Paradox

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The Magazine of the Melbourne University Mathematics and Statistics Society

MUMS

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Words from the Editor

While you are still groaning from the terrible jokes from last issue, another edition of Paradox has arrived with even worse puns and occasionally useful information, such as the brilliant article on how to measure the Earth!

As the editor, my job is to motivate people to write more articles. I thank those who contributed articles for this issue; as for those who haven't, the Administration might tax you for reading future editions of Paradox. So if you have any mathematically related material that vaguely may be called recreational, useful, or plain silly, then send it to us!

Any submissions do not need to be in any specific format, nor be the grammatically correct. For it is the editor's job to edit those who don't edit their own articles. This invariably leads to the question of who edits the editor: if he edits himself then by definition he does not edit himself, but if he does not edit himself then he must edit himself…

You know, paradoxes like this have the power to cut people off mid-sente…

— James Wan

MUMS seminars:

The first week's seminars were a huge success. Here are some more to keep your mind occupied; free food afterwards:

Friday 3 August 1-2:15 pm, J H Michell Theatre (a.k.a. Theatre A): A Mathematician Goes to Hospital, presented by Prof. Terry Mills. Find out how applied maths …applies to hospitals. This seminar is equation free.

Week 3 and 4: The Variational Principle, by Nicholas Sheridan and The Detection of Blood-Doping by Athletes, by Dr. Ken Sharpe. The former concerns finding the shape of the ramp for a ball to roll down in minimal time. Look for notices closer to the dates.

Words from the President

Greetings, and welcome to a new semester and a new issue of Paradox. Following a relatively late AGM last semester, the new MUMS committee has settled in and are charged up to provide you with a steady stream of mathematical goodness. For the first time a biology major has claimed the position of president, which goes to show MUMS welcomes people from all backgrounds. Our Education officer has really outdone himself this seminar season and has arranged quite a number of talks; we're quite figuratively bursting at the seams with lunchtime seminars and I'm confident you'll be able to find at least one that tickles your mathematical fancies.

Also marked down on our events calendar is the University Maths Olympics (UMO). For the uninitiated, the UMO is a fast-paced event that brings together mental and physical[1] agility in a fun-filled hour of problem solving. Even if you're not competing, you can still be part of the experience by being in our audience and get some free food.

Keep an eye out for more information regarding our seminars and events on our noticeboard (opposite the water cooler) and on our website. While you're there, check out our committee and don't feel timid about approaching us and getting to know MUMS. We look forward to seeing new faces in the MUMS room, whether it's to just introduce yourself, find out more about us, or even indulge in a game of chess or scrabble. On that note, I hope you enjoy the latest offerings of our talented Paradox editor, and hope to see you lurking around our events and the MUMS room.

— Alisa Sedghifar

Puzzle 1:

We use integration by parts to find the anti-derivative of $1/x$.

$$\int \frac{1}{x}\mathrm{d}x = \int x' \frac{1}{x}\mathrm{d}x = x\frac{1}{x} - \int -\frac{x}{x^2}\mathrm{d}x = 1 + \int \frac{1}{x}\mathrm{d}x.$$

Hence 0 = 1. What happened?

(Answer see last page.)

[1] That is, walking.

Intelligent Design and Recursive Gods

While we are busy raising religious fanatics who think the Messiah's last name is Potter, elsewhere in the world rages a debate about how life came into being. In the Southern United States in particular, many advocate teaching *intelligent design* as an alternative to Charles Darwin's theory of evolution. For the unfamiliar, the gist of intelligent design is that certain biological features are so complex that it is impossible for them to have arisen naturally, and therefore, must have been designed by some intelligent creator. While most biologists consider intelligent design not to be a scientific theory at all, they are in fact all missing the most important point, which is that intelligent design provides a surprisingly fruitful launchpad for a discussion of some interesting tidbits of mathematics.

First up, what does it mean to say that something is impossible? The tasks described in the *Mission: Impossible* films are clearly not impossible, as Tom Cruise was able to perform them quite easily, though perhaps that has more to do with his immense acting prowess. Some computer games contain a nominally impossible difficulty setting, but most of these, while certainly difficult, are very possible. A notable exception can be found in the obscure game *Penn & Teller's Smoke and Mirrors*, which remarks during the player's inevitable defeat: "Impossible doesn't mean very difficult. Very difficult is getting a Nobel Prize; impossible is eating the sun."

Probability gives a more precise definition – an impossible event is one which is outside the space of observable outcomes. For example, if a fair coin is tossed 100 times, it is possible (though unlikely, with probability approximately 10^{-30}) to observe 100 heads, but it is impossible (with probability exactly zero) to observe 101 heads. It's true that all impossible events have probability zero, but somewhat counter-intuitively, not all events with probability zero are impossible. For example, in an infinite sequence of coin tosses, the probability of observing all heads and no tails is zero, but this is an observable outcome, and hence is not impossible. In fact, the probability of observing any infinite sequence of heads and tails is zero, yet they are all possible since one of them must be the eventual outcome.

Returning to the original statement, if we consider biological features as arrangements of subatomic particles, it is certainly possible for them to randomly occur like that. Furthermore, since there are at most finitely many particles, the probability of this occurring is nonzero – quantum mechanics asserts that a particle has positive (though often minuscule) probability of being anywhere in the universe at any given time, which means I can go to bed hoping all the particles in my body will teleport to uni tomorrow, thereby avoiding all things Connex. Ignoring issues of whether we should accept quantum mechanics if we're going to reject evolution, this means that it is not in any sense impossible for life to have occurred naturally, and arguments for intelligent design really mean to say that evolution is overwhelmingly unlikely, rather than impossible.

