Paradox

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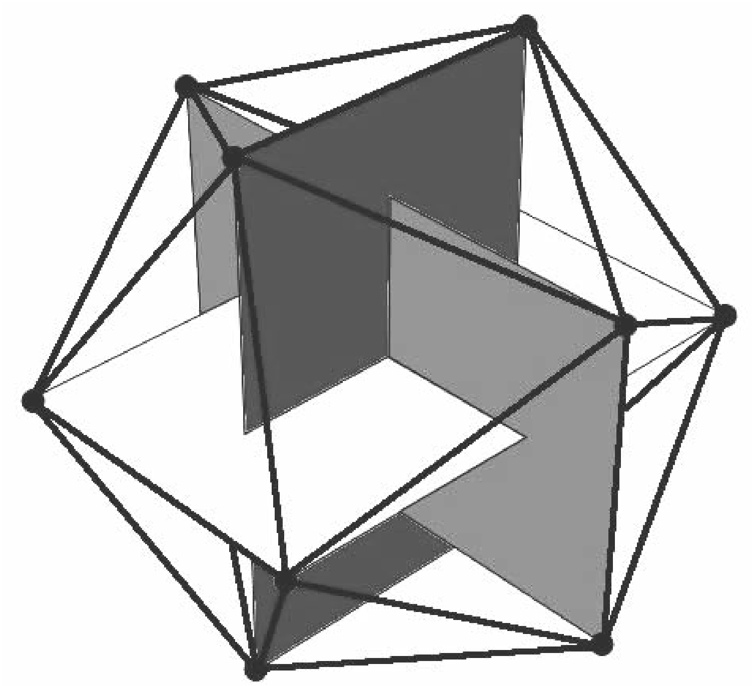

MUMS

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Words from the Editor

Welcome to this edition of Paradox, the magazine of the Melbourne University Mathematics and Statistics Society (MUMS). MUMS has a long history; its aims in 1985 are as valid today as they were back then: to encourage social interaction among maths students, to become a forum of discussion, to provide liaison between students and staff, and to develop an interest in maths outside coursework.

I'd like to stress here that this society is for maths students, not for maths problems. Although we do devote some time to maths problems, as you will see in this edition, for maths is – almost by definition – the smart way of doing things correctly. We have an article on the interesting problem of efficiently weighing coins to find a fake one; another on the rich topic of drawing things with ruler and/*or* compass, showing why you can't trisect an angle, and finally some tips on how to do better on your first year maths subjects (and beyond). As usual, we have loads of jokes and anecdotes that will certainly induce lots of head-shaking.

We also have a room, also known as G24, and people are welcome to relax in there. We don't do maths in the room (unless assignments are due).

There seems to be a consensus that maths is dominated by scary males with little common sense; in fact, this is not entirely true. Well, almost. So if you think you can run things better, please submit an article to Paradox (you can email) on mathematical things that you find interesting, or how to run things better. Also, come to our AGM later in the year (if you have classes in the Richard Berry building, watch out for our posters).

— James Wan

About the front cover
The vertices of a regular icosahedron form 3 mutually perpendicular golden rectangles; if the icosahedron has edge length 2, then the Cartesian coordinates of its vertices are all permutations of the triple $(0, \pm 1, \pm\phi)$, where $\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio.

Words from the President

Welcome. This semester, MUMS has our usual swag of activities lined up to enhance your experience, mathematical or otherwise.

For the uninitiated, MUMS is a society dedicated to exposing students to a variety of fun and interesting maths that is *not* covered in coursework. You can look forward to regular seminars, trivia nights, and our main event, the Puzzle Hunt, from 7 – 11 April. Later in the year, we will also be running the University Maths Olympics (UMO), an interesting sport mixing power walking with mathematics. Publicity for these events will be posted around the maths building, and on our website, http://www.ms.unimelb.edu.au/∼mums/ . To receive email notifications, please subscribe to our mailing list, which is open to anyone. Involvement in MUMS is by no means limited to maths students; we encourage anyone with an interest in maths, no matter how small, to join.

For those wondering what happened to the MUMS room, we have moved! After a number of years inhabiting G06, the MUMS room has relocated to G24 (Richard Berry building), around the corner opposite reception. Please drop by to say hi and hang out with other MUMS regulars.

— Alisa Sedghifar

A Weighty Problem

A problem is 'coined'

Some time ago there appeared this conundrum called the '13 coins problem':

Amongst 13 gold coins there is one that is fake. The only way to distinguish the fake coin from the genuine gold coins is by its weight. Placed at your disposal is a set of two-pan scales. What is the minimum number of 'weighings' that need to be performed on the scales in order to determine which of the coins is the fake?

Probably the main stumbling block to solving the problem is its deceptively small answer – a measly three weighings. It is not hard to show that two weighings are not enough, but constructing a solution for three weighings can take some time.

A more interesting take on the question is whether or not 13 coins are the best you can do with three weighings. Or, more generally:

Given a pile of coins with one of them fake, and given k weighings on a scale, what is the maximum number of coins amongst which you are always able to identify the fake coin?

Even for the case of $k = 3$ the answer is by no means obvious. It is unclear whether it is 13 coins, 14 coins or even something much higher.

'Bits' of information

A good starting point is to establish an upper bound. Could you determine the fake amongst, for example, 1000 coins? Clearly you couldn't, because there is a limit to the amount of *information* that can be attained in just three weighings. So how much information is there? Well, each time you weigh something on the scales it will tell you either

a) the left hand side was heavier,

b) the right hand side was heavier,

or c) the two sides of the scale are equal in weight.

Thus for every weighing you have three 'bits' of information. So, for three weighings, you have $3^3 = 27$ 'bits' of information. But as you are asked to determine which of the n coins is a fake, there are n different outcomes amongst which you must distinguish. Thus as least n bits of information are necessary, and so n is at most 27. We have an upper bound, but not a very good one.

A paradox?

It is here, pondering this problem, that you may realise that maths is throwing up one of its trademark curve-balls. The problem is hard because it has been obfuscated, paradoxically, by making it simpler. This alludes to the fairly common problem solving technique – making the question 'harder' (read: making the initial conditions more restrictive), but in doing so simultaneously making it easier to solve.

To make the question 'harder', we add that you must determine also whether the fake coin is *heavier* or *lighter*. This will reduce the number of coins able to be distinguished in k weighings, but only by very little. This is because once you have determined which coin is fake, you automatically know whether it is heavier or lighter simply by considering one of the 'weighings' that involved that coin. As only one fake coin is present, the side with this coin on it will be heavier if and only if the coin itself is heavier, and similarly if it is lighter.

The only problem would occur if you did not weigh that coin at all, and this only happens if you weighed *every other coin* and determined them all to be equal in weight, concluding that the last coin must be the fake. The presence of this last coin reduces the maximum number of coins you can distinguish by one.

Another way to make it 'harder' is to add the possibility of there being no fake coin at all. Again this makes very little difference to your ability to weigh, and again this is because the only coin to be affected will be the (at most one) coin that you did not weigh at all. Before, if all the other coins weighed the same then you could deduce the last coin was fake, but not anymore. In every other case you are determining the fake coin *directly*, and no such deduction based on the fake coin's existence is required.

