Paradox

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In this edition of Paradox

Regulars

Feature

Articles

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COVER:	Top left: Julian Assange in the old MUMS room; top right: the Wikileaks logo; bottom: Julian Assange

Words from the Editor

Unless you spent the winter holidays lost in the outback,[1] you've undoubtedly heard of Wikileaks. You've also probably seen photos of Julian Assange, the founder and public face of the website, which appeared on the front cover of newspapers worldwide following the recent leak of the 'Afghan War Logs'. You might have even heard that Julian Assange originally comes from Australia. Or that he developed a good portion of Wikileaks while shut up in a house in Melbourne during 2006. Or even that he attended Melbourne Uni between 2003 and 2005. Or even that he was once Vice President of…MUMS.

It probably comes as a surprise to most readers of Paradox that Julian Assange, at the centre of so much media attention over the last few months, is a former MUMS Vice President. Yet it was not so long ago that Julian was striding the hallways of the Richard Berry building, attending classes in Russell Love, or hanging out in the (old) MUMS room. The lack of awareness of Julian's link with MUMS is possibly due to the general aura of secrecy that shrouds Julian's life. Or perhaps it is the fact that Julian only spent a few short years at Melbourne University, and in the end left without ever graduating. Whichever the case, Paradox seeks to set the record straight and provide a full account of Julian's time in MUMS. For those learning of this connection for the first time, and even for those who are not, Paradox hopes you enjoy the article.

Elsewhere in Paradox expect to see articles and features with a more mathematical, and less political, bent. We have more original Paradox comics, some mathematical poetry, articles about Paul the Octopus and everyone's favourite competitive sport Scissors-Paper-Rock, and much more in between.

Finally, Paradox would once again like to extend a warm invitation to readers to submit any items of (mathematical) interest that they stumble across. This can be anything from an amusing quote by a maths lecturer, to an mX article that appeals to mathematics and gets it horribly wrong; all is appreciated, no matter how large or small. Please drop all contributions into the Paradox drop-box, just inside the door to the MUMS room, or alternatively send Paradox an email.

— Stephen Muirhead

[1] As well as the start of the semester playing StarCraft II.

Words from the President

Well, I've survived almost three months as President without being usurped. Nonetheless, we still seem to be in election mode, engaging in such vote-grabbing behaviour as holding trivia nights and games nights, and dishing out ridiculous quantities of food after our seminars. It is also rumoured that cakes have recently been eaten in the MUMS room.

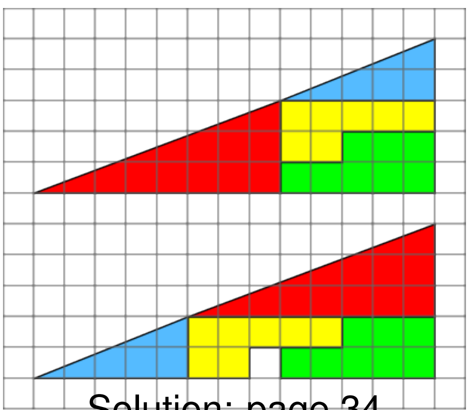
PuzzleHunt, too, was a momentous success, with 821 participants from 242 teams. Congratulations to Killer Chicken Bones, who found SCRT MSG and saved Kevin Rudd from Tony Abbott.[2]

This semester, our weekly seminars will take place at 1pm on Fridays in the Hercus Theatre,[3] followed by refreshments in the MUMS room. Also coming up is the University Maths Olympics, a fun and fast-paced event, culminating in the winners being immortalised indefinitely on our website.

Do check out our website, as well as our Facebook group, and drop by the MUMS room some time for a chat and some games. Finally, I'd like to take this opportunity to draw attention to our thriving tutor lists,[4] which we consider to be an enormous service both to keen students and to battling postgraduates.

— Sam Chow

Puzzle:
How can the missing square be accounted for?



Solution: page 34

[2]But not from Julia Gillard.

[3]In the Physics building.

[4]We have one for Uni and one for VCE. Please see our website for these lists, or email us if you'd like us to add your details.

Mathematical Miscellany

Theorem: A cat has nine tails.
Proof: No cat has eight tails. Since one cat has one more tail than no cat, it must have nine tails.

∞

Q: Why did the chicken cross the Möbius strip?
A: To get to the same side.

∞

Aleph-null bottles of beer on the wall,
Aleph-null bottles of beer,
You take one down, and pass it around,
Aleph-null bottles of beer on the wall.

Lecturer Quotes

'Number theory is pretty self-explanatory: it's the theory of numbers. But not a half. And certainly not root two.'

— Assoc. Prof. John Groves

'If you're interested in topology, there are only two-infinity plus one surfaces. That's amazing! I would have thought there were zillions, but there are only two-infinity plus one!'

— Prof. Arun Ram

'To study all topological spaces is to invite a horror upon yourself from which you can never escape.'

— Craig Westerland

'One of the prerequisites I was not allowed to put in the course description was primary school arithmetic.'

— Assoc. Prof. John Groves,
on why there are no prerequisites for Number Theory.

Maths in the News: The Perfect Handshake

News sources around the world[5] have reported that researchers at the University of Manchester have derived a formula for the perfect handshake, as part of a study for the car company Chevrolet. The formula is:

$$PH = \sqrt{A + B + C},$$

for

$$A = (e^2 + ve^2)(d^2) + (cg + dr)^2$$
$$B = \pi[(4 < s > 2)(4 < p > 2)]^2$$
$$C = (vi + t + te)^2 + [(4 < c > 2)(4 < du > 2)]^2,$$

where:

1. (e) is eye contact (1=none; 5=direct);
2. (ve) is verbal greeting (1=totally inappropriate; 5=totally appropriate);
3. (d) is Duchenne smile – smiling in eyes and mouth, plus symmetry on both sides of face, and slower offset (1=totally non-Duchenne smile (false smile); 5=totally Duchenne);
4. (cg) completeness of grip (1=very incomplete; 5=full);
5. (dr) is dryness of hand (1=damp; 5=dry);
6. (s) is strength (1= weak; 5=strong);
7. (p) is position of hand (1=back towards own body; 5=other person's bodily zone);
8. (vi) is vigour (1=too low/too high; 5=mid);
9. (t) is temperature of hands (1=too cold/too hot; 5=mid);
10. (te) is texture of hands (5=mid; 1=too rough/too smooth);
11. (c) is control (1=low; 5=high); and
12. (du) is duration (1= brief; 5=long).

