad.unimelb.edu.au
EDUCATION OFFICER:	James Saunderson j.saunderson@ugrad.unimelb.edu.au
PUBLICITY OFFICER:	Alisa Sedghifar a.sedghifar@ugrad.unimelb.edu.au
EDITOR OF *Paradox*:	James Wan jim_g_wan@hotmail.com
1ST YEAR REP:	Adrian Khoo my_address_is_this@hotmail.com
2ND YEAR REP:	Stephen Muirhead s_muirhead22@hotmail.com
3RD YEAR REP:	Joanna Cheng j.cheng1@ugrad.unimelb.edu.au
HONOURS REP:	Nick Sheridan n.sheridan@ugrad.unimelb.edu.au
POSTGRADUATE REP:	Norman Do norm@ms.unimelb.edu.au
WEB PAGE:	http://www.ms.unimelb.edu.au/˜mums
MUMS EMAIL:	mums@ms.unimelb.edu.au
PHONE:	(03) 8344 3385

Paradox

EDITOR:	James Wan
LAYOUT:	Stephen Farrar
WEB PAGE:	http://www.ms.unimelb.edu.au/∼paradox
E-MAIL:	paradox@ms.unimelb.edu.au
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Words from the Editor

Welcome to the first edition of Paradox for 2007. As you know, Paradox is the magazine of MUMS, and has existed in its present form for more than 20 years. MUMS is the Melbourne University Mathematics and Statistics Society, which is at least 70 years old. (Mathematics is the study of patterns in everything, and is as old as humanity. That's why you should read this magazine.)

MUMS and Paradox are dedicated to promote interest in maths outside the university coursework. That's why we are having the insomnia-inducing, potentially-cash-earning, Google-powered Puzzle Hunt (see our article for details). There are, as usual, jokes, stories, and problems with cash prizes in this issue. We also have an article explaining how to solve a problem (the Dinitz conjecture) that was open for a long time, hence demystifying mathematical research for those interested.

But remember, Paradox is a magazine written by the students. So if you have any ideas, be them articles on your area of interest, jokes, problems, or just want to get published in a magazine that's being read in no less than 5 unis in Australia (and more overseas), then feel free to send an email to me.

— James Wan

About the cover:
The Norwegian stamps honour Niels Henrik Abel (1802 – 1829). The design is based on his statue in Oslo, which is believed to be the only instance where a mathematician is depicted in the nude. In the statue, he is also seen standing on two vanquished men, who are thought to be the Elliptic function and the Quintic equation.

Words from the President

Welcome to a new year of MUMS, a year of many things; of shoes and ships and sealing wax, of cabbages and kings. Of course, those particular things aren't likely to be administered by MUMS, but we will do our best to organise a fun semester or two for you all.

Coming very soon is the premier event of the MUMS calendar, the Puzzle Hunt, which will be held in the week of 26 – 30 March. For those that haven't entered before, Puzzle Hunt is a week-long puzzle-solving competition you can do from the comfort of your living room. You can find out more or register a team at http://www.ms.unimelb.edu.au/∼mums/puzzlehunt.

While you're outside your living room though, we'll be continuing to run our almost-weekly seminars on all sorts of weird and entertaining topics not likely to be seen in lectures. These seminars don't require any mathematical background beyond approximately Year 10 and come with their own refreshments, and are a great place to meet like-minded people.

We'll also be holding our Annual General Meeting some time in the next month or two, so please come along to vote for your favourite fellow maths students, or run for election yourself! Finally, as always, we will finish the semester off with our biannual trivia night, and come back next semester with, among other things, our world-famous Maths Olympics.

— James Zhao

How to Win a Puzzle Hunt!

For those among you wondering, what on earth is a "Puzzle Hunt", it's an annual event run by MUMS which involves a set of puzzles released daily for five days. (Each puzzle has a story and a question, which you may not notice upon first reading). The puzzles are lateral thinking problems that require ingenuity, insight, and occasionally sheer luck to solve. At the end of the week, a meta-puzzle is released. Combining the answers for all the previous puzzles in a certain way, you should then be able to make the mad dash for the hidden item on campus and win the **$200** first prize.

Now that introductions are done, it's time to get into the real meat of this article: How to solve puzzles. The first and most important step in solving a puzzle is to think logically. Most puzzles will have multiple steps, however each step should follow logically from the previous one. We're not saying that it will be obvious what the steps will be, but once you have your moment of "Zen" and see what to do, it should immediately make sense.

Another key component in being able to solve a puzzle is for your team to have excellent communication, so the idea pool is as full as possible and no one is out of the loop. In 2005, a puzzle titled Christmas was fairly obviously a sudoku, except it used the alphabet with 25 5×5 squares. Very early on, one of our team members (Ray) had worked out that in the story, people were singing "The First No-el". He then knew that the letter L was the one that was left out of the sudoku. However, this information was not passed onto the rest of the team until two days later, and 2 points were thrown away.

Google is everyone's best friend. When given a lot of nonsense, or sentences that look vaguely familiar, just putting words into Google will provide the first step.

Looking for patterns is also always a good place to start too. If you see a pattern that consistent throughout the whole puzzle, it's almost certainly going to be involved somehow.

Lastly, comb the story text and look at the title for thinly veiled hints. Almost every puzzle will have small hints written into the story, and the title is always in some way related to the puzzle. Hints in the story are often fairly easy to pick up, as they seem to slightly disrupt the flow or just stick out like sore thumbs. More often than not, you'll find that searching the text for the little clues and highlighting them provides the inspiration to work out the next step.

So it's easy for us to just write up some hints and tips, but putting these to practice is easier said than done. So now you'll see some guides and solutions on some of our favourite puzzles from previous years.

2005 5.5 Defenestration – by James Zhao

1	43	24	17	36	46	8
45	$\mathcal{O}$	$\mathcal{H}$	39	$\mathcal{X}$	$\mathcal{R}$	31
15	$\mathcal{T}$	$\mathcal{N}$	29	$\mathcal{E}$	$\mathcal{K}$	6
9	12	10	25	40	38	41
44	$\mathcal{D}$	$\mathcal{O}$	21	$\mathcal{O}$	$\mathcal{T}$	35
19	$\mathcal{E}$	$\mathcal{X}$	11	$\mathcal{T}$	$\mathcal{J}$	5
42	4	14	33	26	7	49

At first look, like many of James Zhao's puzzles (which are widely known as being "evil"), it's not obvious at all where to start. The trick to this puzzle is to look for patterns. If you look carefully, you'll notice that no number is repeated. As an extension of that, you've got a 7×7 grid, and the numbers 1 through to 49. Now we're on to something! Surely each number must be used once.

This should sound familiar to most people. With enough team members, at least one of them should realize that it is a magic square, where each column and row must add up to the same sum. A quick check with the supplied numbers will confirm this.

Where to next? Well, logically, completing the magic square must surely be it. But that looks tricky. However, with some more cogitation, you'll notice that numbers opposite each other from the centre of the square have the same sum, as you'll find with many magic squares. Now it isn't all that hard to fill out the rest of the square.