It is this most restrictive version of the problem that we want to consider, where we need to determine the exact nature of the fake coin and where we are unsure if it exists or not. But though this problem is more restrictive, it will actually only reduce our weighing potential by *one solitary coin*.

Now we can attempt to re-apply our upper-bound trick. With k weighings we again have 3^k 'bits' of information. But now we have $2n+1$ different states to distinguish between, as each coin could be heavier or lighter, and also we may have no fake coin at all. So $2n+1 \leq 3^k$, and hence $n \leq \frac{1}{2}(3^k - 1)$, a much tighter upper bound.

But is equality ever possible?

If we apply equality to the $k = 3$ case, adjusting for the extra coin we get from re-loosening the requirements, we find we should be able to find the fake amongst 14 coins, not 13. This is highly suggestive that equality is not attainable.

But let's assume that equality was possible. Now we have 3^k states to distinguish between and 3^k bits of information. The only way to construct an algorithm that distinguishes between the possible states in k weighings is if every weighing reduces the number of possible states down to exactly 3^{k-1}, no matter what the outcome of the first weighing. In other words, each weighing takes the possible states, divides them up evenly into three categories, and then selects one of them. No category can have more than 3^{k-1} states contained in it, as then the remaining $k-1$ weighings would be insufficient.

Now, suppose firstly you weigh x coins against y coins, and suppose the scales balance. Given that there must now be exactly 3^{k-1} possible states, the number of coins remaining must be the number of coins that produce this many states. So equating with the exact maximum number of coins able to be weighed in $k-1$ weighings:[1]

$$\begin{aligned}\frac{1}{2}(3^{k-1}-1) &= \frac{1}{2}(3^k-1)-x-y\\ x+y &= \frac{1}{2}(3^k-3^{k-1})\\ x+y &= 3^{k-1}.\end{aligned}$$

But then $x+y$ is an odd number (if k is not 1), and so we initially weighed up an odd number of coins against an even number of coins, a weighing that will not yield anything useful. We have our contradiction.

Interestingly, the original upper bound is achievable if you are given something with which to balance the number of coins on the scale at the first weighing! For instance, if you are given a coin which you *know* to be true, then you can actually get that upper bound's worth out of your weighings!

Ternary logic

But back to our problem, where our upper bound is now one less at $\frac{1}{2}(3^k-1)$ (after adding on the 'extra' coin from knowing that there exists a fake coin). Great! But is this new upper bound attainable? The case of $k = 3$ (with 13 coins) suggests that it is. Indeed it always is! Let's see why.

First assume you know that the fake coin is heavier than all the rest, and that there exists at least one fake coin. In this case we can actually distinguish amongst 3^k coins. The method is to utilise a ternary number system, and using each weighing to pick out a single digit of the coin's ternary representation. Let the three outcomes of each weighing be 0 (left side heavier), 1 (balanced) and 2 (right side heavier). Then label the 3^k coins from 0 to 3^k-1 in ternary, (for example: 000, 001, 002, 010, ..., 220, 221, 222). Now first weigh the coins with 0 in the first digit against coins with 2 in their first digit, knowing that there will be the same number of coins on each side of the scales. Secondly

[1] If the left hand side is any smaller, then $x+y$ would be larger, and each side of the scale would be $> \frac{1}{2}(3^{k-1}-1)$, which we cannot handle.

weigh coins with 0 in their second digit against coins with 2 in their second digit. Continue in this manner with one weighing for each of the k digits. Suppose the outcomes of the weighings were 0, 1, 2, 0, 2 and 1. Then the fake coin would be coin 012021. Easy.

So we try the same technique not knowing if the fake coin is lighter or heavier. Now we are unable to distinguish between 'mirror-pairs' (e.g. 102 and 120, 010 and 212 etc) because we wouldn't know if it was coin 102 which was heavier or coin 120 which was lighter. So we can take at most one coin from each mirror-pair. But we also know we have a maximum of $\frac{1}{2}(3^k - 1)$ coins, so we need to leave one of the mirror-pairs out completely. So leave out $\{0 \ldots 0, 2 \ldots 2\}$.

Now we just need to ensure that we can make a choice of one from each mirror-pair so that for each digit there is an equal number of 0's as there are 2's. This is essential so that we have an equal number of coins on each side of the scales for each weighing.

The right choices

Proving that such a choice is possible is a simple matter of induction. For $k = 1$ we are only left with the coin labelled 1, so we just pick that. Assume that we can choose them appropriately for $k = t$. Now for $k = t + 1$, we look at the first t digits of each coin, and select them as we would had we been dealing with $k = t$. This selection is fine when we leave out 111…110 and 111…112, as it will not contain any mirror-pairs. And this selection will match up the 0's and the 2's perfectly for the first t-digits. Also, for the last digit, each selection contains the three coin triplet ending with 0, 1 and 2, so the last digit has an equal number of 0's and 2's. Now we are just left with the coins $\{000\ldots001, 000\ldots002\}$, $\{111\ldots110, 111\ldots112\}$ and $\{222\ldots220, 222\ldots221\}$. We then simply pick 000…001, 111…112 and 222…220 to complete the selection.

For example, with $k = 3$ we can pick the following coins:

$$\{001, 010, 011, 012, 111, 112, 120, 121, 122, 200, 201, 202, 220\}.$$

Then we are done. Just do the same sequence of weighings as before, and then see which of the mirror-pairs can be associated with the final weighing output. The sole member of that mirror-pair presenting our choice of coins will be the one that is fake!

So the maximum number of coins you can find a fake coin amongst with 3 weighings is 13, as the original conundrum anticipated. But in order to generalise this rather easy problem, we just had to make it more difficult!

Multiple fake coins

A natural extension to this problem is that where you have some unknown number of fake coins. This problem is left for the reader to try:

Given n coins, where n is even, show that with at most $[\frac{3n}{4}]$ weighings you can divide your coins into two piles, where one pile consists entirely of fake coins and the other entirely of real coins (where $[k]$ is the integer part of k).

Clearly, without any additional information about the number or type of fake coins, you are unable to tell which pile is which. You may end up with all the coins in one pile, in which case you are unsure if all your coins are real, or all your coins are actually fakes!

Even more heights to 'scale'

Another interesting extension is to determine a single fake coin amongst many where your equipment is not a two-pan balance but a set of scales which give you a numerical reading. With 3 weighings it is known that you can find a fake coin amongst 6. But is this the best you can do with 3 weighings? What is the best you can do with k weighings? If you find solutions to these problems, Paradox would like to hear from you!

— Stephen Muirhead

Puzzle 1:
To solve $\sqrt[3]{1-x} + \sqrt[3]{x-3} = 1$, we cube both sides:

$$(1-x) + 3\sqrt[3]{1-x}\sqrt[3]{x-3}(\sqrt[3]{1-x} + \sqrt[3]{x-3}) + (x-3) = 1.$$

Replace the expression in brackets by the initial equation, we get $\sqrt[3]{1-x}\sqrt[3]{x-3} = 1$.
Taking the cube again gives $x^2 - 4x + 4 = 0$, so $x = 2$. Substituting 2 back, one obtains $\sqrt[3]{1-2} + \sqrt[3]{2-3} = -2$, so $1 = -2$.

Puzzle 2:
You and a friend try to guess the year on a coin; the closer guess wins. You get 2 guesses and your friend 1, plus you get to choose who goes first. What is your best strategy?