Confused? So are we... Unfortunately none of the news sources bothered to explain how to apply this formula.

Or perhaps ignorance is bliss.

[5] To name just a few: *The Australian*, 17 July 2010; *The Guardian*, 17 July 2010; *Agence France Presse*, 16 July 2010.

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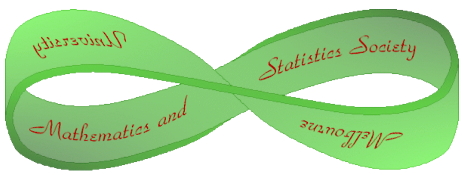
— π.O. [6]

[6] π.O. is a Melbourne poet whose work evokes mathematical themes. His books include *The Number Poems* and *Big Numbers*, available from Readings bookshop.

Paradox Logo: Addendum

The last edition of Paradox[1] featured a short article on the MUMS logo, taking note of the various places the logo turns up in the general course of life. As a reminded, here is the logo:



This edition poses the following challenge as an addendum to the article for last edition:

> What is a good way to draw the MUMS logo mathematically? In other words, what is a surface that closely approximates the MUMS logo?

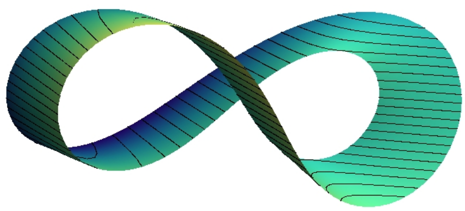
To kick off the competition, Paradox proposes the following surface:

$$(\cos(t)\frac{1 - s\cos(\frac{t}{2})}{1+\sin^2(t)}, \sin(t)\cos(t)\frac{1 - s\cos(\frac{t}{2})}{1+\sin^2(t)}, \sin(\frac{t}{2})(s - \frac{\sin(t)}{4}))$$

$$s \in [-\frac{1}{4}, \frac{1}{4}]$$

$$t \in [0, 2\pi]$$

Which, when plotted with Mathematica, produces:

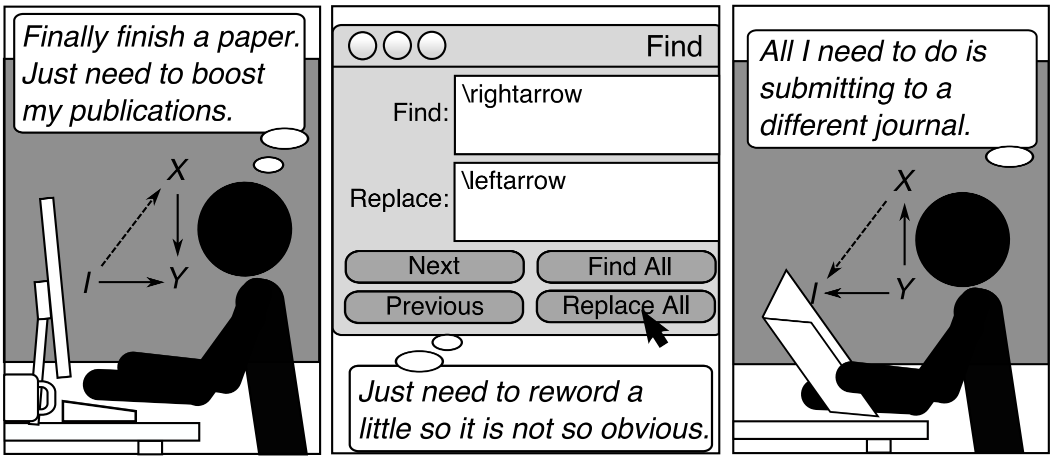


If you think you can do better, send in your surface to Paradox. Entrants will be acknowledged in the next edition.

[1] Issue 1, 2010. Visit the Paradox website for an archived copy.