Surely you must be done by now! Well, no. This is an evil puzzle after all. You've got these letters with matching numbers. You know somehow that you need to work the letters into the puzzle. What's a logical thing to do? Order the letters in the order of the corresponding order of numbers of course!

Now you've got this funny looking string XTHEODOREJKNOTTX. Looking into that, you can see Theodore J Knott. Problem is, who the hell is that? This is when trusty Google comes into play. Google Theodore J Knott and a page off the MUMS website appears first. Theodore J Knott is none other than the legendary superhero of the magazine you're reading right now: Knot Man.

2005 1.4 Division – by James Zhao (again!)

The title seems to suggest that we will require some sort of calculations. Hah, too easy, you may say while expanding out all the brackets. But do not proceed too hastily. Upon close observations, you can see that there are just too many variables for you to do a simple "division". So what should you do? While staring down at these two letter combinations, you begin to notice something. You've seen them, but where?

$$\frac{\dfrac{u^2(b+h+o+s)+s(i+n+1)}{(c+i)(r+1)+f(r+m)+m(g+t)}+\dfrac{b(h+1)+c(l+s)+h(g+o)}{(l+r+t)a+(c+g+s)e}}{\dfrac{(t+s)c+(z+1)n+(p+e)u}{(a+l)(r+u)+(r+t)(e+h)}+\dfrac{(d+r)b+(p+t)m+k+w+y}{(c+p)(d+o)+(n+p)(a+b)}}$$

Indivisible?

Read the hints in the text. "Arts Centre", "Old Geology" and "Chemistry Building" catch your eye. Chemistry you say? Bingo! The letters represent chemical elements. You grab your best friend Google and print out the periodic table. Lots of elements…what should you do with them? You start off by ticking off the ones you have found in the puzzle, but then you notice something. The elements seem to spell out something:

D A N T E

Wow, you've got it! You have finally solved your first Puzzle Hunt puzzle! You go to the puzzle hunt home page and type in your answer, a mere administrative process to confirm your greatness.

"You walk into the food factory and find ten thousand kittens crying."

You fall into a sense of shock and disbelief. You were wrong! How could it be? The answer may lie in the bunch of dots under DANTE that you conveniently forgot about.

L V V I

What could this mean? You are convinced that these represent the letters LV VI. Roman numerals? You look up the Roman numerals and indeed the letter L is one of them. 55 and 6, you tell yourself. 55 and 6 of what? Anyway, who is this Dante guy?

Have no fear, Google is here, or so he says. After reading website after website of material regarding Dante, and after many attempts at taking the 6th word of the 55th sentence, your morale starts to decline. More and more kittens are found crying. How many more food factories are there left, you wonder. It's 1am and everyone is asleep. Your only comfort is the fact that no other teams have solved this puzzle.

One last website, you tell yourself as you open "Dante's Inferno Test". It seems like a little test to show where you would go in the afterlife. The results are in and "You are sent to the First Level of Hell - Limbo". First level?[1] Hmm, you think to yourself. Level one, Lv 1. In this moment of "Zen", you quickly browse the site for other levels of hell, in particular, the 6th. The city of Dis. DIS! It must be it. In this case it was pure luck that we ran upon this site, but hey, clever people make their own luck!

[1]Good for you. Your favourite editor only managed the fifth level.

Well, now that you're all puzzle solving experts, there should be no reason why you wouldn't enter the next puzzle hunt, which starts on Monday the **26th of March** and ends on the **30th of March**. Use our masterful hints, throw in a healthy dose of luck, and you could soon be $200 richer! For more information, including registering your own team and full versions of all the puzzles in previous hunts, go to www.ms.unimelb.edu.au/∼mums/puzzlehunt.

— Han Liang Gan and Ray Komatsu

Colouring the Masses: The Dinitz Conjecture

1 Introduction

One (formerly) popular problem in graph colouring is the Dinitz conjecture. The Dinitz conjecture, now a theorem, was a problem that left mathematicians confounded for about 15 years. Before going any further, it is always best to illustrate the problem by relating it to a real-life, albeit hypothetical, moment. Suppose in a tutorial for (insert the subject code of any maths subject you like), there are 16 students. All the students in the tutorial are seated in a square grid with 4 students on each row and each column. Let's suppose that each student is given a list of 4 colours, where each list may or may not be identical and the studentis told to choose one colour out of the list. Now, the million-dollar (not literally, since it's not a Millennium Problem) question is: if we were to repeat this as many times as possible to different groups of students, is there always one group that has an arrangement in such that for each row and column, there exists no *duplicate* colours?

For the simple case with 16 students you could possibly enumerate all arrangements to find at least one arrangement that satisfies the condition. Of course, with mathematics, one case is not enough. Will this be true if we have n^2 students with a list of n colours each for every $n \geq 2$? If the list of n colours are identical, this is always true as this case corresponds to a *Latin square.* An n-by-n Latin square is a square with n^2 cells where given numbers $\{1, 2, \ldots, n\}$, each row and column must be arranged in such a way that each number exists in a particular row or column only once. Let's say in our previous example, all 16 students are given a list of 4 colours each, each one labeled 1 to 4. Then an

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Figure 1: Example of a Latin square

$\{1,2\}$	$\{2,3\}$
$\{1,3\}$	$\{2,3\}$

Figure 2: An example with unidentical colour lists

example of a colouring that satisfies the condition is shown in Figure 1. This also happens to be an example of a 4-by-4 Latin square.

In general, given that if we have a tutorial class of size n^2 with each student being given identical lists of n colours, then it is always possible to find such a colouring and the solution would be an n-by-n Latin square. Now, if we extend this to the case when the lists of colours are not necessarily identical, then it is a different matter altogether. Consider the case in Figure 2 for example. If we are not careful in our colouring, then it is possible to miss such a solution. If colours 1 and 2 were chosen for the first row, then there is no solution as colour 3 is the only option remaining in the second row.

The previous example essentially illustrates the question proposed by Jeff Dinitz in 1978: while we can clearly see a solution for identical colour lists, does a solution always exist for unidentical colour lists? Like many difficult maths problems, the Dinitz conjecture could be stated simply without requiring deep, specialized knowledge (look no further than the infamous Fermat's Last Theorem). There were quite a number of mathematicians attacking the problem but it wasn't until 1993 that Fred Galvin provided a solution in [3]. Unlike Fermat's Last Theorem which required a monster of a proof, Galvin's solution was simple and required only about two pages, thereby ensuring that paper was conserved. In fact, the simplicity of the solution caused a "why didn't I think of that?" moment in many mathematicians familiar with the problem. One such lament published in a journal is [2].

In the rest of the article, we shall examine Galvin's ingenious proof of the Dinitz conjecture. More surprisingly, not much knowledge is assumed besides basic understanding of proofs by induction, basic algebra and basic graph theory.