How unlikely is it, then? This brings up a philosophical question about probability – what does it mean to talk about the probability of an event which can only happen once? If one asserts that the probability of observing heads on a coin toss is 50%, this means if we toss 1000 identical copies of that coin, we should observe heads on roughly 500 of them. This is known as the frequency interpretation of probability. However, if one asserts that elephants have a 50% chance of becoming extinct in the next 100 years, does this mean that if we made 1000 identical copies of the Earth, then in 100 years' time, elephants should be extinct in roughly 500 of those copies?

The fact that the experiment cannot be performed makes this a somewhat questionable way of thinking about it. Fortunately, Thomas Bayes already thought of this problem, and came up with what is now known as the Bayesian interpretation, which defines probability as the degree to which one believes in the truth of a proposition. This means we can sensibly talk about probabilities for one-off events, but the downside is that probability is now subjective, which is fine for coin tosses where different observers can agree on the facts, but much more prohibitive in contentious topics such as the origin of life. However, since this is the only widely accepted alternative to the frequency approach, it's the best we can do.

Once we estimate a probability, we come across the question of what we can infer from it. Anyone who has studied statistics will be familiar with the 5% threshold – if a hypothesis implies less than a 5% chance of the observed outcome occurring, we say there is significant evidence against it. Usually, this is stated as a double negative – a pharmaceutical developer might find that there is significant evidence against a new medicine not being effective, which means that it is effective. However, nothing in statistics is certain. For every 20 medicines which are significantly effective at the 5% level, we would expect one of them to be a dud. Similarly, if one calculated the probability of life naturally occurring on an Earth-like planet as 10^{-10}, this isn't necessarily evidence against that. If there are 10^{10} Earth-like planets in the universe (which by most accounts is a very conservative estimate), then we would expect life to have naturally occurred on one of them.

The probability that this planet of life is ours would then be 10^{-10}, but under Bayesian probability, this only applies without any additional knowledge of the system – such as the fact that we know life does exist on Earth. Using our previous example of coins, the probability of tossing 100 coins and getting 100 heads is very low, but if we are given that the last 99 tosses were heads, the *conditional probability* given this information is $\frac{1}{2}$. Similarly, the probability that the one planet in the universe with life on it is Earth might be very low, but given that we know Earth has somehow ended up with life, the conditional probability in light of this information is much higher. How much higher? Well, that entirely depends on one's belief of the relative likelihood of evolution and creation, which was the issue in question in the first place.

The very concept of a probabilistic argument against evolution now seems a little silly. However, intelligent design proponents are not the only ones who could do well to learn some mathematical trivia – the common retort of "if a creator created the world, who created the creator?" is also, mathematically speaking, a fallacy. The easy answer is that the creator was created by another creator, who was created by another, resulting in an infinite chain of creators. We don't even need time to extend backwards indefinitely – for example, we could have a system where time began at time 0, the world was created at time 1 by Creator 1, who was created by Creator 2 at time $\frac{1}{2}$, who was in turn created by Creator 3 at time $\frac{1}{3}$, and so on. This might seem a little hard to digest at first, but the logic is perfectly valid. On the other hand, this clearly means I'm guilty of heresy, so I'm glad they don't burn heretics any more.

It also shows that strange things happen at infinity, a fact perhaps more cleverly demonstrated by David Hilbert's *Grand Hotel*, which is a hotel with infinitely many rooms, all of them occupied. In a finite hotel, this would mean no more guests could be accommodated, but here, when a new guest arrives, the front desk can simply ask every guest to move to the room one number higher than their current room, and put the new guest in room 1. Furthermore, even if infinitely many guests arrive, the front desk could ask every guest to move to the room with double the number of their current room, thereby vacating infinitely many rooms for the new guests.

What's the moral of the story? Keep studying maths, it's everywhere and will come in useful in places you least expect. And intelligent design is complete quackery.

— James Zhao

Measuring the Earth

Suppose an alien was to intercept you on your way to university tomorrow, and wanted to know just one thing: how big is the Earth? If you were a physics student, or Eddie McGuire, you'd probably know off the top of your head. And even if you didn't, you'd just reach for your internet-enabled mobile phone and dial up Wikipedia.

"The Earth's radius is $6,378.135$ km," you would declare confidently. But would the alien believe it? Indeed, would you believe it?

The truth is that much of current scientific knowledge is based on faith. We accept scientific facts because we believe that someone, somewhere, has determined them accurately. In this way Science is just another religion – we don't ask for proof, we simply believe. But it doesn't have to be that way. With some ingenuity, and some mathematics, we can measure a lot of things about the world ourselves, no longer having to accept facts merely on faith.

One particularly easy thing to measure by yourself is the radius of the Earth (or its circumference – which is equivalent as we assume that the Earth is a perfect sphere). In fact, Gauss' *Theorema Egregium* ('Remarkable Theorem') tells us that we **must** be able to measure it ourselves, as (Gaussian) curvature is an intrinsic property of any surface, and for a sphere $K = 1/R^2$ (where K is the Gaussian curvature and R is the radius of the sphere).

Here are four simple ways that you can (theoretically) measure the Earth with minimal equipment, one for each type of person. Just pick the label that describes you best, get measuring, and never be forced to rely on faith again.