Great Quotes:

Gauss:

"It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment."

"It may be true, that men, who are mere mathematicians, have certain specific shortcomings, but that is not the fault of mathematics, for it is equally true of every other exclusive occupation."

"There are problems to whose solution I would attach an infinitely greater importance than to those of mathematics, for example touching ethics, or our relation to God, or concerning our destiny and our future; but their solution lies wholly beyond us and completely outside the province of science."

⋆

Pascal:

"The sole cause of all human misery is the inability of people to sit quietly in their rooms."

⋆

The University of Melbourne, Faculty of Science, Board of Review minutes, 1978:

"Computer science had the absolute highest pass rate due to the large numbers in the course."

⋆

If I have seen farther than others, it is because I was standing on the shoulders of giants. – Isaac Newton

In the sciences, we are now uniquely privileged to sit side by side with the giants on whose shoulders we stand. – Gerald Holton

If I have not seen as far as others, it is because giants were standing on my shoulders. – Hal Abelson

Mathematicians stand on each other's shoulders. – Carl Gauss

Mathematicians stand on each other's shoulders while computer scientists stand on each other's toes. – Richard Hamming

It has been said that physicists stand on one another's shoulders. If this is the case, then programmers stand on one another's toes, and software engineers dig each other's graves. – Unknown

Maths Jokes

A mathematical biologist spends his vacation hiking. One day, he encounters a shepherd with a large herd of sheep. One of these cuddly, woolly animals would make a great pet, he thinks.

"How much for one of your sheep?" he asks the shepherd. "They aren't for sale," the shepherd replies.

The math then says: "I will give you the precise number of sheep in your herd without counting. If I'm right, don't you think that I deserve one of them as a reward?" The shepherd nods.

The math biologist says: "247."

The shepherd is silent for a while and then says: "You're right. I hate to lose any of my sheep, but I promised: one of them is yours. Have your pick!"

The math biologist grabs one of the animals, puts it on his shoulders, and is about to leave, when the shepherd says: "Wait! I will tell you what your profession is, and if I'm right I'll get the animal back."

"That's fair enough."

"You must be a mathematical biologist."

The man is stunned. "You're right. But how could you know?"

"That's easy: you gave me the precise number of sheep without counting – and then you picked my dog."

∞

A mathematician believes nothing until it is proven;

A physicist believes everything until it is proven wrong;

A chemist doesn't care;

A biologist doesn't understand the question.

∞

To mathematicians, solutions mean finding the answers. But to chemists, solutions are things that are still all mixed up.

∞

The graduate with a Science degree asks, "Why does it work?"

The graduate with an Engineering degree asks, "How does it work?"

The graduate with an Accounting degree asks, "How much will it cost?"

The graduate with a Liberal Arts degree asks, "Do you want fries with that?"

∞

Biologists think they are biochemists,

Biochemists think they are Physical Chemists,

Physical Chemists think they are Physicists,

Physicists think they are Gods,

And God thinks he is a Mathematician.

∞

Q: Why didn't Newton discover group theory?

A: Because he wasn't Abel.

∞

Top 10.0 reasons to be a statistician

1. Estimating parameters is easier than dealing with real life.
2. Statisticians are significant.
3. I always wanted to learn the entire Greek alphabet.

4. The probability a statistician major will get a job is $> .9999$.
5. If I flunk out I can always transfer to Engineering.
6. We do it with confidence, frequency, and variability.
7. You never have to be right – only close.
8. We're normal and everyone else is skewed.
9. The regression line looks better than the unemployment line.
10. No one knows what we do so we are always right.

∞

Q: What do you get when you add 2 apples to 3 apples?

A: An American senior high school math problem.

∞

Q: What is 8 divided in two parts?

A: Vertically it is 3, horizontally it is 0.

∞

Q: What's the difference between the radius and the diameter?

A: The radius!

∞

I is your imaginary friend.

∞

Definitions:

Dilate: to live long

Free product: things at no charge

Centre of mass: the priest

∞

Beer, Fashion, and Geometric Constructions

Now that I have your attention, let's begin with part 1:

1. Geometric constructions

With the graphics card yet to be invented for another 2000 or so years, how do you suppose the ancient Greeks spent their spare time? Well, for the mathematically inclined, they played a graphical gamed called Straightedge and Compass Construction, with unparalleled elegance and simplicity. Its rules, in modern notation, are as follows:

1. We start with two given points, $(0,0)$ and $(1,0)$. Note that a unit distance is hence given. These points are by default constructible.
2. We may draw a straight line through any 2 constructible points using the straightedge; we may draw a circle with centre at 1 constructible point and passes through another.
3. Intersections of all constructed lines and circles are now constructible.

As you can see, this is a game for people in the following categories: those with an enquiring and precise mind, those obsessed with rigour and neatness, perfectionists, or narcissistic lunatics.[2] However, over the years the game has survived, possibly due to the ascetic beauty and exactness it exudes.

To make a point that at least something can be constructed from the deceptively trivial rules, we bisect a segment AB. Firstly, A and B must have been previously constructed (or given at the start). We can draw a circle centred A and through B, and another centred B and through A. They intersect say at C and D. We draw a line through C and D; by symmetry, this line is the perpendicular bisector of AB.

Note that the straightedge (or ruler) has only 1 edge, is arbitrarily long, and has no markings on it. The compass is collapsible, that is, after drawing each circle, the legs fall back together. So we cannot simply lift the compass and "carry" a distance, but this can be overcome by several algorithms,[3] so after

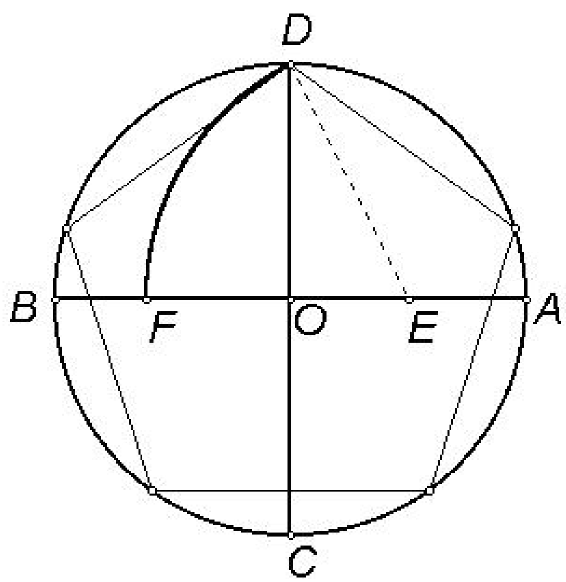
[2]Centuries after people proved certain constructions to be impossible, plenty of avid "circle squarers", barely knowing the above rules, delude themselves with their "important discoveries". Squaring the circle means to construct a square with the same area as a unit circle.

[3]For instance by constructing a parallelogram – the interested reader may want to give this a go.

all we can use it as a normal compass. Also note that a point is constructible if it can be obtained by the above procedure in a finite number of steps.

Using these algorithms, it should be fairly straightforward to bisect an angle, draw parallels or perpendiculars to a line, transfer segments and angles, and divide a segment into n equal pieces. (For the last one, think parallel lines.)