2 Taking a Closer Look

When faced with a mathematical problem, the most important step before one attempts it is to find the right model that simplifies the visualization of the problem. The model must be able to capture the intricacies of the problem and more importantly, it determines the entire approach of the solution. In other words, when you've got a good plan, things are more likely to work out. It's just like planning a trip to the Maldives for example: the first step is to select a mode of transport to determine how you're going to get to your destination. In our case, the destination is a complete solution of the problem. Evidently, the mode of transport is also important as one could *possibly* take a (water-proof) car to the Maldives but get there with a much longer travel time than say, an aircraft. So it is the same with proofs: some methods yield the same results with a more elegant argument.

In the case of the Dinitz conjecture, the problem would be best represented using graphs. First, we start off with some notations that will be used; hopefully, the reader doesn't groan at the sight of these notations. A standard notation for a graph is given by $G = (V, E)$ where V represents the set of vertices and E represents the set of edges in a graph G. For a vertex $v \in V$ the *degree* $d(v)$ is the number of edges connected to (or *incident* to) the vertex.

A *directed graph* $\vec{G} = (V, E)$ is a graph where every edge $e \in E$ has an orientation. Figure 3 shows an undirected graph and an example of its directed counterpart. We denote the edge $e = (u, v)$ to mean the direction is going $u \to v$ in a directed graph. To account for the directions, we would require the concept of an *indegree* $d^-(v)$ and an *outdegree* $d^+(v)$. The indegree specifies the number of incident edges directed towards the vertex v and the outde-

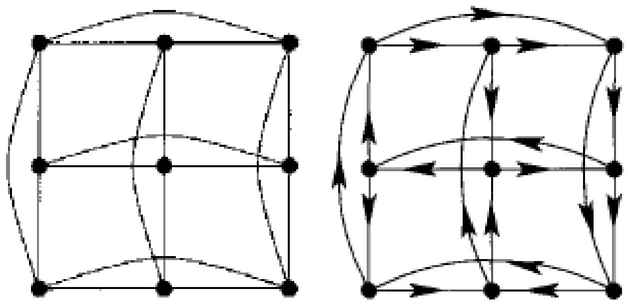

Figure 3: Example of an undirected and a directed graph

gree specifies just the opposite. Additionally, the total sum of the indegree and the outdegree of a vertex v must be the degree of the vertex itself, thus $d^-(v) + d^+(v) = d(v)$. Note that for every undirected graph, there are $2^{|E|}$ directed graphs since there are two possible directions for every edge in an undirected graph. A subgraph G_A is a graph made up of vertices of set $A \subseteq V$ and contains all edges of G between the vertices of A. An *induced subgraph H* of G is a subgraph induced by some vertex set A and $H = G_A$ if the induced subgraph is G_A.

In our case, we can model the problem in terms of a grid-like graph that is shown in Figure 3. The undirected graph on the left in Figure 3, for example, would model a class of 9 students. The edges between the vertices in the graph would correspond to the relationship between the vertices in the graph. Let's generalize this and denote such a graph of n^2 vertices as S_n. The graph in Figure 3 would then be S_3. Each vertex belonging to graph S_n would clearly have a degree of $2n-2$, since the vertex has an edge to the other vertices on its own row and column.

3 The First Piece of the Puzzle

The crux of the entire proof actually hinges on the following lemma:

Lemma 3.1 *Let $\vec{G} = (V, E)$ be a directed graph and suppose that for each vertex $v \in V$ we have a colour set $C(v)$ that is larger than the outdegree, $|C(v)| \geq d^+(v)+1$. If every subgraph $\vec{G}$ possesses a kernel then there exists a list colouring of G with a colour from $C(v)$ for each v.*

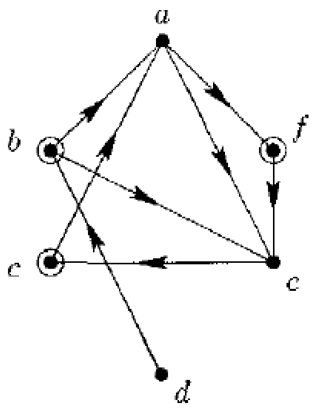

Figure 4: Example graph to illustrate kernels

Lemma 3.1 can be proven using induction but the proof is omitted here. Essentially, if you could somehow orient the graph S_n into a directed graph so that each vertex now has an outdegree of $n-1$ then we can show that the graph requires exactly n colours by the lemma and we are done. Note that this can be done since the vertices in the undirected graph S_n have a degree of exactly $2n-2$, so a natural choice would be to set both the indegree and outdegree of the directed graph $\vec{S}_n$ to $n-1$. But first, what is a kernel? Basically in a directed graph, the kernel K is a subset of vertices such that the following two conditions are satisfied:

- the set K is independent in the graph;
- for all other vertices u in the graph that do not belong to K, there is a vertex v belonging to K such that there exists the edge $u \to v$.

Let's look at the example in Figure 4. Here we find that the set $\{b, c, f\}$ satisfies the conditions and hence is a kernel. On the other hand, the set $\{a, c, e\}$ is not a kernel because edges exists between the vertices in the set and they form a cycle.

Now, suppose an orientation for the graph S_n is found. The next step would be to prove that any induced subgraph of the new oriented version $\vec{S}_n$ has a kernel. Clearly, we can see that if this condition holds then we can apply Lemma 3.1 to put the Dinitz conjecture to rest.

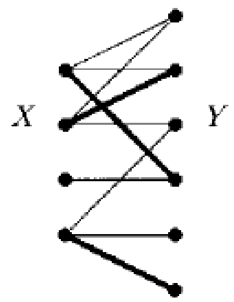

Figure 5: Example of a bipartite graph with a matching

4 Is Your Marriage Stable?

In order to prove that any induced subgraph in graph S_n has a kernel, we first would have to take a little detour. A *bipartite* graph is a graph such that the vertices can be split to two distinct sets X and Y where edges only exist between vertices in X and Y but not within each set itself. More formally, the graph $G = (X \cup Y, E)$ can be coloured with precisely two colours. Figure 5 shows an example of a bipartite graph.

A matching M in graph G would be the set of edges in set M such that no two edges share the same end-vertex. The bold lines in Figure 5 would constitute one possible matching. So, what does matching have to do with stable marriages? Consider the following problem. Suppose you were to run your own match-making agency, where marriages are of the heterosexual kind. Let X be the set of men and Y be the set of women. A matching in this case would be equivalent to an edge $uv \in E$ for any $u \in X$ and $v \in Y$; and since we want to ensure that no bigamy is committed, the matching condition holds. Furthermore, consider that every man and woman have their preferences of the ideal mate. Each man has a list of women with the candidates on his list ranked from highest to lowest. The women also have their own respective lists of preferred male candidates.