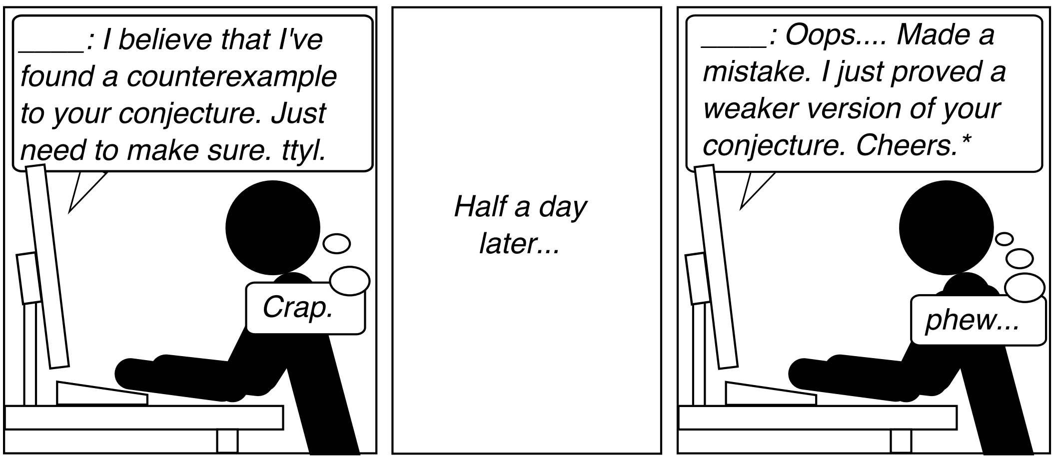
Paradox **Comics**

COPAPER - a way to double your number of publications

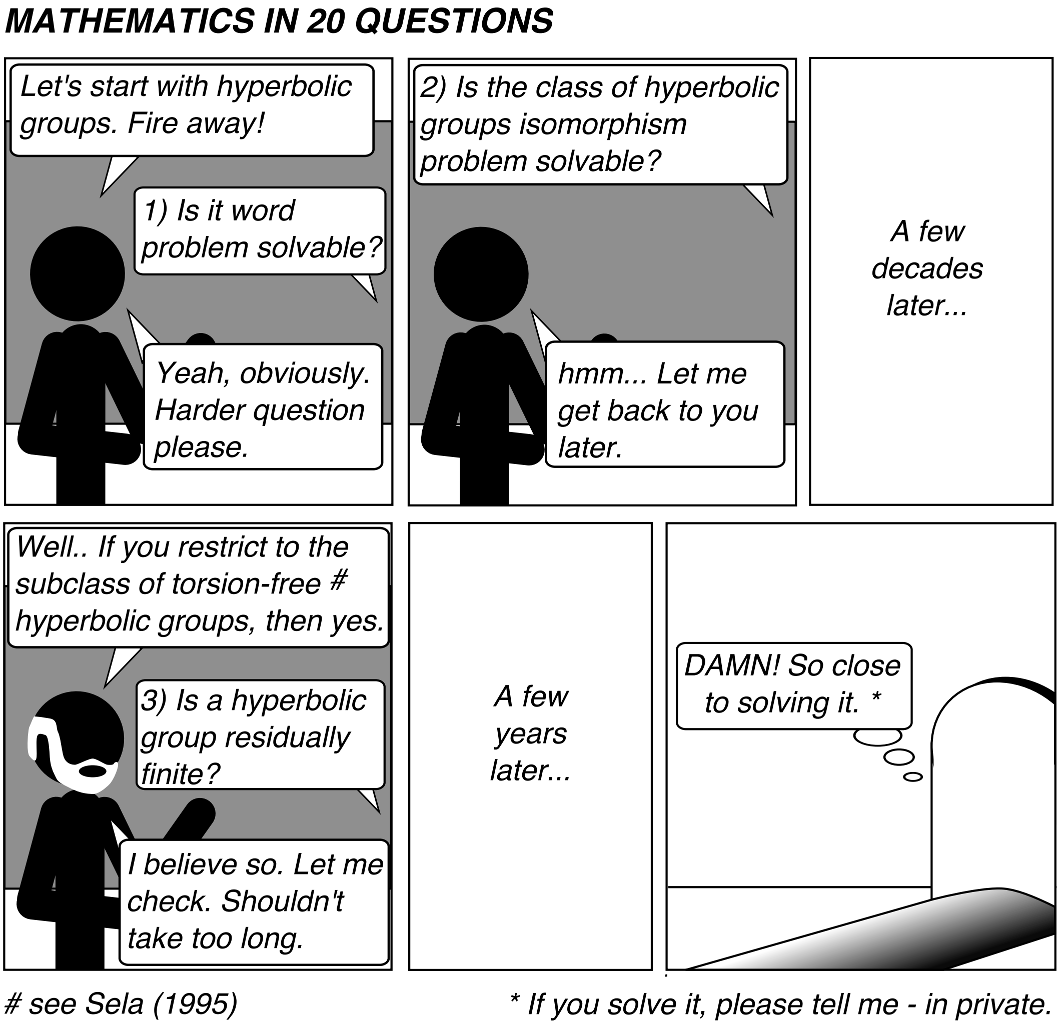


Gotta love category theory.

OOPs.....



** Actually, someone else already found a counterexample to the stronger version.*



— Tharatorn Supasiti

Magic Square

A magic square that is 'magic' under both $+$ and $\times$ (including diagonals):

46	81	117	102	15	76	200	203
19	60	232	175	54	69	153	78
216	161	17	52	171	90	58	75
135	114	50	87	184	189	13	68
150	261	45	38	91	136	92	27
119	104	108	23	174	225	57	30
116	25	133	120	51	26	162	207
39	34	138	243	100	29	105	152

Rock Paper Scissors: Luck or Skill?

> In order to appreciate the subtle and sometimes profound tactics associated with Rock Paper Scissors, one must first understand the fact that strategy does indeed play a role in the game. Acceptance of this fact differentiates the unranked novice from the player of a higher order.[1]
>
> — Douglas and Graham Walker,
> Leaders of the World RPS Society

In the game of Rock Paper Scissors (RPS), two players simultaneously choose one of *rock, paper* or *scissors* to play, with rock beating scissors beating paper beating rock. If both players choose the same weapon, then we have a *stalemate* and they try again. A game of pure chance, or so it would seem…

We'll begin by studying the game from a theoretical perspective, assuming that players can play 'randomly' with chosen probabilities. Let player 1 play rock, paper and scissors with respective probabilities p_1, p_2, p_3, and let player 2 play rock, paper and scissors with respective probabilities q_1, q_2, q_3. We have:

$$p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = 1.$$

So we have a pair of strategies $(p_1R + p_2P + p_3S, q_1R + q_2P + q_3S)$. We define payoffs: 1 for a win, 0 for a stalemate, -1 for a loss. Moreover, we assume that each player wants to maximise their expected payoff. Such a set of strategies is a *Nash equilibrium* (NE) if no player can – given other players' current strategies – increase their expected payoff.[2] In simple terms, a Nash equilibrium is a set of strategies – one for each player – so that nobody has an incentive to deviate. So let's find all Nash equilibria for (two-player) RPS.

Suppose we're in NE. First of all, the sum of expected payoffs is 0:

$$E(\pi_1) + E(\pi_2) = E(\pi_1 + \pi_2) = E(0) = 0.$$

[1] Douglas Walker and Graham Walker, *The Official Rock Paper Scissors Strategy Guide* (2004). The article draws on this book throughout.