Some of the most aesthetically pleasing objects to construct are the regular polygons. The square is rather trivial. The regular hexagon is also trivial (mark off radii in a circle); from this we also get the equilateral triangle. The regular pentagon, however, is not as easy, and following diagram shows how it can be done.



We draw two perpendicular diameters of the circle, AB and CD. Now construct E, the midpoint of AO. Using E as centre and ED as radius, draw an arc which intersects BO at F. Now DF is the required side length of an inscribed regular pentagon.

Note the appearance of the golden ratio in the diagram: $OD/OF = FA/OA = \frac{\sqrt{5}+1}{2}$. We also have a bonus result: OF is the side length of an inscribed regular decagon. The proof of both constructions come from the knowledge that $\sin 18° = \frac{\sqrt{5}-1}{4}$.

So polygons of sides 3, 4, 5 and 6 can be constructed. It is easy to double the sides of any constructed polygon: you either bisect an angle or an arc. The Greeks also found how to construct the 15-gon: what you do is construct an equilateral triangle and a regular pentagon in the same circle, sharing vertex A. Now, the angle (measured from the centre) between first vertex away from A of the pentagon and the triangle is $120° - 72° = 48°$. Bisecting this angle gives the required central angle for a 15-gon.

In fact, a more general argument is possible. Say we can construct a regular n-gon and an m-gon, where m and n are coprime. We can certainly add angles together. Working in degrees, we add the two central angles together and get $\frac{360(m+n)}{mn}$. There exists k such that $k(m+n) \equiv 1 \bmod mn$; so copy the angle k times around the circle, and we get an angle of $\frac{360}{mn}$. So an mn-gon is constructible.

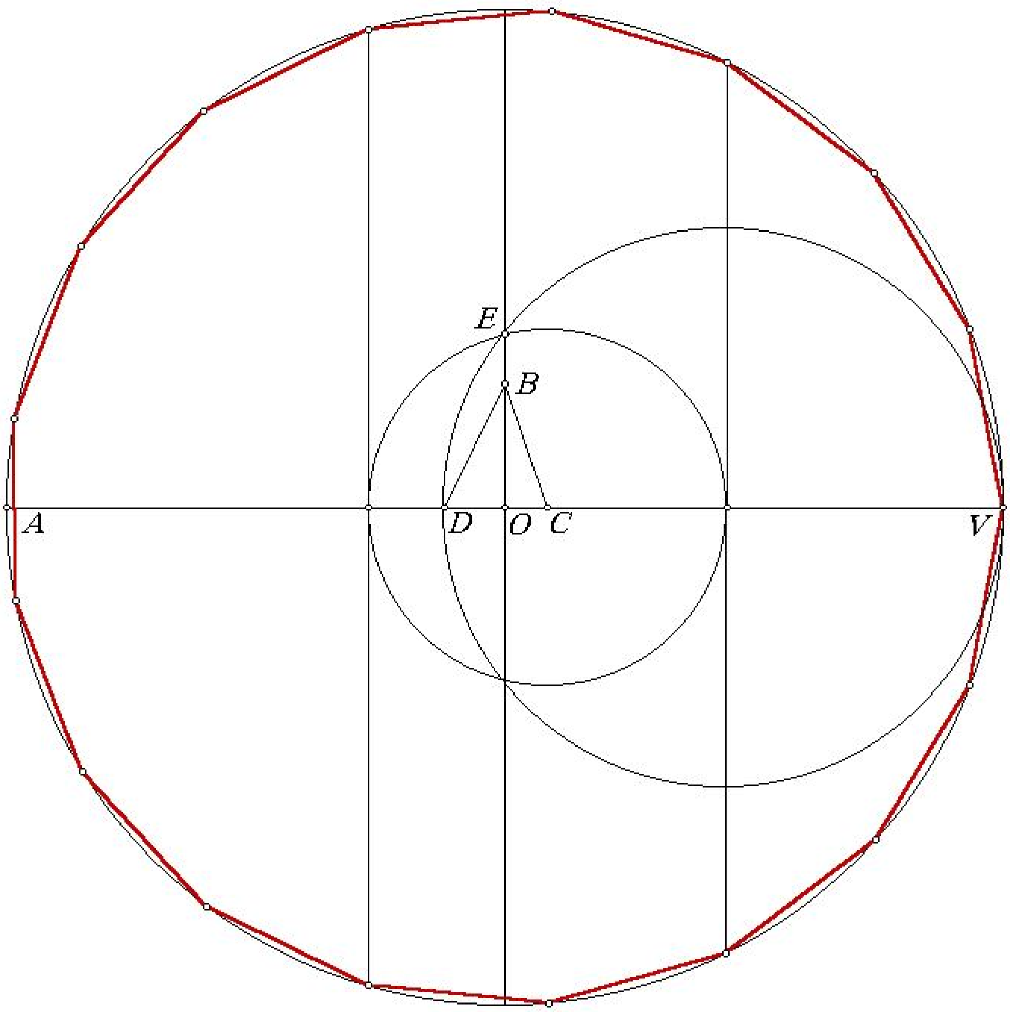
Of course, the condition of m and n being coprime is a bit strong for the Greeks, because from their initial set of constructible polygons, we can only get n-gons where $n = 2^a 3^b 5^c$, where b and c are either 0 or 1.

Well, are there any other constructible regular polygons? The problem remained open for thousands of years until the 17 year old Gauss, pitying the mere mortals who went before him, decided to give it a go, by first studying the regular heptagon. He solved the problem when he was 18.[4] He showed that the 17-gon is constructible. Legend goes that he requested a heptadecagon to be engraved on his tombstone, which was not carried out, possibly due to the number of sides rendering it similar to a circle, or the fact that this is only a legend. However, there is a monument of Gauss in his hometown, Brunswick, with a 17-pointed star. Gauss showed five years later that the n-gon is constructible if and only if[5] $n = 2^a \prod p_i^{a_i}$, where a_i is either 0 or 1, and p_i are Fermat primes: primes of the form $2^{2^k} + 1$. The only known Fermat primes are 3, 5, 17, 257 and 65537; Euler showed that $2^{2^5} + 1$ is not prime by quickly factorising it as 641×6700417. A (not necessarily correct) construction for a regular 65537-gon was first given by J. Hermes in 1894; he spent 10 years completing the 200-page manuscript. If you decide to carry it out, it'll look very round.

One of the simplest constructions for the regular 17-gon is given below; you'll need to know that $BO = \frac{VO}{4}, \angle OBC = \angle OBV/4, \angle CBD = 45°$; C is the centre of the smallest circle.

[4] It is commonly quoted as 19, but that was when he decided to publish to result.

[5] There is doubt on the "only if" bit, for he did not include a proof. However, Wantzel (see below) was able to complete the proof.



Pierre Laurent Wantzel (1814-1848) was a French mathematician. In 1837, when still an engineering (!) student, he was the first to publish a proof of the impossibility of constructing a polygon not in the form above. He was also the first to show that the problems of doubling the cube and trisecting the angle were impossible using only compass and straightedge. Those were two of the three great geometric problems of antiquity; the third, squaring the circle, was left to Lindemann in 1882, who showed π is transcendental.