Let's call the ranking of the set of vertices adjacent to vertex $v \in X \cup Y$ as $N(v)$. There is a list of rankings for every vertex in the bipartite graph G and we have $N(v) = \{z_1 > z_2 > \ldots > z_{d(v)}\}$ where z_1 is the top choice and so on. With these, we introduce the notion of a *stable* marriage. A marriage is stable if the following condition holds: for every man $u \in X$ and woman $v \in Y$, provided both u and v are matched to a partner, u is not married to a woman

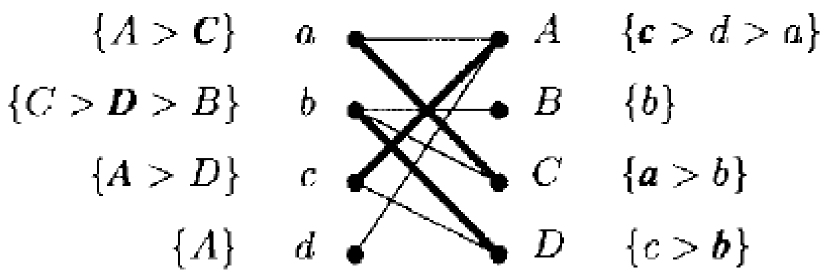

Figure 6: Example of a matchings given by the lines. The stable matching is given by the bold lines. The rankings are given by the sets next to each vertex.

$v' \in Y$ when he prefers v and v is not married to another man $u' \in X$ when she prefers u. The match uv is called a *blocking pair*, and if this edge is not present in a given matching, it would lead to an unstable matching. Figure 6 provides an illustration of a matching. While there is a unique matching M with four edges given by the set $\{aC, bB, cD, dA\}$, unfortunately M is not a stable matching. Here, cA constitutes a blocking pair.

The stable matching problem was solved by Gale and Shapley in 1962. A generalized version of the problem was solved much earlier in 1952 and was used for medical residents' matchings. Gale and Shapley provided an algorithm that can be applied to the stable matching problem, subsequently known as the (no prizes for guessing) Gale-Shapley algorithm. The algorithm proceeds in a few rounds and basically goes like this:

- In the first round, each man proposes to the woman he prefers most, regardless of whether she had been proposed to by another man.
- From the proposals received, each woman selects the proposer she prefers most, rejecting all other proposals. Women who do not receive any proposals in this round do nothing.
- Now, in the second round, each man already engaged, that is if their proposal from the previous round was accepted, do nothing. The men who were rejected previously make a proposal to the next highest woman on their respective lists.
- Women who are not engaged would then select the man highest in their respective rankings and women who did not receive any proposals do nothing.

- The rounds continue. Any man whose last proposal is rejected (ouch!) is then dropped from consideration (double ouch!). The algorithm continues until either all men have been matched or dropped from consideration.

The algorithm will terminate because eventually either each man in the set will be matched or dropped from consideration. Furthermore, the important thing about the Gale-Shapley algorithm is that the algorithm will always produce a stable matching. Looking at the steps in the algorithm again, it is clear that each man is always guaranteed that his present mate will always be better than any other mate lower in ranking to the present one. Since this is precisely the stable marriage condition, any solution produced is guaranteed to be stable.

If we run the algorithm on the example in Figure 6, then we will have $M = \{aC, bD, cA\}$ and d is the rejected male and B remains unmarried. This matching is given by the bold lines. Note that it is not possible to match d to B because both has not considered each other, guaranteeing their singlehood after this episode. However, everyone else lives happily ever after as their marriages are guaranteed to be stable, at least according to the algorithm.

5 From Marriages to Colouring

Now that we have all the pieces, let's put the puzzle together. First, we denote the vertices of the graph S_n with $(i, j), 1 \leq i, j \leq n$. A vertex (i, j) is adjacent to vertex (r, s) if and only if $i = r$ or $j = s$. For any Latin square L with numbers $\{1, 2, \ldots, n\}$ we denote the entry in cell (i, j) as $L(i, j)$. As suggested in Section 3, we then convert graph S_n into a directed graph $\vec{S}_n$. The orientation is done in such a way that

- for horizontal edges, if $L(i, j) < L(i, j')$ then set the edge as $(i, j) \rightarrow (i, j')$,
- for vertical edges, if $L(i, j) > L(i', j)$ then set the edge as $(i, j) \rightarrow (i', j)$.

Basically, we orient smaller numbers to larger ones horizontally and we do the opposite vertically. Figure 7 shows an example for a 3-by-3 Latin square and the corresponding graph $\vec{S}_3$. Observe that once this orientation is done, the

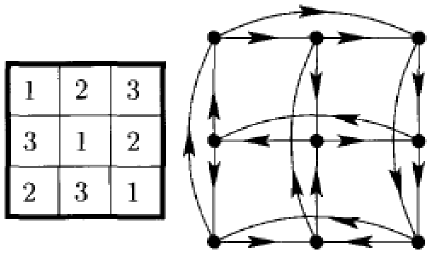

Figure 7: A 3-by-3 Latin square and its corresponding orientation

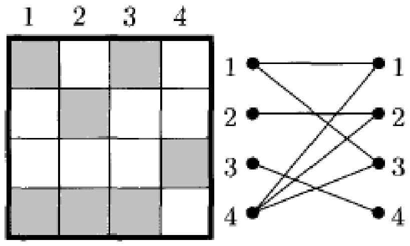

Figure 8: An example set of A and the corresponding bipartite graph

outdegree of each vertex (i, j) is $d^+(v) = n - 1$. To see this, consider if we take a random cell $L(i, j) = k$, then we find that are exactly $n - k$ cells in row i that contain an entry larger than k while $k - 1$ cells in column j have an entry smaller than k. The sum of these would then be equal to the outdegree of each vertex.

Since we have established the fact that we can orient the graph such that each vertex (i, j) has an outdegree of $n - 1$, it remains to be shown that every induced subgraph of $\vec{S}_n$ possesses a kernel (recall that this is a condition in Lemma 3.1). Let's consider a subset $A \subseteq V$ where V is the set of vertices in the graph $\vec{S}_n$. We can construct a bipartite graph $G = (X \cup Y, A)$ such that X is the set of rows and Y is the set of columns in the corresponding Latin square. For every $(i, j) \in A$, we would then represent it as an edge ij in the bipartite graph G with $i \in X$ and $j \in Y$. Figure 8 illustrates an example with the set $A = \{(1, 1), (1, 3), (2, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$. The cells of A are shaded in the Latin square next to the bipartite graph.

Due to the orientation of S_n, a natural ranking is induced on the neighbourhoods in graph G. Observe that this orientation sets $j' > j$ in $N(i)$ if $(i, j) \rightarrow (i, j')$ in $\vec{S}_n$ and $i' < i$ in $N(j)$ if $(i, j) \rightarrow (i', j)$. (Try working out the rankings in Figure 8.) Thus, given such a ranking, the graph G would have a stable matching M which is a subset of A. M is in fact the kernel we were looking for, since it is independent in A as none of its vertices shares a common end-vertex. Furthermore, if we pick a vertex $(i, j) \in A \backslash M$, then as M is a stable matching, either there exists $(i, j') \in M$ with $j' > j$ or $(i', j) \in M$ with $i' > i$, which in $\vec{S}_n$ would be $(i, j) \rightarrow (i, j') \in M$ or $(i, j) \rightarrow (i', j) \in M$, thus proving that M is a kernel. By Lemma 3.1, since any subgraph induced in $\vec{S}_n$ has a kernel, then as long as our colour set is at least of size $d^+(v) + 1 = n$, we can always find a colouring for the graph S_n. Hence, the proof is complete.