1 For the jetsetter

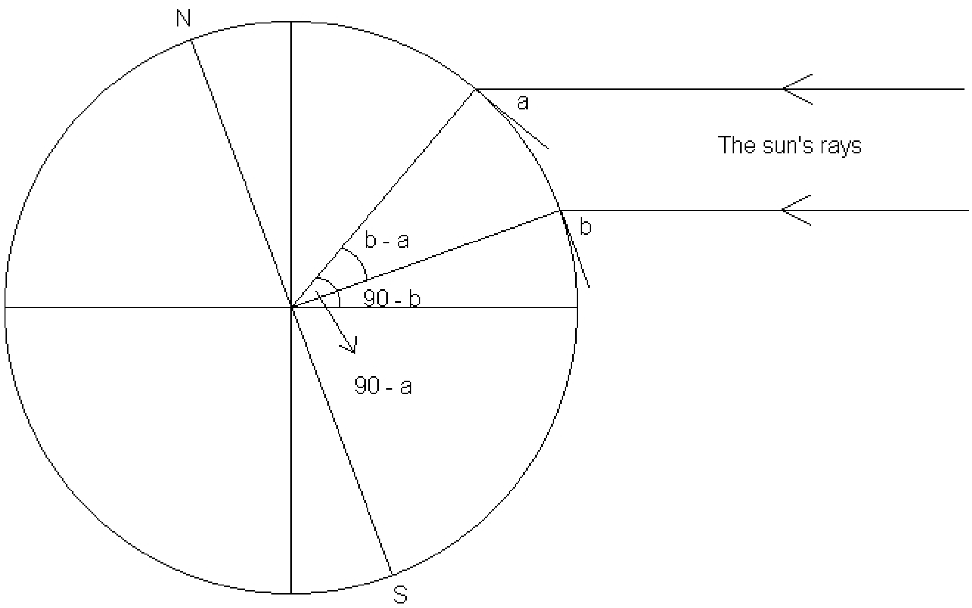
What you'll need: a pole, an atlas (or a car), and a year to spare (or a lot of money for flights)

Accuracy rating: some people seem to trust this method like they do with our politicians, but you suspect there is a reason that most don't

If you're a jetsetter, you like traveling, so why not incorporate a quick measurement of the Earth into your next trip. In fact, the very first person to accurately measure the size of the Earth (Eratosthenes, of the Sieve of Eratosthenes fame) used this technique. His measurement was about 41200 km, only about 3% too large, a pretty good effort in 200BC.[2]

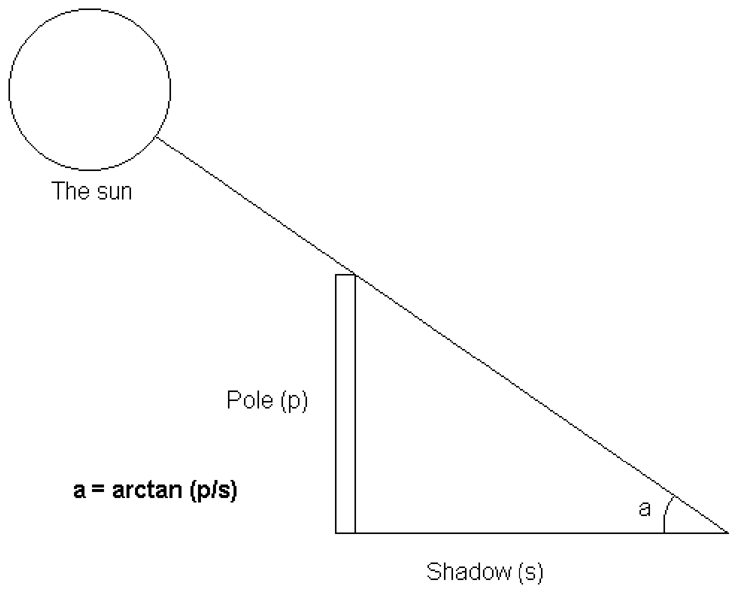
The key idea is that because the Earth's surface is curved, the sun's angle of elevation changes as you move north to south, and it changes in direct proportion to the distance around the Earth that you've traveled. This assumes a number of things – firstly that the sun is far enough away that its rays travel parallel to each other, and secondly that the sun appears point-like due to its distance. You'll need to keep these in mind when you measure the position of the sun.

The method consists of the following. Start in a city of your choice and measure the maximum angle of elevation that the sun rises from the horizon on a certain day (angle a in the figure). To do this, plant a pole in the ground and measure the length of its shadow at its shortest point, then apply some geometry (see figure over the page). Next, exactly one year later, take another measurement in a different city which is directly north/south of the first (angle b in the figure), with the extra condition that the second city must be on the same side of the Earth (the side facing the sun) as the first on that particular day. If you look at an atlas and pick cities not too far apart, you should do fine.



[2]Various figures are given, and discrepencies result because he gave the answer in ancient Greek units, and we are not certain how long those units are.

Now, the difference between the measured elevations (angle b minus angle a) will be the angle change measured from the centre of the Earth. Therefore, this angle difference divided by 360 degrees will be the fraction of the Earth's circumference you have traveled around. For example, if the elevation of the sun is 58 degrees in, say, Paris and 64 degrees in a city due south of Paris, say Barcelona, then Paris and Barcelona are $(64 - 58)/360 = 1/60$ of the way around the Earth's circumference.



But just knowing this is not enough, for you also need to know how far apart the cities are, and this can be difficult to measure. Eratosthenes did it by getting the Greek army to pace their march. You can take advantage of your car's odometer or alternatively just use a map (though you'll have to rely on the accuracy of the map, which may also use a previously measured curvature of the Earth). Once you know how far apart the cities are, and you know the fraction of the Earth's circumference you travel going between them, you can complete the calculations. For instance, if Paris and Barcelona are 650 km apart, then the Earth's circumference is $60 \times 650 = 39000$ km.