A few more words on our hero here. Wantzel was a child prodigy, surpassing his teacher at an early age; at nine, the teacher sent for him when he encountered a difficult surveying problem. At the age of 15, he edited a book on arithmetic (making new proofs on the way; arithmetic was much more impressive back then). He was the first to come first in the entrance examination to the École Polytechnique and the École Normale. In 1845, he gave a new proof of the impossibility of solving all algebraic equations by radicals. He died at a young age, possibly because of overwork; he was said to be "alternatively abusing coffee and opium".

The polygon proof is fairly straightforward if you know about cyclotomic polynomials and some Galois theory,[6] but let's not bother with it here. Instead, we show that the simplest non-constructible polygon is indeed the heptagon.

To proceed, we must see what can be done using the rules of the game. We note that the constructible numbers (defined as either of the coordinates of constructible points) form a field. Clearly, we can add or subtract 2 constructible numbers. To construct ab, where a and b are constructible numbers, we simply form a triangle with shorter sides 1 and a, and a triangle similar to it with a shorter side b. Then another side is ab. We can also obtain $\frac{1}{a}$ "similarly".

So the set of all constructible points at least contains $\mathbb{Q}$. We can also take the square root of a: construct AB of length $a+1$, and draw a circle using AB as the diameter. Let C on AB be 1 unit from A, then the distance from C to the circle and perpendicular to AB is $\sqrt{a}$. A proof again comes from similar triangles. Now when we draw circles and lines through constructible points as per the rules, we may find their intersections using coordinate geometry, that is, find the new constructible points. Now the equation of a circle is second order, so by solving the simultaneous equations, we find that the points of intersection require the usual operations, plus only one more operation, the taking of square roots. Hence we can describe this field: take $\mathbb{Q}$, we append all the square roots of its (positive) elements to it to form a larger field; now take this field and append all the square roots... Hence any constructible point must have, as the technical term goes, a degree extension of 2^n over $\mathbb{Q}$. It is then intuitive that any number with a minimal polynomial of degree $\neq 2^n$ is not constructible. A more detailed and rigorous treatment can be found in the third year Algebra subject, or whatever fancy name it will be called under the New Melbourne Model, if it survives at all. For cubics which concern us here, there exists more elementary but rather lengthy proofs of the above fact. So for example, a number like

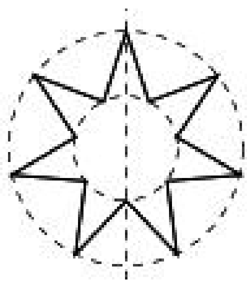
$$\sqrt{\frac{1}{2}+\frac{1}{8}\sqrt{7+\sqrt{5}+\sqrt{6(5+\sqrt{5})}}}$$

is constructible (it happens to be $\cos 3^\circ$), but $\sqrt[3]{2}$ is not, rendering the problem of doubling the cube intractable.

[6] Basically showing that the order of the polynomial must be a power of 2; and conversely, the cyclotomic extension has a cyclic Galois group, and its chain of subgroups corresponds to a tower of quadratic extensions.

Now suppose the heptagon is constructible, then clearly so is $x = 2\cos\frac{2\pi}{7}$. Let $t = \frac{2\pi}{7}$. Then, applying the double angle formula twice, we get $2(2(\frac{x}{2})^2 - 1)^2 - 1 = \cos 4t$. But $\cos 4t = \cos 3t = 4\cos^3 t - 3\cos t = 4(\frac{x}{2})^3 - 3\frac{x}{2}$, so equating them gives $x^3 - x^2 - 2x - 1 = 0$. It is easily checked that this has no rational roots, and hence is irreducible. But its degree over $\mathbb{Q}$ is not a power of 2, contradiction.

So how do we get around this problem, for after all, doesn't the heptagon feature prominently on the Australian flag?[7] The star on the flag is reproduced below; the ratio of the radii is $\frac{4}{9}$. Incidentally, there were only 6 points on the stars on the original flag, representing the six states. The seventh was added in 1908 to represent the Territory of Papua and future territories.[8] The first way get around the problem is to approximately construct a heptagon, which will do if we want to print off a flag, or something. The other way is the bend the rule of the game. Neusis construction will do the trick, as we will see. Another way to do it exactly is via the rich art of paper folding!



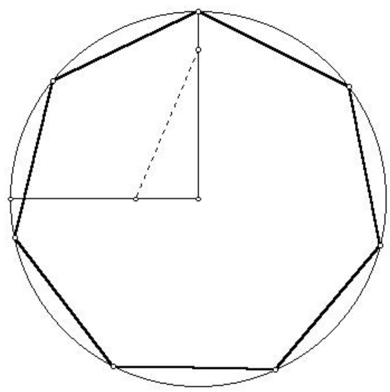
Now to approximate a heptagon, we could find a value close to $2\sin\frac{\pi}{7}$. An early attempt was made by Heron (or Hero) of Alexandria (circa 10 - 70 AD), an engineer and geometer. He is credited with the invention of the first steam-powered device, the first wind-powered device (operating an organ), and the first vending machine (a coin which gives a set amount of holy water, using a lever activated by the weight of the coin until it fell off). He also made an entirely mechanical play about ten minutes long, powered by a binary-like system ropes and simple machines. He is best known for the formula for the area of a triangle which bears his name, but this was not his invention, for Archimedes knew the formula.

Anyway, Heron used the value of $\frac{\sqrt{3}}{2}$, which is about 0.2% off. By using continued fractions, one can obtain the easily constructible value of $\frac{13}{15}$, which is

[7]But not on the wimpy New Zealand flag.

[8]Northern Territory was separated from South Australia in 1911, ACT was formed in the same year, and curiously Jervis Bay Territory was considered separate in 1989.

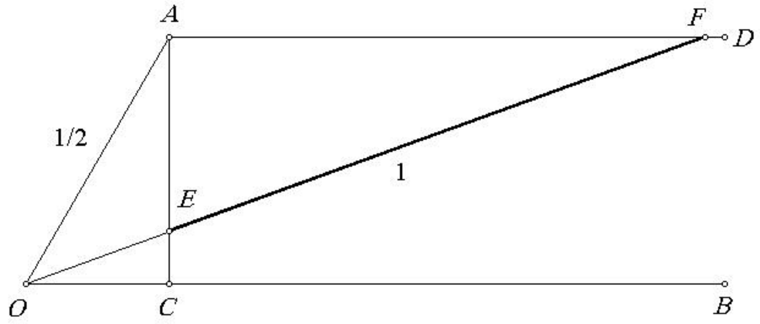
only 0.13% off. This profoundly useless observation was allegedly first discovered by yours truly, at the age of 8, and to this day remains the pinnacle of his intellectual achievement. To carry out the construction, we set up perpendicular radii on the circle, and from the centre, mark off $\frac{4}{5}$ of one radius and $\frac{1}{3}$ of another. The distance between the two points is approximately the side length of an inscribed regular heptagon. Of course, one can resort to continued fraction convergents of larger denominators, or quadratic approximations (for instance, $\frac{5\sqrt{2}-1}{7}$ is even better).