6 Wrapping Up

When looking at the proof of the Dinitz conjecture, one can't help but marvel at the simplicity of the proof. The entire proof only requires two concepts, that of Lemma 3.1 and of stable marriages. Sometimes, seemingly difficult problems can be broken down to easier constituent parts, with insight putting them all together. More importantly, it shows that one needs to have a balance between specialized and general mathematical knowledge. This proof of the Dinitz conjecture is beyond doubt a thing of beauty to those who behold it. So get a pencil and paper and start working on a problem now! Who knows, you might be fortunate enough to prove an interesting open problem.

— Paul Tune

References

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[2] D. Zeilberger, "The Method of Undetermined Generalization and Specialization Illustrated with Fred Galvin's Amazing Proof of the Dinitz Conjecture", American Math Monthly, vol. 103, no. 3, pp. 233-239, March 1996.

[3] F. Galvin, "The List Chromatic Index of a Bipartite Multigraph", Journal of Combinatorics, Series B, vol. 63, pp. 153-158, 1995.

[4] M. Aigner, G. M. Ziegler, "Proofs from the Book, 3rd Ed.", Springer, 2003.

Maths jokes

How professors prove it

Proof by example: Give the $n = 2$ case. The other 255 cases are analogous.

Proof by deferral: “We'll prove this later in the course”. “Trivial.”

Proof by vigorous hand waving: Works well in a classroom or seminar setting. Cloud-shaped drawings frequently help here.

Proof by cumbersome notation: Use lots of $P, p, \rho, \varrho, \ldots$ without any hint of their meanings.

Proof by reference to inaccessible literature: Cite a simple corollary of a lemma found in a privately circulated memoir of the Icelandic Mathematical Society, 1876.

Proof by importance: A large number of useful consequences follow from the proposition.

Proof by evidence: Long and diligent search has not revealed a counter example.

Proof by mutual reference: In reference A, Theorem 5 is said to follow from Lemma 3.1.4 in reference B, which is shown to follow from Corollary 6(e) in reference C, which is an easy consequence of Theorem 5 in reference A.

⋆

To a mathematician, real life is a special case.

⋆

Denis Diderot was a French encyclopedist and philosopher. He once stopped at the Russian court in St Petersburg. His wit and charm soon drew a large following among the nobles – and so did his atheist philosophy. That worried empress Catherine very much.

Leonhard Euler, a devout Christian, was working at the Russian court at that time. It is said[2] as that the empress asked him for help in dealing with the threat posed by Diderot.

[2]This story is most likely false, given that Diderot was actually a capable mathematician.

Euler was introduced to Diderot as a man who had found an algebraic proof for the existence of God. With a stern face and serious tone, the mathematician confronted the philosopher: “Monsieur, $(a + b^n)/n = x$, therefore God exists. Any answer to that?”

Diderot was dumbstruck. He was laughed at by his followers, and soon returned to France.

⋆

When the logician’s little son refused to eat his vegetables for dinner, the father threatened him: “If you don’t eat your vegies, you won’t get any ice-cream!” The son, frightened at the prospect of not having his favorite dessert, quickly finished his vegetables.

After dinner, impressed that his son had eaten all his vegetables, the father sent his son to bed without any ice-cream.

⋆

A father who is concerned about his son’s bad grades in maths decides to register him at a Catholic school. After his first term there, the son brings home his report card: he’s getting full marks in maths.

The father is pleased, but wants to know: “Why are your math grades suddenly so good?”

The son explains: “When I walked into the classroom the first day, and I saw that guy on the wall nailed to a plus sign, I knew this place means business!”

⋆

A mathematician, an engineer, and a computer scientist are riding in a car, when suddenly the engine stops working.

The mathematician: “We came past a gas station 3.42 minutes ago. Someone should go back and ask for help.”

The engineer: “I should have a look at the engine. Perhaps I can fix it.”

The computer scientist: “Why don’t we just open the doors, slam them shut, and see if everything works again?”

True stories

It used to bother John Littlewood (1885–1977) a good deal that he seemed to dream at night of solutions to the problems he was working on, but could never remember the details in the morning. He resolved to address this: he put a notebook and pencil by his bedside. That night, he had a particularly lucid dream in which all the pieces of the solution of his problem were plainly laid out and explained. He forced himself to wake up and wrote down all his thoughts, then he went blissfully back to sleep. In the morning, he was excited to read what he had recorded, confident that he would now be able to write an important paper. What he read on his notebook was, "Higamus, bigamus, men are polygamous. Hogamus, bogamous, wives are monogamous."

R H Bing (1914–1986) was a topologist at the University of Texas. Many people thought he preferred his initials to his first and middle names. Bing in fact *did not have* a first and middle name. His full name, as recorded on his birth certificate, was R H Bing. One year Bing applied for a visa. He filled out the application form. The form was returned to him, and he was told that the State Department would not accept initials. Bing wrote back, explaining patiently that his name was " 'R only' 'H only' Bing." In a few weeks he received his visa in the mail. It was made out to "Ronly Honly Bing."

A popular story about Edward Titchmarsh (1899–1963) is that he once announced a course that would be a "double series" of lectures — lasting not one academic year but two. The first half of this marathon course ended in April, and there was a six month break before the next half began. Titchmarsh strode into the room that day, picked up a piece of chalk, and said, "Hence, ..."

Historical Quotations about Prime Numbers

Below is a chronological collection of imaginary thoughts and sayings (the real part is their names) from some selected famous people regarding their opinion about the series of the prime numbers. Each thought was devised trying to reflect their particular way of reasoning, or at least, by the way we remember their lasting works.

1. (b.470 B.C.) Socrates: "3 is prime, 5 is prime, 7 is prime, 9 is. . . sorry, this leads me to the discovery that I only know that I know nothing."

2. (b.384 B.C.) Aristotle: "1 is not prime, by definition. 2 is an unnatural prime, 4 is an unnatural prime, and 6 is an unnatural prime. All other natural primes cannot be unnatural primes."

3. (b.780) Mohammad ibn al-Khwarizmi: "3 is prime, 5 is prime, 7 is prime. . . 9 is not prime, but don't tell it to the infidels."

4. (b.1225) Thomas Aquinas: "3 is prime, 5 is prime, 7 is prime and 9 is prime. God can do anything."

5. (b.1285) William of Occam: "3 is a prime, 5 is a prime, and 7 is a prime. Why bother with non-prime numbers when the primes can do everything?

6. (b.1451) Christopher Columbus: "3 is prime, 5 is prime, 7 is prime. According to some ancient manuscripts 9 is not a prime number, but in the New World that I am going to discover, there are surely lots of them."