[2] Here we have a two-player game, but the definition holds for any number of players. Often we speak of a *Nash equilibrium in pure strategies* when everyone always makes a particular choice, and a *Nash equilibrium in mixed strategies* when at least one player makes different choices with probabilities greater than 0. We don't need to make such a distinction here.

Lemma: $E(\pi_1) = E(\pi_2) = 0$.
Proof: Suppose $E(\pi_1) < 0$. Then player 1 can improve his/her payoff by 'copying' player 2's strategy (ie. set $p_1 = q_1, p_2 = q_2, p_3 = q_3$), thereby achieving an expected payoff of 0. But P1 cannot have an incentive to deviate, since we're in NE. Hence $E(\pi_1) \geq 0$. Similarly, $E(\pi_2) \geq 0$. As $E(\pi_1)+E(\pi_2) = 0$, the result follows.

Now consider the expected payoffs of P1's pure strategies:

$$E_1(R) = q_3 - q_2, E_1(P) = q_1 - q_3, E_1(S) = q_2 - q_1.$$

None of these can be greater than 0, since we're in NE and $E(\pi_1) = 0$ (otherwise P1 would have an incentive to deviate). Thus,

$$0 \geq q_3 - q_2, 0 \geq q_1 - q_3, 0 \geq q_2 - q_1.$$

So $q_1 \geq q_2 \geq q_3 \geq q_1$, which means $q_1 = q_2 = q_3$. Similarly, $p_1 = p_2 = p_3$.

It's easy to check that this is indeed NE. Both players currently have an expected payoff of 0. If P1 were to deviate, the payoff would be

$$p_1E_1(R) + p_2E_1(P) + p_3E_1(S) = 0,$$

since $q_1 = q_2 = q_3$. Similarly, P2 would not improve his/her payoff by deviating from the NE.

Hence there is a unique NE: $(\frac{1}{3}R + \frac{1}{3}P + \frac{1}{3}S, \frac{1}{3}R + \frac{1}{3}P + \frac{1}{3}S)$. In other words, both players use each weapon with equal probability.

Why then are RPS enthusiasts so steadfast in their belief that there is skill in the game?

Well, one explanation is that **people can't choose randomly**. That is, if you faced the World RPS Champion and tried to play the strategy $\frac{1}{3}R + \frac{1}{3}P + \frac{1}{3}S$, you might give away some physical 'tell' regarding your next throw. For instance, players may exhibit tension around jaw muscles prior to throwing a rock, or tend to shift stance when they change throws.

In fact, a common strategy against experienced players is to try to broadcast false tells. The logical counter is to try to ignore your opponent. If both players ignore each other, we get the theoretical game described earlier. But what if you can't completely ignore your opponent? How would it affect your opponent psychologically if you were to wear a belt that said, 'ROCK'?

Moreover, while it's hard enough to throw randomly for one throw, try throwing randomly for a series of throws! Consider taking three such throws. Perhaps, in an attempt to throw randomly, you'll tend to throw one of each. Perhaps, since it requires less effort, you'll throw more rocks. Our brains are just no good at generating randomness! Consider for instance the following exercise[3]. You have 100 two-digit numbers. By sampling 10 randomly, estimate the mean of the 100 numbers. In fact people's estimates are on average 30-40% higher than the actual mean: people tend to select the larger numbers!

> Aside from the demeaning notion that chance plays a role, RPS played by humans does not conform to the law of averages. Probability would dictate that over the long run roughly one third of all throws ever made would be Rock, one third Paper, and one third Scissors. After careful observation of numerous championships, in which literally thousands of throws were made, the World RPS Society knows this is not the case.
>
> — Douglas and Graham Walker

What, then, if we could generate randomness, or at least the type of randomness generated by a computer (pseudorandomness)? In 2003, a player, guided by the computer program Deep Mauve, was allowed to compete at the World Championships. Deep Mauve would provide a pseudorandom series of throws to the player, who would play them out. It could have been bad luck, or it could have been tells, but the player performed dismally, failing to progress past the qualifying round.

So there we have it: even if we could generate 'random' series of throws, we would still need to ensure that we didn't give away any tells. Frankly, I'd rather flip a coin.

— Sam Chow

> To me, RPS is both a game and sport, as well as a way of life.
>
> — Master Roshambollah

[3]This was actually used in the first tutorial for 620-168 Experimental Design and Data Analysis.

Anyone for Octopus?

If you were even remotely interested in the FIFA[1] World Cup, then you would be aware of the feats of Paul the Psychic Octopus. Exhibited at a Sea Life Centre in Oberhausen, Germany, Paul rose to fame by correctly predicting the outcome of eight World Cup matches, including each of Germany's seven matches in the tournament as well as the World Cup Final:[2]

Round	Team 1	Team 2	Paul's Choice	Result
Group	Germany	Australia	Germany	Germany (4:0)
Group	Germany	Serbia	Serbia	Serbia (1:0)
Group	Ghana	Germany	Germany	Germany (1:0)
2nd Round	Germany	England	Germany	Germany (4:1)
Quarterfinal	Argentina	Germany	Germany	Germany (4:0)
Semifinal	Germany	Spain	Spain	Spain (1:0)
3rd Place	Uruguay	Germany	Germany	Germany (3:2)
Final	Netherlands	Spain	Spain	Spain (1:0 AET)

[1] Read 'soccer' if you don't know what this stands for.
[2] AET = After extra time.

In case you were wondering, Paul's 'psychic' label does not refer to the method by which Paul communicated his prediction to the human population. Instead, the prediction was facilitated by placing two clear plastic containers bearing the flags of the respective counties playing in each match into his tank. To give Paul an incentive to pick one or the other, a mussel was placed in both boxes and the prediction was deemed to be the one Paul ate first. Critically, there seemed to be no provision for Paul to be able predict a draw, despite – as football's detractors no doubt will point out – their relative abundance in the sport.