But the limitation on two cities directly North/South of each other is quite restrictive to your traveling. A slight variation on this technique removes this constraint, and delivers greater accuracy, as long as you are willing to stay in each city for a whole year. What you can do is measure each city's latitude by averaging the angle of elevation of the sun at the June and December solstices. As long as you stay on the same side of the equator (and if you don't you can make minor adjustments to the calculations), the difference in these latitudes will again be the angle change measured from the centre of the Earth. From there we proceed as above.

2 For the romantic type

What you'll need: a city by the sea and near the equator, a stopwatch, and a skyscraper observation deck

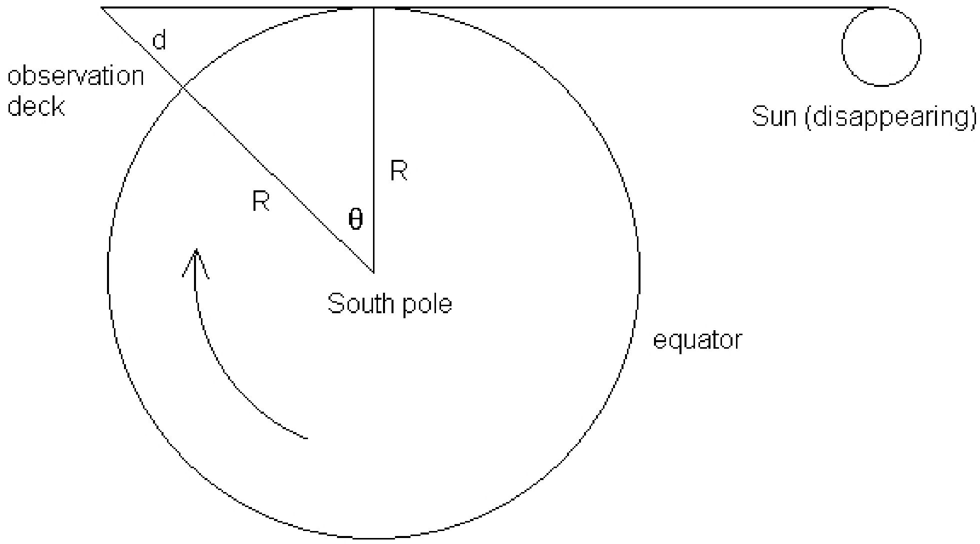
Accuracy rating: more reliable than our politicians – it's about as reliable as comedians

If you're the romantic type, you like nothing better than watching the sunset; and for this measurement you get to watch the sunset, not once, but twice in the same day! How is this possible? Well, the curvature of the Earth means that sunset will occur later at some height in the sky than it will at ground level. Knowing that the Earth rotates once every 24 hours, we can use this time difference to calculate the Earth's radius.

You'll need to find a city near the equator where you'll be able to see the sunrise/sunset. Singapore would be perfect but there are other candidates out there. You'll also get the best results if you take the measurement at either of the equinoxes, as this is the only time the equator lies in the same plane as the sun's rays (if you can't then don't worry about it). Once you've found your city, all you need to do is pick a skyscraper that is close to the shorefront, and get ready to watch the sunset (or sunrise). What you need to do is to measure the difference between the time of sunset/sunrise at the top of the observation deck, and the time of sunset/sunrise at ground level close to the skyscraper. The closer to the sea you are for this the better – it's worth moving a little from the base of the skyscraper if you can get down lower. If you are facing the East, you'll need to go up the tower first and wait for the sunrise. If you are watching the sunset, you will need to stay on ground level first. You'd better be quick though, because if your tower is an average skyscraper you'll have roughly 2 minutes to get to the top (you might want to have a friend already up there helping you out!).

Now, say you measure a time difference of t seconds between the sunrise/sunset in each location. As the Earth takes 24 hours to rotate fully, during t seconds it rotates $\frac{t}{24\times60\times60}$ of a full circle, which is an angle of $\theta = 2\pi\frac{t}{24\times60\times60}$. If the skyscraper observation deck is d meters above sea level, we also know that

$\cos\theta = \frac{R}{R+d}$ (see figure). Therefore, the radius of the Earth is $\frac{d}{\sec\frac{\pi t}{43200}-1}$.



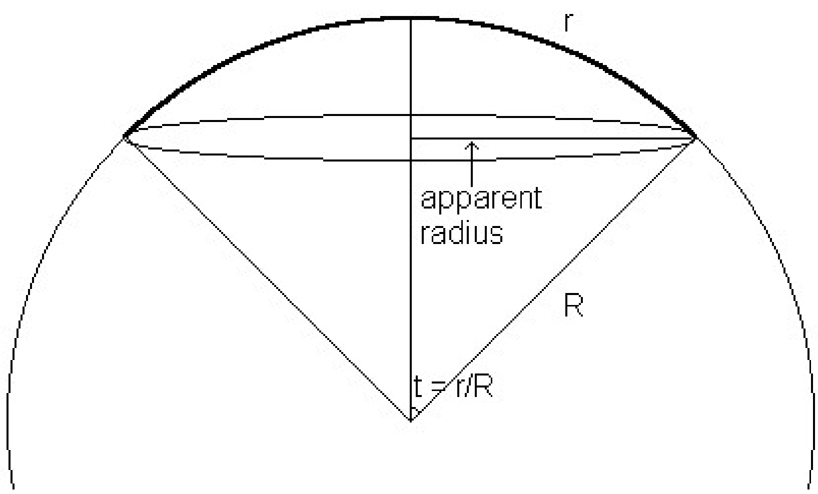
This is a pretty neat method of calculating the Earth's radius. If you conducted this experiment on the equator, with the ground level measurement taken from exact sea level, you'd end up with a surprisingly accurate result. Unfortunately, the limitation to cities on the equator makes this method quite restrictive. However, we can make this method valid in many more cities in the world by combining it with a measurement of latitude (see 'For the jetsetter' for a description on how to do this). What you in fact measure by timing sunsets is the radius of your city's circle of latitude. That is why a measurement on the equator gives the radius of the Earth. However, armed with the knowledge of your latitude (l degrees), the radius of the Earth (R meters) is the measured radius of your circle of latitude (by the observation of the sunsets) divided by $\cos l$, simple. Now all you need is a city with skyscrapers right on the beach. Surfers' Paradise anyone?