7. (b.1532) Menoccio Scandella[3]: "In the beginning there was a chaos, and out of the chaos came the primes: 1, 2, 3. Out of these numbers came the composites like the holes in a fermented cheese."

8. (b.1548) Giordano Bruno: "I don't care if 1 is prime or not, if 2 is prime or not, if 3 is prime or not. All I care is that there are more stars in the heavens than primes in the earth."

9. (b.1564) Galileo Galilei: "I, Galileo, son of the late Vincenzio Galilei, swear that I never said that the prime numbers are useless. What I said was that you cannot count lunar craters by counting 2, 3, 5, 7, . . . "

10. (b.1732) George Washington: "What's the difference between prime numbers and composite numbers? There are no real differences, remember: *E pluribus unum*.[4]"

11. (b.1809) Abraham Lincoln: "3 is a prime, 5 is a prime, 7 is a prime and 9 should be a prime. But keep in mind that you can fool all the primes some of the time, and some of the primes all of the time, but you cannot fool all of the primes all of the time."

[3] His view of God and angels appearing from chaos likes "worms from cheese" got him into trouble with the Inquisition, which lead to his death in 1600.

[4] From many, one.

12. (b.1818) Karl Marx: "2 is a proletariat prime, but 4, 6 and 8 are also composite proletariats. Composites of the world unite; you have nothing to loose but your chains!"

13. (b.1834) Dimitri Mendeleev: "3 is a prime, 5 is a prime, and 7 is a prime, but 9 is a noble prime that deserves a separate row in the periodic table of the primes."

14. (b.1845) George Cantor: "$\aleph_3$ is a transfinite prime,$\aleph_5$ and $\aleph_7$ are also transfinite primes; however, it is my conjecture that $\aleph_9$ is the first transfinite composite between $\aleph_7$ and $\aleph_{11}$. Keep in mind that any normal number contains all the digits of all the primes."

15. (b.1869) Mahatma Gandhi: "3 is prime, 5 is prime, 7 is prime, but 9 is not prime; in this incarnation."

16. (b.1872) Bertrand Russell: "3 is prime, 5 is prime, 7 is prime, 9 is a paradox."

17. (b.1881) Pablo Picasso: "1 is a prime, 2 is a prime, 3 is a prime, 4 is a prime... Give me a museum and I'll fill it!"

18. (b.1887) Erwin Schrödinger: "3 is a prime, 5 is a prime, and 7 is a prime, however, 9 has a dual prime-composite state that can collapse to one side or the other depending on the observer's cat."

19. (b.1917) John F. Kennedy: "1 is not a prime number and 9 is not a prime number? Then ask not what the primes can do for you, ask what you can do for the primes."

20. (b.1918) Billy Graham: "3 is prime, 5 is prime, 7 is prime, 9 is an unfortunate mistake of the devil. But if it repents, it will be saved!"

21. (b.1929) Martin Luther King: "What? You say that 2 is the only even prime number? I have a dream that one day this number will rise up and live out the true meaning of its creed. We hold these truths to be self evident that all numbers are created equal."

22. (b.1946) George W. Bush: "3 is prime, 5 is prime, 7 is prime, and 9... well, any odd number can be prime as long as it is not 9."

— Ernesto Pérez Acevedo (Puerto Rico)

Inequalities

I'm starting to like inequalities already. No, not the socio-economic kind of inequalities, this is about the real stuff you don't often learn at uni.

All of you would know basic inequalities – we see such comparisons every day, whether it be comparing prices, or thinking about whether it's time to wash those plates because there aren't enough clean ones for your dinner. But inequalities aren't just about that; with it we can compare very complex algebraic expressions and getting very interesting results indeed.

The usual high-school inequality has a single variable: something along the lines of $2x-1>13$. The pinnacle of high-school inequalities, at present, seems to be something like $f(x)>k$, whereupon out comes the graphing equipment and the answer can thus be read off.

A common fallacy is that while you can add and multiply inequalities (provided they go the same way), the same does not go for subtraction and division. $3>2$ and $4>1$ does not mean that $3-4>2-1$. This is because, as $3>2$ means $3=2+a, a>0$ and $4>1$ means $4=1+b, b>0$, we really know nothing about $a-b$.

Now, how would we prove stuff like, for all real numbers a, b and c, $a^2+b^2+c^2\geq ab+bc+ca$?

Well, squares are non-negative.

Realising that the inequality is equally expressed as $a^2-ab+b^2-bc+c^2-ca\geq 0$, we can proceed to express the left hand side as a sum of squares:

$$\begin{aligned}
a^2-ab+b^2-bc+c^2-ca &\geq 0\\
2a^2-2ab+2b^2-2bc+2c^2-2ca &\geq 0\\
(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2) &\geq 0\\
(a-b)^2+(b-c)^2+(c-a)^2 &\geq 0
\end{aligned}$$

Such an argument can be used to solve many inequalities involving real variables. Take, for example, prove for all real t, that: $t^4-t+\frac{1}{2}>0$.

Of course we could use calculus, but note that the left hand side is $t^4 - t^2 + \frac{1}{4} + t^2 - t + \frac{1}{4} = (t^2 - \frac{1}{2})^2 + (t - \frac{1}{2})^2 > 0$.

But why is the inequality $>$, not $\geq$? Well, consider the equality case, $(t^2 - \frac{1}{2})^2 = 0$ and $(t - \frac{1}{2})^2 > 0$. This gives $t = \frac{1}{2}$ and $t = \sqrt{\frac{1}{2}}$, a contradiction.

Other inequalities are solved quite differently.

Suppose you were a lucky contestant at a game show, and you had won a prize. Three bins sit before you, one containing \$10 notes, one containing \$20 notes, and the third containing \$100 notes. You are allowed to choose 100 notes from one bin, 10 from another, and 1 from the last. How would you choose your notes?

The greedy contestant would, quite obviously, choose 100 \$100 notes, 10 \$20 notes, and 1 \$10 note, making a total of \$10210. The game show producers, on the other hand, would much rather you chose 100 \$10 notes, 10 \$20 notes, and 1 \$100 note, reducing your winnings to \$1300. But how do they know that such arrangements are the greatest and least amounts the contestant could take?

Enter the rearrangement inequality. It states that:

If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, then $a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b_i + a_2b_j + \cdots + a_nb_k \geq a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$, where the middle is any permutation of the sequence $\{b_i\}$.

Intuitively this is true, and we can prove this more rigorously by thinking about swapping the order of pairs of products: if $w \geq x$ and $y \geq z$, then $wy + xz \geq wz + xy$.

In true uni style, this is left as an exercise for the reader (hint: turn inequalities into equalities with variables).

Once such a result is obtained, we can 'swap partners' between pairs $a_ib_l + a_kb_j$ to $a_ib_j + a_kb_l$ ($i > k, j > l$), all the time increasing our total, until we get the left hand inequality. The right hand inequality is similarly done.