Paul's run of successful predictions quickly attracted media attention, and his predictions both delighted and angered his supporters. After Paul forecast that Germany would lose to Spain in the semifinals, several Germans expressed a desire to eat him, prompting the Spanish Government to offer him asylum. Paul was retired from his career of prediction after the World Cup.

Many of you will no doubt be sceptical that an octopus is capable of making predictions about the outcome of soccer matches, let alone predictions influenced by psychical means. However, the sceptic is obliged to offer an alternative explanation for Paul's success.

Luck

The first explanation that we might put forward is that Paul was simply lucky, or in a more mathematically correct manner of speaking, that his choice was random. This means we assume that Paul's 'prediction' on each occasion consisted of a selection between two outcomes that were equally likely: Team 1 wins or Team 2 wins (recalling that the setup did not cater for a draw). It is a very simple calculation to show that the probability of Paul making eight successful predictions was $\left(\frac{1}{2}\right)^8 = \frac{1}{256} \approx 0.0039$, which most people would agree is very unlikely. What we have just calculated is called a *p-value* for the test of Paul's clairvoyance, and is the probability that the the results would be replicated under the Null Hypothesis (in this case, that Paul was picking the boxes at random). Statisticians usually reject the Null Hypothesis whenever the *p-value* is less than 0.05.[3] We can therefore confidently reject the hypothesis that Paul was simply lucky.

[3]Sometimes 0.01 is preferred instead; it just depends on how confident you want to be that the Null Hypothesis should be rejected.

Bias

A second explanation is that Paul's 'predictions' were actual non-random selections that were affected by bias. There is discussion of several sources of bias on Paul's Wikipedia article, ranging from intricate and improbable, to not very determinative. The suggestion that Paul has a bias for horizontally striped flags is particularly useless as all of the flags available for Paul to pick except Australia and England had horizontal stripes that run the entire width of the flag, and all except Uruguay have more or less the same horizontal thirds background scheme.

While there is evidence that Paul's species, *Octopus vulgaris*, is colourblind,[4] it is still possible that Paul can distinguish flags from one another in grayscale by either their shape or intensity. This leaves room for the observation that, since Paul picked Germany to win most of its matches, he may have had a bias towards the German flag (which would make sense given that he lives in Germany, so is probably most exposed to the German flag. Though you might have thought his English heritage – Paul was born in England – would have had an influence here). Let us make a fanciful assumption that he is twice as likely to pick the German flag than any other flag. The probability that the actual predictions were made is therefore $\left(\frac{2}{3}\right)^5 \times \left(\frac{1}{3}\right)^2 \times \frac{1}{2} \approx 0.0073$ which is still very low. This approach will never show that Paul's prediction were not statistically significant, as no matter how high the bias towards Germany, the fact that Paul picked against Germany (twice in fact) means that the *p-value* does not rise much above this value. Indeed the formula is:

$$
\begin{aligned}
p(x) &= \frac{1}{2}x^5(1-x)^2 \text{ where } x \text{ is the probability of picking Germany.} \\
p'(x) &= \frac{1}{2}\left(5x^4(1-x)^2 - 2x^5(1-x)\right) \\
&= \frac{1}{2}x^4(1-x)(5-7x)
\end{aligned}
$$

So $x = 1$ and $\dfrac{5}{7}$ are the critical points.

Of course $p(1) = p(0) = 0$, whereas $p(\frac{5}{7}) = \frac{50000}{1647086} \approx 0.0076$, so we can be comfortable that 0.0076 is the maximum. Since the *p-value* in no case rises above 0.05, we can reject a bias for the German flag as an explanation for Paul's clairvoyance.

[4] http://jeb.biologists.org/cgi/reprint/70/1/49.pdf.

Prediction Rigging

The inadequacy of the explanations addressed so far might lead us to entertain a further explanation: that Paul's keepers rigged the setup. They could have done this, for instance, by 'flavouring' one of the mussels to make it more attractive to Paul. His keepers could then bias Paul towards the favourite for the match, according to betting odds such as the following:[5]

Match	Odds for Team 1 Win	Odds for Draw	Odds for Team 2 Win
Ger/Aus	1.50	3.88	7.16
Ger/Ser	1.61	3.72	5.77
Gha/Ger	7.43	4.25	1.45
Ger/Eng	2.93	3.01	2.59
Arg/Ger	2.29	3.23	3.18
Ger/Spa	2.75	3.29	2.55
Uru/Ger	3.98	3.63	1.87
Net/Spa	3.74	3.33	2.03

While research into octopus mussel preference is beyond the scope of this article, let us pretend that the owners managed to prepare the bait so that Paul's choice reflected the probabilities in accordance with the betting market. There are a couple of caveats to this methodology. First, the data above comes from the starting time of each match, while Paul's prediction was made earlier and the market could have shifted in the interim. Indeed there was evidence of some punters relying on Paul to place their bets, and hence distorting the market in favour of his predictions, though these are unlikely to have been numerous enough to produce a significant effect.[6] Second, all bookmakers factor in a gap between the 'fair' payoff for a bet and the payoff they actually offer – an 'overround' in betting parlance – in an attempt to ensure that they make a profit. What this means is that if we try to naively reverse engineer the probability estimates for each outcome in a match from the odds listed above, the probabilities would sum to greater than one. To account for this we may do the following, if o_1, o_d and o_2 are the returns on 1 currency unit in the table

[5] http://www.betexplorer.com/soccer/international/world-cup-2010-south-africa/?round=6&group=0

[6] http://www.telegraph.co.uk/news/worldnews/asia/kazakhstan/7898730/Kazakh-bookmakers-furious-at-Paul-the-Octopus.html

above and i is the winning team then,

$$P(i) = \frac{\frac{1}{o_i}}{\frac{1}{o_1} + \frac{1}{o_d} + \frac{1}{o_2}}$$

will give the probability estimate of the market. From this we may calculate the probability of a Team 1 or 2 win given that there is no draw by the familiar conditional probability formula:

$$P(i \mid d') = \frac{P(i \cap (i \cup (-i+3)))}{1 - P(d)}$$

Plugging the gambling data into Excel we get:

$$\sum_{j=1}^{8} P_j(i \mid d') \approx 0.0068$$

where i is the winning/predicted team and j is the match number. Again this is an extremely low probability, one that does not show Paul's feats to be any less extraordinary. So far, it's not looking good for the non-believers. Interestingly, since the *p-value* in the 'rigging' scenario is actually lower than the *p-value* in the 'bias' scenario, the sceptic is better served by claiming that Paul has a preference for the German flag than by claiming that the whole setup is rigged!