3 For the guy who likes to walk around in circles

What you'll need: a seriously flat surface, a rope, and a rolling pace measurer

Accuracy rating: Cannot be taken seriously, much like the former Iraqi Defense Minister ("Yes, the American troops have advanced further. This will only make it easier for us to defeat them.")

If you're the guy who likes to walk around in circles, you, well, like to walk around in circles. And that's exactly what you'll be doing in this method. When you walk around a circle, you expect the distance you have walked to equal 2π times the radius. This is how it works on paper, and you'd expect it to be the same in the real world. However, on a curved surface such as the Earth, things are somewhat different. On a sphere, as your circle gets larger, the ratio of circumference to radius will get less. Imagine a circle with centre on the North Pole and radius of $1/4$ the Earth's circumference. This circle would in fact be the equator, and so would equal the Earth's circumference in length. So here the ratio of circumference to radius is not 2π but 4! But you don't have to walk the whole way around the Earth to see this effect. Even relatively small circles will show a decreased circumference/radius ratio. We can use measurements of circumference and radius to calculate the radius of the Earth.



If the Earth has a radius of R meters, then a circular walk with radius r meters will have an apparent radius of $R \sin \frac{r}{R}$ meters (see figure). Thus the circumference (C meters) of this walk will be $C = 2\pi R \sin \frac{r}{R}$. Unfortunately this equation is difficult to solve for R, so we use a Taylor series approximation: $\sin x = x - x^3/3! + x^5/5! - \cdots$.

Now, as $\frac{r}{R}$ is small, $C = 2\pi R(\frac{r}{R} - \frac{r^3}{6R^3} + \cdots) \approx 2\pi r(1 - \frac{r^2}{6R^2})$. So $R \approx \sqrt{\frac{\pi r^3}{3(2\pi r - C)}}$.

In terms of putting this into practice, you're going to need a huge flat surface and a lot of skill with walking while holding a taut rope. In order to get even remotely accurate results you'll need to have a radius of a few hundred meters. That means your rope will have to be extremely thin and light in order to keep it taut. All in all this is a pretty rudimentary way to measure the radius of the Earth.

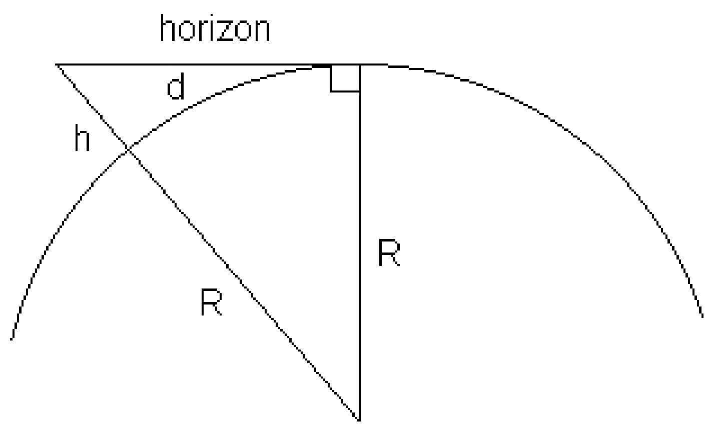
4 For the mountaineer

What you'll need: a city by the sea, a mountain, and an altimeter (a good sports watch will have one)

Accuracy rating: like a trusty old mathematician (for instance, Euler), this rarely goes wrong

If you're the mountaineer, you like climbing mountains, and this method will require you to do just that. The idea is that the Earth's curvature makes the distance to the horizon vary on the height above the Earth's surface. From the top of a mountain you can see further than at ground level – common sense. In order to measure the Earth's radius it would be sufficient to measure the distance to the horizon from a certain height, but this is very difficult to do practically. A more intelligent way to go about the problem is to line up two objects with the horizon, and measure the distance between them.

At a height of h meters, the distance to the horizon is $d = \sqrt{(R+h)^2 - R^2} = \sqrt{2Rh + h^2}$ meters (see figure). As $h \ll R$, we can say that the distance to the horizon is $\sqrt{2Rh}$ meters without great loss of accuracy.



Now, after you have located a suitable city (it must have a reasonably high mountain nearby), you need to pick a notably tall landmark in the city which must have a sharp point on top (a bridge tower or the spire of a small skyscraper is ideal) and start your trek up the mountain. What you are looking for is the exact altitude when the tip of your landmark **perfectly** lines up with the horizon. This must be done with a good degree of accuracy. Once you have lined up your landmark with the horizon, you need to measure the altitude at this point, and you also have to work out the distance between the point where you are standing and the tip of the landmark you are lining up. This can be tricky to do accurately. For a rough measurement, simply work out the horizontal distance between you and the landmark using a map and Pythagoras' theorem (however, if you are far from the landmark, then this may be inaccurate as you need to take into account the curvature of the Earth). A more accurate measurement can be attained by triangulating with another object near the landmark (if a bridge is your landmark then a second tower would be perfect). Triangulation involves forming similar triangles using the two landmarks and two reference objects near to you. By measuring the lengths of the triangle formed by the two objects near to you, you can work out the lengths of the triangle formed by the landmarks.