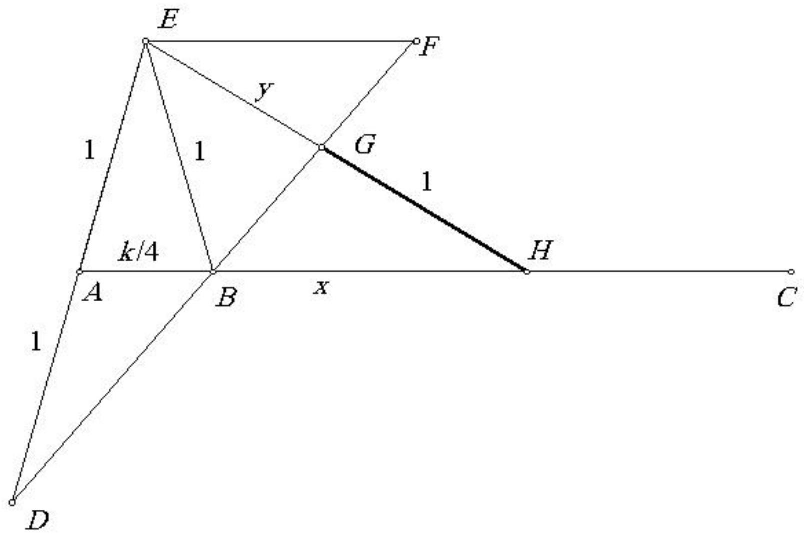
A nonagon (enneagon) can be constructed using a similar method to the heptagon, using $\frac{13}{19}$ as an approximation (the error is 0.05%). Note that one can easily prove that a nonagon cannot be contructed exactly using the ruler and compass, using a similar method to the heptagon case; this means an angle of $120°$ cannot be trisected, settling the problem of angle trisection.

To construct a heptagon exactly, we can use "Neusis" construction. The word approximately means "verging", and this is its additional feature: the ruler has a unit length marked on it. Now, the ruler may pass through a point, such that each end of the unit length falls on a constructed line or circle, and then a line may be drawn along the ruler. It can be seen that in general, there are 4 ways to verge from a point to two intersecting lines, hinting that degree-4 equation is to be solved. Indeed this is the case, and it turns out we can solve all cubics and quartics.[9] Below we give the Neusis constructions for 1) trisecting an angle, 2) finding cube roots, and 3) drawing the regular heptagon. Note that now we can also draw the nonagon. The full power of Neusis construction is currently not known, despite its origin tracing back to at least 400 BC.

[9] To solve a quartic, we only need the cubic formula, which is too large for this footnote; so whatever constructs cubic roots can also solve quartics.

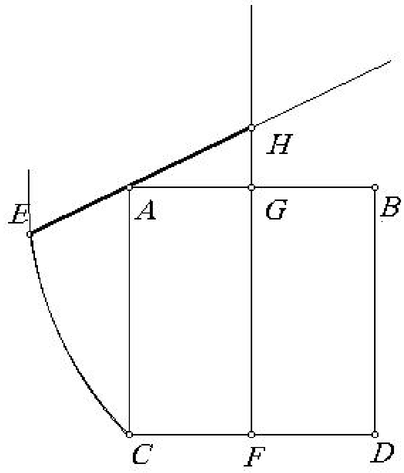


1) This method is due to Pappus. To trisect angle AOB, where $AO = \frac{1}{2}$, construct AD parallel to OB, AC perpendicular to OB. Now use the marked ruler to find E and F, where $EF = 1$. Then FOB is the required angle. To prove this, construct (in your mind and not on paper, if you wish) the midpoint of ED, and join that up with A; now do an angle chase.

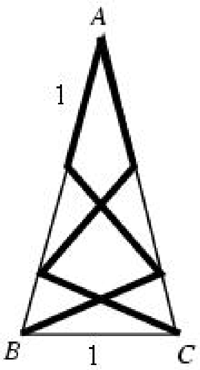


2) This is due to Nicomedes. To find $\sqrt[3]{k}$, let $AB = \frac{k}{4}$, and construct C, D, E, F accordingly (EF is parallel to AB). Now verge from E such that $GH = 1$; $BH = x = \sqrt[3]{k}$. To prove this, note that by Pythagoras, $(1+y)^2 = (1^2 - (\frac{k}{8})^2) + (x + \frac{k}{8})^2$. As GFE and GBH are similar, $y = \frac{k}{2x}$. Now solve.

3) $ABCD$ is a unit square. EC is an arc with centre B. We verge from A, so EH is 1 unit. Then angle AHB is an internal angle of a regular heptagon. To see this, let $AH = x$, and the angle be t. Then the cosine rule tells us that $2x^2(1 - \cos t) = 1, x^2 - 2x\cos t = 1$. Eliminating x yields the minimal polynomial for $\cos\frac{5\pi}{7}$, which can be readily recognised from the minimal polynomial for $2\cos\frac{2\pi}{7}$ above.



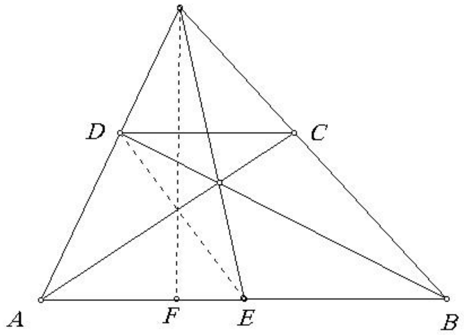
There is also a "match stick" construction for the regular heptagon, and the proof is surprisingly an angle chase. The angle BAC is $\frac{180}{7}^\circ$.



Of course, one may believe that the original rules of the game are so simple that they cannot be made any simpler. However, it is surprising to know that we can neglect either the compass or the straightedge, and given a slightly different set of starting conditions, we can achieve the same constructions!

The Steiner construction only involves the ruler, and a starting unit circle, plus the points $(0,0)$, $(2,0)$ and $(0,2)$. (Steiner has been considered the greatest pure geometer since Apollonius of Perga.) Now it can be checked that this starting set, plus the ruler, gives a field (that is, the constructible points are closed under the four basic operations).

A proof is based on the following diagram, which states that when AB is parallel to CD, E thus constructed is the midpoint of AB. The interested reader may wish to prove this and its converse (assuming E is the midpoint, show that CD is parallel to AB). As the starting set quickly yields 3 equally spaced horizontal lines and 3 equally spaced vertical lines, any line meets a set of them to give a segment with a midpoint, so we can draw parallels. The above process can be repeated so that $AF = \frac{1}{3}AB$. Using these tricks we are able to carry out the four operations and get $\mathbb{Q}$.



So we just need to be able to take the square root to achieve what can be done using the compass and ruler. That's when the circle comes in. We make the observation that

$$\sqrt{z} = \frac{z+1}{2}\sqrt{1-(\frac{z-1}{z+1})^2},$$

now $|\frac{z-1}{z+1}| < 1$, so it can be thought of as (or actually transferred to be) an x-coordinate on the circle; then the expression in the square root is simply the y-coordinate.

Analogously, the amazing Mohr-Mascheroni construction uses only the compass, although a proof here is much more complicated. Mohr was the first to prove it in 1672, and his work was unknown to Mascheroni. One needs to first show that you can reflect a point using just a compass (try it). This allows it to carry distances. We also figure out how to multiply or divide a segment by n. We then show that the intersection of lines or circles joining compass points have a compass constructible intersection. This requires the solutions of Napoleon's problem (divide a circle into 4 equal arcs using just the compass), solved by the emperor's friend Mascheroni (who's better known for the constant named after him and Euler). This implies we can carry distances, so the points form a field. And so all ruler and compass points are compass points, so we get the same field as the former.