Attrition Bias

But before we get ahead ourselves and abandon a rational view of the world, there is another possible explanation: attrition bias. Attrition bias is a form of selection bias, and occurs where the apparent size of a sample pool is unwittingly reduced, most commonly by failing to notice unsuccessful experiment samples. An example of attrition bias is newspaper phone polls: the only people that contribute to the polls are those with a strong opinion on the subject; the 'failed' samples – those not calling up to vote – are not even considered.

It should come as no surprise that, at the same time Paul was choosing his mussels, there were may other animals making predictions involving World

Cup matches, few of whom had a perfect record.[7] Thus observations about the general ability of animals to predict soccer matches suffers from severe attrition bias: that the only animals we pay attention to are the ones who are successful, leading to a false confidence in the ability of animals in general to predict such events. So just how many animals[8] predicting soccer matches at random do we need before it becomes more likely than not that at least one will predict the results of eight matches correctly? The situation is a binomial distribution:

$$\begin{aligned}
\frac{1}{2} &= \sum_{i=1}^{n} \binom{n}{i} \left(1 - \left(\frac{1}{2}\right)^8\right)^{n-i} \left(\left(\frac{1}{2}\right)^8\right)^i \\
\frac{1}{2} &= 1 - \binom{n}{0} \left(1 - \left(\frac{1}{2}\right)^8\right)^{n} \left(\left(\frac{1}{2}\right)^8\right)^0 \\
\log\left(\frac{1}{2}\right) &= n \log\left(1 - 2^{-8}\right) \\
n &\approx 178
\end{aligned}$$

So Paul isn't really one in a million, you might say that there's an even chance he's 1 in 178.[9]

We have at last reached a satisfactory explanation: that Paul's success is remarkable only in light of the effects of attrition bias. At the end of the day, Paul's feat, assuming he was indeed selecting at random between the boxes, is much the same as tossing eight consecutive heads; it is an unexpected event, but not necessarily one that would make you conclude that the coin is biased or... psychic.

—Narthana Epa

[7] `http://en.wikipedia.org/wiki/Paul_the_octopus`. Incidentally, Paul himself does not have a perfect record in predicting the outcome of soccer matches: he made two incorrect predictions in the Euro 2008 tournament (although a keeper claims that this was a different octopus). I would include this in the main article if it was not such a buzzkill.

[8] Strictly speaking the predictors need not be animals, but remember, strictly speaking, humans are animals too.

[9] Marida, José was perhaps the first to publish this figure, albeit in Spanish and without calculations so you may rest assured that the author conducted at least some original research: 'El pulpo Paul', *Prensa Libre*, 15 July 2010, 19.

MUMS the Word: Julian Assange, Wikileaks, and the Fight to End Government Secrecy

Julian Assange speaking to Chris Anderson from TED, July 2010
www.ted.com/speakers/julian_assange.html

The Melbourne University Mathematics and Statistics Society (MUMS) has long sheltered mathematics students of diverse backgrounds and interests. Many alumni have had colourful pasts, and equally bright futures. Yet, for sheer notoriety, Julian Assange stands alone. Described as 'one of the most intriguing people in the world',[1] Julian is simultaneously lauded and demonised for his role as founder, and public face, of the anonymous whistle-blowing website Wikileaks. For many, Wikileaks is a beacon of hope in the fight to ensure government and corporate accountability, and Julian – who works full time on Wikileaks, yet does not even draw a salary – worthy of a Nobel Peace Prize. To others – the US Government chief among them[2] – Wikileaks poses a genuine threat to national security, with some sources suggesting that Julian is in real danger of arrest by the US State Department, or worse.[3] To members of MUMS, however, Julian is fondly remembered in a different capacity: as a former Vice President of the Society.

[1] Nikki Barrowclough, 'The Secret Life of Wikileaks Founder Julian Assange', *The Age*, May 22 2010, http://www.theage.com.au/technology/technology-news/the-secret-life-of-wikileaks-founder-julian-assange-20100521-w1um.html.

[2] But also more recently, and to a lesser extent, human rights groups such as Amnesty International: see http://www.guardian.co.uk/world/2010/aug/10/afghanistan-war-logs-wikileaks-human-rights-groups.

[3] http://www.democracynow.org/2010/6/17/wikileaks_whistleblowers.

Even though Wikileaks has leaked thousands of private and classified documents since its inception in 2006 – including the Australian Government's 'Blacklist' of websites, and the 'Climategate' emails[4] – it is only this year that Julian and Wikileaks have shot to world-wide attention. In April, Wikileaks released a video entitled 'Collateral Murder' that showed footage of US soldiers in Iraq killing unarmed civilians,[5] including two Reuters journalists. The footage had previously been classified by the US Government, and Reuters were repeatedly denied access to it despite invoking the US's Freedom of Information laws. When Wikileaks was sent an encrypted version of the footage by an anonymous source,[6] a team of volunteers quickly decrypted, edited and published the footage, unperturbed by massive protest from the US Government.

Suddenly, all major news outlets were carrying features on Wikileaks, and Julian, as the public face of the website, was the focus of much of the attention.[7] Every aspect of Julian's life has since been dissected and analysed: his singular past – as a child he attended 37 different schools, and later on became a prominent computer hacker,[8] and was once prosecuted in Australia for intruding into private computer networks;[9] his secretive lifestyle – he has no fixed address, and often disappears for months on end without trace; and his enigmatic character – he apparently works days on end without sleep, once spent two months in a room in Paris without leaving (other people brought him food), and is so absent-minded as to occasionally forget to bring luggage

[4]Emails from within the University of East Anglia's Climate Research Unit that reveal alleged misconduct in the release of climate change data by the Unit.