Suppose your landmark is h meters high, you line it up with the horizon when you are at t meters above sea level, and the distance between you and the landmark is d meters. Then, $d = \sqrt{2Rt} - \sqrt{2Rh}$, so $R = (\frac{d}{\sqrt{2t}-\sqrt{2h}})^2$.

This method is the most accurate of the four. The only potential unreliability is the measurement of the distance between you and the landmark, but with a bit of effort the triangulation can be made extremely accurate, as any surveyors would tell you. If you only have time to measure the Earth once, use this method.

— Stephen Muirhead

"If you don't know [how] to add fractions, you don't know how to think." – Prof. Barry Simon while lecturing a freshman class at Caltech.

Terrible maths jokes

⋆ For his epitaph, Erdös suggested, "I've finally stopped getting dumber."

⋆ In the topological hell, beer is packed in Klein's bottles.

⋆ Q: Why couldn't the möbius strip enroll at the school? A: They required an orientation.

⋆ Q: Do you know any catchy anagram of Banach-Tarski?

A(1): Hack at brains.

A(2): Banach-Tarski Banach-Tarski ...

⋆ Integral calculus is an anagram of calculating rules.

⋆ At the end of his course on mathematical optimisation, the professor sternly says to his students: "there is one final piece of advice I'm going to give you now: whatever you have learned in my course – never try to apply it to your personal lives!"

"Why?" the students ask.

"Well, some years ago, I observed my wife preparing breakfast, and I noticed that she wasted a lot of time walking back and forth. So, I went to work, optimised the whole procedure, and told my wife about it.

"Before I applied my expert knowledge, my wife needed half an hour to prepare breakfast. And now, it takes me less than fifteen minutes..."

⋆ **Definitions:**

Lecturer: one who talks in someone else's sleep.

Lecture: an art of transferring information from the notes of the lecturer to the notes of the students without passing through the minds of either.

Disjoint: what I am about to smoke in dis moment.

Hausdorff: two distinct points can be "housed-off" in open houses such that the houses are disjoint.

Abelian: a thousand melian. As in:

Lecturer: Today we will be studying Abelian groups.

Student (waking up): What? I hardly know two groups!

$\star$ The world is divided into two classes: people who say "the world is divided into two classes", and people who say "the world is divided into two classes: people who say "the world is divided into two classes", and people who say "the world is divided into two classes: "people who say...

$\star$ There are two groups of people in the world: those who believe that the world can be divided into two groups of people, and those who don't.

$\star$ I asked a statistician for her phone number, and she gave me an estimate...

$\star$ $43^7 = 271818611107 = (2 + 7 + 1 + 8 + 1 + 8 + 6 + 1 + 1 + 1 + 0 + 7)^7$

$\star$ This actually happened in a lecture. Professor: expand $(x - a)(x - b)(x - c) \cdots (x - z)$.

Student: 0. (Look at the third last term.)

$\star$ Problem: find $\lim_{x\to 0} \frac{\sin 7x}{5x}$.

Actual answer: $\frac{\sin 70}{50}$.

$\star$ 8 out of 5 people do not understand fractions.

$\star$ A world without geometry is pointless.

True stories...

$\star$ The mathematician Stefan Bergman (1895-1977) once went to the beach, and being cold, he came out of the water and decided to change into his street clothes. He wandered off in the wrong direction of the parking lot, but his friends paid no attention as they were used to this sort of behaviour. He came back, clearly not in his own clothes, and exclaimed, "You know, there is the most unfriendly woman in our car!"

$\star$ The mathematician Abram Besicovitch once drove for hours to an old friend's home, and began a detailed discussion of mathematics. The friend invited him to stay for lunch. Afterwards they resumed the discussion. Five or six hours later, the friend asked him for stay for dinner, and Besicovitch readily assented. The friend suggested him to phone his wife, lest she be worried. Besicovitch said, “No, she is not worried. She is waiting in the car.”

⋆ The mathematician Pete Casazza was once assigned to teach a large calculus lecture. He had assumed the task many times before and was tired of it. So he arranged another person to meet the class on the first day. Cassazza sat near the front, pretending to be a student. He found many faults with the presentation and raised various comments. The lecturer became increasingly frustrated, and finally threw down his chalk in exasperation, “All right. If you think you can do a better job then you teach the class.” He then stormed out of the room, and Casazza took over.

⋆ Norbert Wiener one day began his lecture by saying: “Today we will learn some applications of Fourier analysis to number theory. The basic unit of number theory is the prime number. A prime number is a positive integer that has no divisors except for 1 and itself. Here are some examples of primes. . . ” Wiener proceeded to write out the first 200 primes. Then he said, “The first result is this. If $\{p_1, p_2, \ldots\}$ are primes, then

$$\sum_j \frac{1}{p_j} = \infty.$$

This is obvious by inspection.”

⋆ Norbert Wiener was proud of his Chinese ability. Once he was invited to lecture in China, and he began by some words in Chinese. What he actually said was along the lines of “the cow is green”. The audience listened politely. Once, Wiener, A. Weil (who also knew some Chinese) and S. S. Chern (actual Chinese) were in the same elevator. Wiener and Weil jabbered away in Chinese during the ride. After they got off, Chern turned to a graduate student and said, “Can you please tell me what language they were speaking?”