Well, apart from game shows, how is this inequality useful in mathematics? Let's look at our original inequality: prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Without loss of generality, let $a \geq b \geq c$. Let $\{a, b, c\}$ be our first and second sequence. Then the rearrangement inequality gives the result straight away, as $a^2+b^2+c^2$ is the largest obtainable total, while $ab+bc+ca$ a mere permutation of the second sequence.

Surely, however, we can do something with the powers in the expression? Note the square in the terms in the LHS that is not apparent in the RHS?

Enter Muirhead's inequality.

Firstly the terminology. Sequence A *majorises* sequence B (of the same number of real elements) if, when organised in non-increasing order, the respective cumulative sums in Sequence A are greater than or equal to the respective cumulative sums in Sequence B; further the total sum of the sequences are equal.

For example, $\{9, 6, 5.5, -4.5\}$ majorises $\{5, 5, 4, 2\}$ since

$$\begin{aligned} 9 &\geq 5 \\ 9+6 &\geq 5+5 \\ 9+6+5.5 &\geq 5+5+4 \\ 9+6+5.5-4.5 &= 5+5+4+2 \end{aligned}$$

Muirhead's inequality tells us that, if sequence $\{a_1, a_2, \ldots, a_n\}$ majorises sequence $\{b_1, b_2, \ldots, b_n\}$, then for any real x_i,

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n},$$

where $\sum_{sym}$ means the symmetric sum, that is, taking all possible permutations of x_i in the sum.

In particular, if we let $x_1 = a, x_2 = b, x_3 = c$, and note that $\{2, 0, 0\}$ majorises $\{1, 1, 0\}$, then

$$\sum_{sym} a^2 b^0 c^0 \geq \sum_{sym} a^1 b^1 c^0$$

$$\Rightarrow 2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca.$$

There is a factor of 2 on each side because we are adding $3! = 6$ permutations of a, b, and c.

Of course, the versatility of Muirhead's inequality doesn't stop at simple inequalities. For a challenge, try the following:

(1996 Iranian Maths Olympiad) For all positive real numbers x, y, and z, prove that:[5]

$$(xy + yz + xz)(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(x+z)^2}) \geq \frac{9}{4}$$

(hint: expand and simplify.)

Another way to utilise the squares in the left hand side of our original inequality is to consider the Arithmetic Mean-Geometric Mean (AM-GM) inequality. It states that if $x_1, x_2, \ldots, x_n > 0$, then $\frac{1}{n}(x_1 + x_2 + \cdots + x_n) \geq \sqrt[n]{x_1 x_2 \cdots x_n}$.

Now if we set $x_1 = a^2, x_2 = b^2$, and $x_3 = c^2$, then we see that

$$\frac{a^2+b^2}{2} \geq \sqrt{a^2b^2} = ab, \frac{b^2+c^2}{2} \geq \sqrt{b^2c^2} = bc, \frac{c^2+a^2}{2} \geq \sqrt{c^2a^2} = ca.$$

Adding them together yields our familiar inequality.

Of course, the AM-GM inequality is much more versatile than this. E.g. try to prove that, for positive a and b, $\frac{a+nb}{2} \geq \sqrt[n+1]{ab^n}$, or that, for positive a, b, and c, $\frac{a+b+c}{3} \geq \sqrt{\frac{ab+bc+ca}{3}} \geq \sqrt[3]{abc}$.

The Geometric Mean and the Arithmetic Mean all come from a long line of Means denoted the "Power Means":

In a set of numbers $\{x_i\}$, the Mean of power p is of the form:

$$M_p(\{x_i\}) = \sqrt[p]{\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}},$$

where M_0 is defined to be the Geometric Mean.

Theorem: If $p \geq q$, then $M_p \geq M_q$ for the same sequence $\{x_i\}$ in the positive reals.

[5]Without Muirhead's inequality, this is somtimes considered the hardest inequality problem ever to appear in a high school level competition. You can claim $5 from MUMS for solving it.

This is useful whenever dealing with powers, or finding a bigger mean than the person next to you.

Consider that $M_2 \geq M_1$ for $\{a, b, c\}$, $\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3}$, squaring both sides and rearranging gives $a^2 + b^2 + c^2 \geq ab + bc + ca$, which looks altogether very familiar.[6]

Now try these:

- For positive reals a, b, c and d, prove that:
  $$4(a^3 + b^3 + c^3 + d^3)^2 \geq (2ab + c^2 + d^2)(2ac + b^2 + d^2)(2ad + b^2 + c^2)$$
- If a_i are positive reals which add up to 1, prove that
  a) $\sum_{i=1}^{n} \frac{1}{a_i} \geq n^2$ (hint: use M_{-1} and M_1).
  b) $\sum_{i=1}^{n} \frac{a_i}{1-a_i} \geq \frac{n}{n-1}$.

Many of you may know about the Cauchy-Schwarz-(Buniakowski) inequality:

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \geq (x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2.$$

Proof: let $(x_1, x_2, \ldots, x_n)$ be vector $\vec{x}$ and $(y_1, y_2, \ldots, y_n)$ be vector $\vec{y}$, then $|x|^2|y|^2 \geq |x|^2|y|^2 \cos^2\theta = |x \cdot y|^2$. Now we just expand it out.

How is this useful? Let $x_1 = a, x_2 = b, x_3 = c, y_1 = b, y_2 = c, y_3 = a$, then

$$(a^2 + b^2 + c^2)^2 \geq (ab + bc + ca)^2,$$

and taking the square root of this (non-negative) expression yields our old result.

For more uses of Cauchy-Schwarz, try these:

- If reals a, b, c, d are such that $a^2 + b^2 + c^2 + d^2 = 100$, show that
  $$\frac{1}{a^2} + \frac{4}{b^2} + \frac{9}{c^2} + \frac{16}{d^2} \geq 1$$
  (hint: multiply one side by $a^2 + b^2 + c^2 + d^2$ and the other side by 100.)

[6] The RHS only hold for positive reals by the theorem, but one can easily see that this extends to all reals.

- (IMO 2005) Let x, y, z be positive real numbers such that $xyz = 1$. Prove the inequality

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0$$

  (hint: this is hard, and you may need to use Cauchy-Schwarz several times, as well as a few other inequalities.)

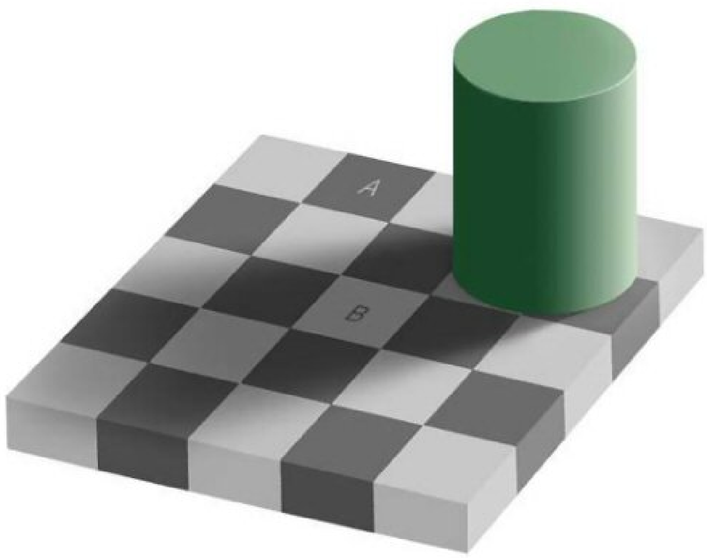
- Prove, for $a, b, c > 0$,

$$\frac{a}{b + 2c} + \frac{b}{c + 2a} + \frac{c}{a + 2b} \geq 1.$$

Of course, many of these inequalities are quite high-powered and rarely would one expect to use them in daily life (or even at uni). However, there is still a chance that you may one day find yourself calling upon any one of these while at the shops, making bets about averages, or deciding the largest amount of money you can win from a game show.