[5]Though there has been some suggestion that Wikileaks selectively edited the footage to conceal the presence of weapons, including a rocket-propelled grenade launcher.

[6]A US Army intelligence analyst named Bradley Manning was later arrested and charged with leaking the footage, after being turned in by an acquaintance to whom he'd boasted of the leak.

[7]The spotlight on Julian and Wikileaks recently intensified after its release in late July of the 'War Logs': a collection of 90,000 classified US army documents relating to the war in Afghanistan, which some commentators have labelled the most significant leak since the Pentagon Papers during the Vietnam War.

[8]Though Julian himself has rejected this term; he believes that it nowadays carries connotations of fraudulent conduct that should not be associated to his activities.

[9]Julian pled guilty to twenty-five charges, but escaped with a small fine. The judge at his sentencing found 'no evidence that there was anything other than a sort of intelligent inquisitiveness and the pleasure of being able to – what's the expression – surf through these various computers': Raffi Khachadourian, No Secrets: Julian Assange's mission for total transparency, *The New Yorker Magazine*, June 7 2010, `http://www.newyorker.com/reporting/2010/06/07/100607fa\_fact\_khatchadourian`, page 7. Among the many excellent articles written about Julian Assange, the piece in the *New Yorker* stands out as providing the most enlightening look at Julian's life and work.

when he travels.[10] Yet one aspect of Julian's past has thus far received scant attention: his link to MUMS.

There is much to suggest that MUMS played a role in Julian's transformation from a young, persecuted activist, to a global superstar. Julian first came to Melbourne University in 2003 as a mature-age student, studying physics and mathematics at undergraduate level. He had never undertaken tertiary education before – his education, from the start, had been a combination of home-schooling and self-directed study[11] – and his decision to return to university was precipitated by a tumultuous period in his life, during which he had entertained doubts about his future direction. According to the *New Yorker*, Julian's decision to study physics and mathematics was above all spurred by a hope that 'trying to decrypt the secret laws governing the universe would provide the intellectual stimulation and rush of hacking'.[12]

From the start Julian was evidently a bright student with much promise. Norman Do, a former editor of Paradox and a student in one of Julian's early tutorials, recalls that Julian was 'obviously more mature than most of the other students, very intelligent, self-motivated and extremely curious'. Unfortunately, Julian's time at Melbourne University was characterised by a growing disenchantment with academia. Given that much of Julian's life had been dedicated to undermining institutions, it was perhaps unsurprising that a university would not be immune to his critical eye.

Julian's ire was quickly roused by the Applied Maths Department, apparently over research links between the Department and the US military.[13] Given his

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[10] *New Yorker*, above 8, page 8.

[11] According to the *New Yorker*, above 8, 'Assange's mother believed that formal education would inculcate an unhealthy respect for authority in her children and dampen their will to learn. 'I didn't want their spirits broken,' she told me. In any event, the family had moved thirty-seven times by the time Assange was fourteen, making consistent education impossible. He was home schooled, sometimes, and he took correspondence classes and studied informally with university professors': at page 6.

[12] Above 8, page 8.

[13] According to an interview Julian gave with *the Age*, this research involved improvements to the 'Grizzly Plough', a military bulldozer that, in Julian's words, aims to 'move at 60 kilometres an hour, sweeping barbed wire and so on before it, and get the sand and put it in the trenches where the [Iraqi] troops are, and bury them all alive and then roll over the top: above 1. While the Department's Micromechanics of Granular Media Group does list the US Army Research Office and the US Army Corps of Engineers as past sources of their funding, it has since emerged from discussions with staff of the Department that Julian's account of the Department's research is inaccurate, and that no such research on the 'Grizzly Plough' took place: http://www.mgm.ms.unimelb.edu.au/index.php.

activist roots, it is hardly surprising that Julian would find the ethical implications of military research distasteful. Julian has been equally candid about his distaste for the Physics department, and 'career physicists' in general. On a blog written in 2006, he describes the physicists at a conference he attended as 'snivelling fearful conformists of woefully, woefully inferior character',[14] and remarked that 'there was just something about their attire, and the way they moved their bodies, and of course the [Defence Department] bags on their backs didn't help much either. I couldn't respect them as men.'[15]

Yet, if Julian was disillusioned by the staff at the University, it appears he found solace in his fellow students. And it was in MUMS that Julian discovered a group of like-minded, bright, and self-motivated students to whom he could relate. Julian didn't take long to get involved, and in his first semester at Melbourne University he was elected Second-Year Representative. Julian quickly established himself as an important contributor to the Society: enthusiastic, vocal at meetings and spending much of his free time hanging out in the MUMS room. Just six months later he had risen to the position of Vice President, a position that he would hold until 2005.

It appears that throughout this time Julian saw MUMS as an outlet for his energy and intelligence. Damjan Vukcevic, President at the time, recalls that Julian would willingly engage MUMS members in all manner of discussions, 'whether it be on mathematics homework, philosophy, art or politics.' His most significant contribution to the Society, and one for which he is most fondly remembered by current members, was establishing the Melbourne University Puzzle Hunt, a competition now in its seventh year and with a worldwide following that grows larger with each incarnation. According to MUMS members of the time, the seeds of the Puzzle Hunt were sown one day when Julian sent an email to the MUMS committee commenting on how impressive he found the MIT Mystery Hunt, and how he thought MUMS could pull off something similar.