⋆ Charles Brown Tompkins (1921–1971) was a professor at UCLA. He was known to tipple during the day excessively. He would sit at the desk in his office with the door open. If a student appeared, he would crawl under the desk until the student went away. One day he was sitting at the desk and flipped over backwards, hitting his head. He was found dead by the janitor a few days later.

Great quotes:

Dirac, when asked about his views on poetry, responded, “In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in poetry, it’s the exact opposite.”

“When a philosopher says something that is true then it is trivial. When he says something that is not trivial then it is false.” – Gauss

“God made the integers, all else is the work of man.” – Kronecker

“There was more imagination in the head of Archimedes than in that of Homer.” – Voltaire

“A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.” – Banach

Mathematical graffiti:

Room 105, Richard Berry Building, where the last word was added: “`Turn off the lights and shut the door if you are the last to leave` Australia”.

Bathroom, Richard Berry Building: in the middle of a graffiti battle, the word “standard” was inserted: “Learn your *standard* errors”.

Subway station: “I’ve just developed a proof that demonstrates that, where $n > 2$, the equation $a^n + b^n = c^n$ cannot be solved with integers. Unfortunately, my train is coming.”

Bathroom, University of Washington: “have you ever noticed that there’s no attempt being made to find really large numbers that aren’t primes? I mean, wouldn’t you like to see a news report that says ‘today the Department of Computer Sciences at the University of Washington announced that $2^{58111625031} + 8$ is even. This is the largest non-prime yet reported.’ ”

A “sign” at Princeton University which stayed up for three years: “In case of fire remove all clothing. Don fire-proof togas that will be provided.”

Solutions to Problems from Last Edition

We thank all people who submitted solutions.

Question 2 was solved by Rick Tankard, who may now collect $3 from the MUMS room. Sam Chow has noted that with question 2, by demanding exact answers one can ask for $P(\pi)$ and obtain the coefficients, as π is transcendental.

Paul Tune solved question 4, and correctly noted that when $x = \pi$, we get $-\infty$ and when $x = 0$, we get $\log 2$. Paul may collect $5 from the MUMS room. Special mention goes to Nick Sheridan, who solved question 4 while trekking in the wilderness of Peru, proving every detail from scratch.

Question 5 ($7) was solved by Yi Huang.

Q1 Lucy usually takes the train and arrives at the station at 8:30 am, where she is immediately picked up by a car and driven to work.

One day she takes the early train, arrives at the station at 7:00 am, and begins to walk towards work. The car picks her up along the way and she gets to work 10 minutes earlier than usual. When did Lucy meet the car on this day?

Solution: Many people used a large number of variables to solve this problem, in the end messing it up and failing to obtain the answer. It in fact can be solved with 4 variables.

However, the question is elegant in that it only requires two pieces of information: that she arrived 10 minutes earlier, and that the car was getting to the station at 8:30. The 7:00 am train is irrelevant.

The car drives 10 minutes less than it usually does, then it drives 5 minutes less in each direction, so Lucy gets in the car 5 minutes early, or at 8:25 am.

Q2 P is a polynomial of any degree with non-negative integer coefficients. You are asked to determine P by only asking for $P(x)$ at 2 x values. How do you do it?

Solution: Ask for $P(1)$. This gives an upper bound for the coefficients. Now pick $10^k > P(1)$, and ask for $P(10^k)$. The coefficients will be nicely displayed in the answer, separated by 0's.

Q3 In $\triangle ABC$, the bisector of external $\angle A$ meets BC at D. Find the length of

AD.

Solution: After much tedious work in the field of Euclidean geometry, one obtains $AD^2 = bc((\frac{a}{c-b})^2 - 1)$. To do this, you need the angle bisector theorem and Steward's theorem. These together imply that, when the internal bisector has length d and the external bisector d', and that they divide BD into segments of lengths x, y, z, we have: $d^2 = bc - xy$ and $\frac{c}{b} = \frac{x}{y} = \frac{a+z}{z}$; also note that the two bisectors meet at right angles. We leave the details to the diligent reader.

Q4 Express in closed form $\cos x - \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x - \ldots$

Solution: Note the expression is equal to $\Re(e^{ix} - e^{2ix}/2 + \cdots) = \Re(f(e^{ix}))$, where $f(y) = y - y^2/2 + \cdots$.

Now note that $f(y)$ is the antiderivative of $1 - y + y^2 - \cdots = 1/(1+y)$. Hence $f(y) = \log(1+y)$.

We get $f = \log(2\cos(x/2))$, and so to study the behaviour at the radius of convergence (π), we only need to study $\log(\cos(\pi x/2))$, which has radius of convergence 1.

Abel's convergence test states that a series with coefficients monotonously decreasing to 0 and radius of convergence 1 converges at $|x| = 1$ with the possible exception of $x = 1$. Clearly, the series for $\log(\cos(\pi x/2))$ has this property (for it relates to the series of $-\tan x$, which satisfies this property; this can be verified by differentiation). Hence it converges on $|x| = 1$; as when $x = 1$ it takes the same value as $x = -1$, so it too converges.

Q5 Find the volume of the square antiprism with side length 1.

Solution: When a nice diagram of the solid is drawn, we note that the planes containing each of the 4 corner triangles plus the places of the squares form a truncated square pyramid. The bottom face of the truncated pyramid has side length $\sqrt{2}$ because its midpoints are distance 1 apart, and the top face has length 1.