On the next issue, we might continue with parts 2 and 3 of the article. Maybe.

— James Wan

True Stories

Richard Feynman was once called in on a consulting job, and was presented with some very complicated blueprints of a nuclear reactor which he could not understand. He stared at them for several minutes, then, to avoid embarrassment, he pointed to something and said, "is that a window?" The engineers in the room got excited and said, "oh, we see what you mean. The ventilation should go here. Then the pressure can dissipate there. If we do that, then the fission…" Soon the problem was solved, and they credited it to Feynman's advice.

Feynman also offers this advice: "if anyone asks me a question, I always say, 'differentiate under the integral sign.' More than half the time this will solve the problem. And, even if it doesn't, they will think you are a really smart guy."

Paul Dirac once made a mistake in a question which he wrote on the blackboard. A student raised his hand and said, "Professor Dirac, I do not understand equation (2)." Dirac continued to write on the board. The student assumed that Dirac had not heard him, and repeated what he just said. There was no reaction. A student in the first row intervened, "Professor Dirac, that man is asking a question." Dirac paused, and replied, "oh, I thought he was making a statement."

Dirac was once in his garden, when postman came with a delivery. The postman asked him, "is professor Dirac in this house?" Dirac replied, "no."

J . J. Sylvester once sent a paper to the London Mathematical Society for publication. He included a cover letter asserting that this was the most important result in the subject for 20 years. The Secretary replied that he agreed with Sylvester's assessment, but that Sylvester had already published the result five years earlier.

Sylvester once gave a speech on the conciseness of mathematical expressions: one can express pages of thought in just a few symbols. Thus, his comments would be painfully brief. He finished three hours later.

Useful Theorems and Methods for First-years

This article aims to give a very brief introduction to some theorems and methods which I have found somewhat useful in first-year mathematics. They are: Pappus' theorem for surface areas and volumes, determinant formula for the cross product, Descartes' rule of sign, and synthetic division for linear factors.

There are generalisations for some of the methods. To find out more, such as proofs of these theorems, I suggest you do your own research. For example, the proof of Pappus' theorem is straightforward, provided you know how to locate a centroid. The proof of Descartes' rule is more involved.

Pappus' theorem for surface areas/volumes of surfaces/solids of revolution

These are two very simple and somewhat intuitive equations.

Pappus' theorem for volumes states that the volume V of a solid of revolution, generated by rotating a plane region R about an axis, is give by the distance traveled by the centroid of R times the area of R (of course, the axis is coplanar with R and does not lie in its interior). So, if a is the distance from the axis to the centroid, and A is the area of R, then

$$V = 2\pi a A.$$

For example, the volume of a torus formed by rotating a circle of radius r around an axis a away from its centre is $2\pi a \times \pi r^2 = 2\pi^2 a r^2$.

Alternatively, the last result can be derived by directly applying the formula for the volume of a solid of revolution, but this requires many more steps.

Pappus' theorem for surface areas is an analogous equation; it states the surface area of a surface of revolution (S) is L, the arc length of R, times the distance traveled by the centroid:

$$S = 2\pi a L.$$

For example, the torus has a surface area of $2\pi r \times 2\pi a = 4\pi^2 a r$.

3×3 determinant formula for the cross product

There is a bonus for remembering how to calculate the determinant of a 3×3 matrix. The cross product of 2 vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$

is given by:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Descartes' rule of sign

A very simple rule which tells you how many real roots a polynomial may have.

The rule is as follows: let $P(x) = a_n x^n + \cdots + a_1 x + a_o$ where a_i are real, then the number of positive roots of $P(x)$ is equal to the number of times the signs of 2 consecutive coefficients change; or less than that by a multiple of 2. Note that roots with multiplicity k are counted as k roots.

For example, $P(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5$ may have 4, 2, or 0 positive roots, because the coefficients change sign 4 times (between x^5 and $-x^4$, $-x^4$ and $3x^3$, $9x^2$ and $-x$, $-x$ and 5).

Also, the graph of $P(-x)$ is that of $P(x)$ reflected about the vertical axis, so applying Descartes' rule to it gives the maximum number of negative roots. So we see that $P(-x) = -x^5 - x^4 - 3x^3 + 9x^2 + x + 5$ has exactly 1 negative root (because it can't have a negative number of roots). By the same argument, any polynomial with only 1 sign change has exactly 1 positive root.

Another example. If $P(x) = x^3 + 9x + 5$, then it has no positive root by the rule. By considering $P(-x)$, we see that it has 1 negative root. As $P(x)$ is a cubic, it must have 2 more roots, both being complex. So we discovered the nature of the roots without any computations.

Synthetic division for linear factors

Suppose you want to divide a linear factor into a polynomial (for instance, you may need to do this to simplify an integrand). This can be done using long division. However, a faster algorithm known as synthetic division might save you a bit of time.

The algorithm is best shown with an example. Note the coefficient of x in the linear factor must be 1 before applying the algorithm; if not, we can write $ax + b$ as $a(x + \frac{b}{a})$.

Divide $x - 2$ into $x^3 - 5x^2 + 3x - 7$.

We first write out the coefficients of the dividend in a row, and put the underlined root of the linear factor after them.

$$\begin{array}{ccccc} 1 & -5 & 3 & -7 & \underline{2} \end{array}$$

Drop the first number down to the 3rd row:

$$\begin{array}{ccccc} 1 & -5 & 3 & -7 & \underline{2} \\ & & & & \\ 1 & & & & \end{array}$$

Multiply the rightmost number in the 3rd row by the underlined number, and put the result in the 2nd row of the next column:

$$\begin{array}{ccccc} 1 & -5 & 3 & -7 & \underline{2} \\ & 2 & & & \\ 1 & & & & \end{array}$$

Add up the 2 numbers in that column, and put the result in the 3rd row of that column:

$$\begin{array}{ccccc} 1 & -5 & 3 & -7 & \underline{2} \\ & 2 & & & \\ 1 & -3 & & & \end{array}$$

$$\begin{array}{ccccc} 1 & -5 & 3 & -7 & \underline{2} \\ & 2 & -6 & -6 & \\ 1 & -3 & -3 & -13 & \end{array}$$

The coefficients of the quotient and remainder are shown in the 3rd row. So

$$\frac{x^3 - 5x^2 + 3x - 7}{x - 2} = x^2 - 3x - 3 - \frac{13}{x - 2}.$$

— Wilson Ong

Puzzle 1 solution: an equation raised to a power is a consequence of the original equation, but is not necessarily equivalent to it, as it may produce more solutions.
Puzzle 2 solution: you guess 1 year above and 1 year below your friend's guess.

Solutions to Problems from Last Edition

We had a large number of correct solutions to the problems from last issue. Collectively, all problems were solved. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

The occupants of room 610, physics building may collect \$5 for solving problems 1 and 3.

Hui-Shyang Lee may collect \$17 for solving problems 2, 3, 4 and 6.

Wonki Noh may collect \$8 for solving problems 3 and 4.

Gus Schrader may collect \$9 for solving problems 3 and 6.

Geoffrey Gebert may collect \$3 for solving problem 3.

Tharatorn Supasiti may collect \$3 for solving problem 3.

Zhenda Yin may collect \$3 for solving problem 3.