Or, if nothing else, you can be quite sure that for real numbers a, b and c, $a^2 + b^2 + c^2 \geq ab + bc + ca$.

— Matthew Ng



In the above figure, the squares a and b actually have the same colour.

Solutions to Problems from Last Edition

1. P is polynomial with positive coefficients. Prove that if $P(x)P(\frac{1}{x}) \geq k^2$ for $x = 1$, then this property holds for all $x > 0$

   Solution: Let $P(x)$ be $a_0^2+a_1^2x+\ldots+a_n^2x^n$. Then, when $x > 0$, $P(x)P(\frac{1}{x}) = (a_0^2 + a_1^2x + \ldots + a_n^2x^n)(a_0^2 + a_1^2\frac{1}{x} + \ldots + a_n^2\frac{1}{x^n}) \geq (a_0^2 + a_1^2 + \ldots + a_n^2)^2 = P(1)^2 \geq k^2$ by the Cauchy-Schwarz inequality.

2. *a*) Prove $\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos \frac{x}{2^k}$ and *b*) find the value of $\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2}$.
   Solution: a)

$$\cos\frac{x}{2}\cos\frac{x}{4}\ldots\cos\frac{x}{2^n}\sin\frac{x}{2}\sin\frac{x}{4}\ldots\sin\frac{x}{2^n} = \frac{1}{2^n}\sin x\sin\frac{x}{2}\ldots\sin\frac{x}{2^{n-1}}.$$

   After cancellations, $\prod_{k=1}^{\infty} \cos \frac{x}{2^k} = \lim_{n\to\infty} \frac{\sin x}{2^n \sin \frac{x}{2^n}}$. Evaluating the limit gives the result.

   b) Note that $\arctan \frac{1}{2k^2} = \arctan(2k + 1) - \arctan(2k - 1)$, so the sum telescopes (when you write it out explicitly, things cancel), giving the answer $\frac{\pi}{4}$.

3. For a biased coin with probability p of landing on H on any toss:

   - Even if you don't know p, how can you use the coin to achieve fair results with probability $\frac{1}{2}$?
   - What is the probability of getting an even number of H in n tosses?
   - What is the expected value of tosses needed to get 2 successive H?
   - What is the expected value of tosses needed to get 2 successive H or T?

   Solution: a) toss the coins twice until the results differ; then use the first one. This is because HT and TH are equally likely.

   b) Let P_n be the probability after n throws. Then

$$P_n = (1 - p)P_{n-1} + p(1 - P_{n-1}) = (1 - 2p)P_{n-1} + p, P_0 = 1.$$

   Solving the recurrence gives $P_n = \frac{1}{2}((1 - 2p)^n + 1)$.

   c) We use conditional expectations: $E = 1 + (1 - p)E + p(1 + (1 - p)E)$. The 1 is for the first toss, the second term for restarting the experiment if a T is encountered, and the last term for the 1 more toss if H shows up, or restarting the experiment if T shows up. This gives $E = \frac{1+p}{p^2}$.

d) Combining the expectataions for H and T using the previous method, one can show that $E = \frac{2+p-p^2}{1-p+p^2}$.

4. A point is distance a, b and c away from the vertices of an equilateral triangle, where $a^2 + b^2 = c^2$. What is the side length of the triangle?

   Solution: there are 2 cases: the point P can be inside or outside the triangle. Rotate the whole figure centred at a vertex, say B, by $60°$ so $A' = C$, and we observe that $\triangle BPP'$ is equilateral, and $\triangle APP'$ is right by Pythagoras. The side length can then be found by using the cosine rule on the triangle with the $150°$ angle. The answer is $\sqrt{c^2 \pm \sqrt{3}ab}$.

5. Consider 2 adjacent faces, F_1 and F_2, of a regular dodecahedron. B is one of the vertices at which they meet, A is the vertex closest to B on F_1 but not on F_2, and C is one of the vertices on F_2 furthest away from A. Find the size of $\angle ABC$.

   Solution: choose a vertex V of the dodecahedron; three faces meet there: F_1, F_3 and F_5. The three faces next closest to V are F_2, F_4 and F_6; let F_2 share edges with F_1 and F_3, etc. Then ABC forms three vertices of a hexagon which is completed by identical triangles similarly defined on the other two pairs of faces. Then by rotational symmetry (with centre at V), all angles of the hexagon are the same, so $\angle ABC = 120°$.

Paradox **Problems**

Sadly, there were very few submissions to last issue's problems, and even worse, as most of them were made by people affiliated with MUMS, they are not eligible for the cash prizes.

So please note, the increased prize amount for the questions is not due to their extra difficulty, but because MUMS got a new source of funding. Also, bear in mind that anyone who submits a clear and elegant solution many *each* claim the indicated amount. Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room in the Richard Berry Building.

Question 1 was solved correctly by Ruth Luscombe, who can now collect \$3 from the MUMS room.

Anton Troynikov correctly solved part 1 of question 3, and he can collect \$2 from the MUMS room.

1. (\$3) Lucy usually takes the train and arrives at the station 8:30 am, where she is immediately picked up by a car and driven to work.

   One day she takes the early train, arrives at the station at 7:00 am, and begins to walk towards work. The car picks her up along the way and she gets to work 10 minutes earlier than usual. When did Lucy meet the car on this day?

2. (\$3) P is a polynomial of any degree with non-negative integer coefficients. You are asked to determine P by only asking for $P(x)$ at 2 x values. How do you do it?

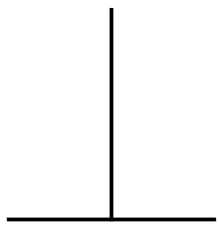
3. (\$5) In $\triangle ABC$, the bisector of external $\angle A$ meets BC at D. Find the length of AD.

4. (\$5) Express in closed form $\cos x - \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x - \cdots$

5. (\$7) Find the volume of the square antiprism (a solid formed by 2 parallel square faces, one of which is rotated by $45°$ with respect to the other around the axis through their centres, and the vertices then joined up with equilateral triangles) with side length 1.

6. (\$10) Find

$$\prod_{\text{all primes p}} \frac{p^2+1}{p^2-1}.$$



The 2 lines in fact have the same length.

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