Each triangle has height $\sqrt{3}/2$, with the top vertex being horizontally $\frac{\sqrt{2}-1}{2}$ away from the bottom square. Using Pythagoras, this gives a vertical height of $h = 2^{-\frac{1}{4}}$.

The volume of the shape is the volume of the truncated pyramid minus the volumes of the 4 triangular pyramids. The former is computed using the formula $\frac{1}{3}h(S+T+\sqrt{ST})$, where $S=1$ and $T=2$ are areas of the square faces. The volume of 1 triangular pyramid is $\frac{1}{3}\frac{1}{2}abh$, where $\frac{1}{2}ab = \frac{1}{2}\cdot 1 \cdot \frac{2-1}{2} = \frac{1}{4}$ is the area of the base triangle.

Putting these together, we obtain volume = $\frac{1}{3}(2^{\frac{1}{4}}+2^{\frac{3}{4}})$, or equivalently $\frac{1}{3}\sqrt{4+3\sqrt{2}}$.

Q6 Find

$$\prod_{\text{all primes p}} \frac{p^2+1}{p^2-1}.$$

Solution: We use Euler's product formula (this can be proven by a sieve-like process):

$$\zeta(s) = \prod_p (1-p^{-s})^{-1} = \prod_p \frac{p^s}{p^s-1}.$$

So $\zeta(2s) = \prod_p (1-p^{-2s})^{-1} = \prod_p (1-p^{-s})^{-1} \prod_p (1+p^{-s})^{-1}$, which gives $\prod_p (1+p^{-s})^{-1} = \prod_p \frac{p^s+1}{p^s} = \frac{\zeta(s)}{\zeta(2s)}$.

Multiplying the two results together, we get $\prod_p \frac{p^s+1}{p^s-1} = \frac{\zeta(s)^2}{\zeta(2s)}$.

We recall that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$. Hence the answer is $\frac{90\pi^4}{6^2\pi^4} = \frac{5}{2}$.

(Iran 1996) Prove that, for $x,y,z>0$, $(xy+xz+yz)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \geq \frac{9}{4}$.

Solution: We multiply both sides by $(x+y)^2(y+z)^2(z+x)^2$, expand everything out, cross multiply, and cancel terms. This yields, on the LHS,

$$4\sum_{sym} x^5y^1z^0 + \sum_{sym} x^4y^1z^1 + \sum_{sym} x^2y^2z^2,$$

and on the RHS,

$$3\sum_{sym} x^3y^3z^0 + 2\sum_{sym} x^3y^2z^1 + \sum_{sym} x^4y^2z^0.$$

Note how the first term on the LHS majorises the first and last terms on the RHS altogether (as $(5,1,0)$ majorises $(3,3,0)$ and $(4,2,0)$), so the inequality holds. We only need to prove that it holds also for the remaining terms. Dividing them by $2xyz$, we get

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2,$$

which is true by taking $t = 1$ in Schur's inequality ($x^t(x-y)(x-z) + y^t(y-z)(y-x) + z^t(z-x)(z-y) \geq 0$).

(Proof of Schur's inequality: without loss of generality, $x \geq y \geq z$, then $(x-y)(x^t(x-z) - y^t(y-z)) + z^t(x-z)(y-z) \geq 0$.)

Paradox **Problems**

Below are some puzzles and problems for which cash prizes are awarded. Bear in mind that anyone who submits a clear and elegant solution may *each* claim the indicated amount (unless two solutions are the same, in which case only the first submission will be rewarded). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G06) in the Richard Berry Building.

The first three problems are more like puzzles, and no mathematical knowledge is required! To solve the last three, some mathematics may help.

1. ($2) I am thinking of one of three numbers: 1, 2 or 3. You may ask me exactly one yes-no question to find out what number it is, and I will answer truthfully (yes, no or I don't know). What do you ask?

2. ($2) You are blindfolded before a table, and on the table there are some coins, exactly 247 of which are head up. How can you divide all of the coins into 2 piles such that each has the same number of heads facing up?

3. ($4) Two candles each burn for exactly 1 hour at uneven rates. Measure 45 minutes with the candles and some matches.

4. ($5) Let a, b and c be the side lengths of a triangle with fixed perimeter $2s$. As a, b and c vary, what is $\limsup\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$?

5. ($10) In a regular heptagon $A_1A_2\cdots A_7$, prove that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$.

6. (\$10) We have n letters with n corresponding envelopes. Suppose we put each letter in an envelope randomly, what is the probability that none of them goes into the correct envelope?

Puzzle 2:

We evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ by telescoping.

First way: $\frac{1}{2}\sum_{n=1}^{\infty}(\frac{1}{2n-1} - \frac{1}{2n+1}) = \frac{1}{2}(1/1 - 1/3 + 1/3 - 1/5 + \cdots) = \frac{1}{2}$.

Second way: $\sum_{n=1}^{\infty}(\frac{1}{1} - \frac{2}{3} + \frac{2}{3} - \frac{3}{5} + \cdots) = 1$.

Which is the right answer?

Answer to puzzle 1: we have to put in the limits, so the 1 disappears; otherwise we need to put constants on each side.

Answer to puzzle 2: the first one is correct; in the second one, the terms do not tend to 0, and so the series does not converge.

"The pleasure we obtain from music comes from counting, but counting unconsciously. Music is nothing but unconscious arithmetic." – Leibniz

"I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery." – Descartes

Paradox would like to thank Wilson Ong, James Zhao, Stephen Muirhead, Kate Mulcahy and Alisa Sedghifar for their contributions to this issue.