Duc Truong may collect \$5 for solving problem 4.

Alan Chang may collect \$5 for solving problem 4.

Kate Mulcahy may collect \$5 for solving problem 5.

Sarah Traine may collect \$6 for solving problem 6.

Michael Couch may collect \$6 for solving problem 6.

1. You have a compass whose legs are set at a fixed distance apart. How can you draw 2 circles of different radii on paper?

Solution: tear out a bit of paper, fold it a few times so its thickness is non-negligible, and its other dimensions sufficiently small. Now put it on the remaining paper, put the sharp end of the compass on this elevated creation, and draw a circle on the remaining paper. This circle will have a different radius.

We received a very creative (sometimes even unphysical) set of solutions from room 610, physics building. It includes: modify the compass or introduce another compass; stretch the paper uniformly in all directions, draw a circle, then unstretch the paper; make a point on the paper for a 0-radius circle; draw a straight line using 1 leg of the compass, creating a circle with infinite radius; any drawn circle has finite width so there are infinitely many different circles between the inner and outer edges.

We accept their solution of piercing a hole in the paper with the compass and lowering 1 leg into the hole.

2. Can you tile a 18×18 board using only T-shaped pieces each made of 4 squares?

Solution: No. To see this, we alternatively colour the squares of the board black and white. Then each T-shape would cover either 1 black and 3 white squares (type A), or 1 white and 3 black squares (type B). Suppose that a tiling exists, then we have x tiles of type A and y tiles of type B. Clearly, $x + y = \frac{18^2}{4} = 81$, and the total number of blacks squares is $x + 3y = \frac{18^2}{2} = 162$. The two equations yield $2y = 81$, so y is not an integer, contradiction.

3. Prove that for reals $x + y + z = 1$, $xy + yz + zx \leq \frac{1}{3}$.

Solution: (from Hui-Shyang Lee)

$$
\begin{aligned}
1 = (x+y+z)^2 &= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \\
1 &= 3(xy + yz + zx) + \frac{1}{2}(x-y)^2 + \frac{1}{2}(y-z)^2 + \frac{1}{2}(z-x)^2 \\
1 &\geq 3(xy + yz + zx) \\
\frac{1}{3} &\geq xy + yz + zx
\end{aligned}
$$

There were many solutions for this question, with techniques ranging from the arithmetic mean-geometric mean inequality to finding the maximum using calculus.

4. A bug crawls along the edges of a cube; at each vertex it has probability of $\frac{1}{3}$ of going to any adjacent vertex. When it reaches the vertex opposite its starting one, enlightenment is achieved. What is the average number of edges it must crawl on to do this?

Solution 1: we use conditional expectation. Start from 1 vertex, we must take at least 2 steps, in which there is $\frac{1}{3}$ chance of returning to the start, and $\frac{2}{3}$ chance of getting to 1 edge away from enlightenment. Let E be the expected value, then we have $E = 2 + \frac{E}{3} + \frac{2}{3}F$, where F is the expected value of reaching enlightenment from 1 edge away. We need to take at least 1 step to do so, but $\frac{2}{3}$ of the time we get back to 1 step from the start. Hence $F = 1 + \frac{2}{3}(E-1)$. Solving for E, we get 10 as our answer.

Solution 2: (from Alan Chang) call the vertices 1 edge away from the start B. Then it is clear that we can get to B with probability 1 in the first step. From there, there is $\frac{2}{9}$ chance of getting to enlightenment in 2 steps, and $\frac{7}{9}$ chance of returning to B in 2 steps.

Hence, the average = $\frac{2}{9} \cdot 3 + \frac{7}{9} \cdot \frac{2}{9} \cdot 5 + \cdots = \frac{2}{9} \sum_{n=0}^{\infty} (\frac{7}{9})^n (2n+3) = 10$.

5. Any right triangle contains an isosceles triangles whose areas is at least α times the area of the original triangle. Find the maximum value for α.

Solution: after fiddling with the geometry of the situation, we see that there are two types of contained isosceles triangles that are larger than the other types:

Type A: one leg lies on the hypotenuse of the right triangle, the other leg coincides with the latter's second longest side.

Type B: the base coincides with the hypotenuse, and the opposite vertex lies on the second longest side of the right triangle.

As we are only interested in the ratio of areas, let the second longest side of the right triangle be x and the shortest side be 1. We see that type A has area $\frac{x^2}{2\sqrt{x^2+1}}$, and type B has area $\frac{x^2+1}{4x}$. They intersect at a point which gives the minimum area, $x = \frac{1+2^{\frac{1}{3}}}{\sqrt{3}}$. It can be checked that a right triangle with this given proportion cannot contain a larger isosceles triangle. Hence the best value of α, using this value of x, is $2^{-\frac{1}{3}}$.

6. Find $\int_0^{\pi} \frac{x}{1+\cos^2 x} \mathrm{d}x$

Solution: let the integral be I. We make the sneaky substitution, $x = \pi - t$, then $I = \int_0^{\pi} \frac{\pi - t}{1+\cos^2(\pi - t)} \mathrm{d}t = \int_0^{\pi} \frac{\pi}{1+\cos^2 x} \mathrm{d}x - I$, so

$$I = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1+\cos^2 x} \mathrm{d}x = \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos^2 x} \mathrm{d}x.$$

The last integrand is equal to $\frac{\sec^2 x}{\tan^2 x + 2}$. Substituting $t = \tan x$ gives $I = \pi \int_0^{\infty} \frac{1}{t^2+2} \mathrm{d}t = \frac{\pi^2}{2\sqrt{2}}$.

Paradox **Problems**

Below are some puzzles and problems for which cash prizes are awarded. Anyone who submits a clear and elegant solution may claim the indicated amount (unless two solutions are the same, in which case only the first submission will be rewarded). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

1. ($2) A die is thrown until a 6 is obtained. What is the probability that a 5 is not obtained before that? (Do this without a series.)
2. ($3) A $3 \times 3 \times 3$ cube can be cut up into 27 unit cubes. What is the minimum number of straight cuts required to do this, if you are allowed to move pieces around in between cuts?
3. ($3) Show that the medians of a triangle can also form a triangle, with area $\frac{3}{4}$ that of the original one.
4. ($4) Find the volume enclosed by the graphs of $|x| + |y| = 1, |y| + |z| = 1, |z| + |x| = 1$.
5. ($5) In an equilateral triangle ABC, point Q is on BC, and AQ meets the circumcircle of the triangle at P. Prove that $1/PB + 1/PC = 1/PQ$.
6. ($5) Starting with one amoeba, every second it splits into either 0, 1, 2, or 3 amoebae with equal probability. What is the probability that the population eventually dies out? What if it can only split into 0, 1 or 2 amoebae with equal probability?
7. ($6) It is well known that no four distinct integer squares can be in arithmetic progression; it is obvious that three can. Find a way to generate all of them.
8. ($5) Find 3 ways to write 3 as the sum of 3 3rd powers of integers.
9. ($6) In our article on constructions, an explicit formula for $\cos 3^\circ$ was given. Find an explicit formula for $\cos 1^\circ$; the simplest submission wins.

Paradox would like to thank Alisa Sedghifar, Stephen Muirhead, Wilson Ong and Kate Mulcahy for their contributions to this issue.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