Initial scepticism about the prohibitive amount of work required was soon drowned out by Julian's unbridled enthusiasm, and it wasn't long before the idea got off the ground. Fellow organisers recall that Julian's 'programming prowess was invaluable in the early years of the Puzzle Hunt', that Julian was 'responsible for our media strategy which resulted in the huge participation levels; a lot more people signed up for Puzzle Hunt than we anticipated', and

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[14] *New Yorker*, above 8, page 8.
[15] *The Age*, above 1.

that Julian had the main ideas for the storyline – unsurprising given the complexity of the security breaches effected by the protagonist. The Puzzle Hunt website credits Julian with 'plot/script' and 'general nonsense', as well as designing six puzzles (more than anyone else). Julian was again involved in the second incarnation of the Hunt in 2005, and is credited on the website with designing two puzzles.

Apart from the Puzzle Hunt, Julian's other lasting contribution to MUMS was a donation of books to the MUMS library. Apparently Julian also promised the Paradox editor of the time that he would eventually contribute an article to the magazine. Perhaps Paradox should hold him to that promise!

What, then, do MUMS members of the 2003-05 era remember of Julian? And does it correspond to the media's portrayal of him? For a start, Julian's secrecy and cautiousness towards revealing personal information was already highly developed at the time of his involvement in MUMS. After being elected to the Committee in 2003, Julian refused to allow his photo to be put on the MUMS website (as is the custom), citing security reasons. In its place a photo of an alien was substituted. Julian was also known to enter the MUMS room with a variety of strangers in tow, declining to introduce them to others. Further, Julian was also displaying early signs of his incredible focus and singular work ethic. One former member recounts how Julian 'often used to remark about not having slept for the last few days. This seemed par for the course for him.'

Finally, Julian's anti-authoritarian streak was also readily apparent, manifesting in a playful enjoyment of chaotic disruption. One former member relates a story about the day when Julian's 'curiosity got the better of him, and he decided to release a random valve on the side of the chemistry building. He came to the MUMS room afterward, saying that there had been a massive noise and a cloud of smoke and for a few seconds, he thought he was in heaven!' Julian would also delight in catching people unaware by nominating them for President at MUMS AGMs; one year he nominated five different people. As a past President of the Society simply puts it: 'there were some interesting adventures'!

That Julian was already demonstrating the character traits associated with his role in Wikileaks is hardly surprising if the timing of Julian's involvement in MUMS is considered. Julian left Melbourne University, without graduating, at the end of 2005. Wikileaks was officially founded in 2006. So, can MUMS be said to have had an influence on Julian's subsequent development of Wikileaks? Certainly, when Wikileaks was in its fledgling stage, Julian is known to have discussed some of his ideas with MUMS members. In 2006, a year in which Julian worked round-the-clock on the website, he was locked in a house near the University, but still attended social events run by the Society. Julian was also actively seeking to recruit MUMS members around this time. At the end of 2006, just as Wikileaks was being launched, Julian made the following frantic plea for help (extracted):

> Are you interested in being involved with a courageous project to reform every political system on earth – and through that reform move the world to a more humane state? Wikileaks, a project I've been working on, is in the middle of an exponential media cascade. From a single blog reference four days ago to 51,000 google pages now and articles out next week by Washington Post, Science, New Scientist, Forbes, etc. people have even translated the philosophy into German and Spanish! It's great! But we weren't going to launch for at least two months. Now we have only 22 people trying to usher in the start of a world-wide movement. We don't have time to reply to most reporter's emails, let alone the interview requests - and I leave for Africa in under a week! We need help in every area, admining, coding, sys admining, legal research, analysis, writing, proofing, manning the phone, standing around looking pretty, even making tea.

Aside from such calls for direct involvement, the clearest link between Wikileaks and MUMS is the Melbourne University Puzzle Hunt. First of all, as mentioned above, Julian Assange was a crucial factor in the initial creation of the Puzzle Hunt. But the links between the Puzzle Hunt and Wikileaks go deeper. One of the most challenging aspects of the Puzzle Hunt is data sifting; a typical puzzle will start with a lot of raw data, which the puzzler must sort through in order for the puzzle to make sense. Wikileaks has similar problems with raw data: since all leaks are completely anonymous, Wikileaks has no way of verifying the authenticity or usefulness of a leak without further analysis. The data contained in a leak must be sifted if the leak is to reveal its true worth. Daniel Mathews, a former President of MUMS, takes the point even further: 'I think it would be fair to say that [Julian] saw Wikileaks, in some ways, at some times . . . as a political version of the Puzzle Hunt, with great social implications.' Further evidence of this can be found in an email Julian once sent to all Puzzle Hunt participants (extracted):

> Hello Puzzle Hunters. I am Julian, founder of the Melbourne University Puzzle Hunt and president of the Wikileaks advisory board.
>
> I am looking for good people, courageous people, intelligent people to help develop and run an international leaked document analysis & essay competition.
>
> Wikileaks is only new, but we have already broken major stories in the international press that have achieved significant reforms likely to save tens of thousands of lives. Our problem? We're drowning in leaked documents.
>
> Across the world there are other notable analytical, mooting and essay competitions. Competition in most of these cases is what we might describe as 'mere competition'; the motivational elements extend to social and professional standing, competition camaraderie and the pleasure of discovery and creation, but together we can create a much more interesting competition; a competition where teams of bright people form an engine for justice, a competition where:
>
> 1. The basis is of real substance and interest in the form of never before released leaked documents of potentially significant political importance. 2. Discovery and creation are augmented by the nature of the material and its moral calling. These are real puzzles with real discoveries to be found. 3. In addition to traditional or academic honors, there is the ultimate honor: to have a positive impact on civilization through one's labours and for this to be internationally recognised.
>
> Proposed awards: over-all winner, lightning (24 hour), best analysis, best critical analysis, best news story. Where 'best' is defined as 'whose insights contribute most to humanity'.

Sadly no such competition ever got off the ground. Nonetheless, the Puzzle Hunt stands as both a testament to the intelligence and passion that Julian brought to MUMS, and as an enduring link between current MUMS members and Julian's work in Wikileaks.

To conclude, it is clear that Julian left an indelible mark on MUMS. Equally, perhaps, it can be argued that MUMS left a mark on Julian. As evidence I cite the following email sent by Julian, after he had left MUMS, to Kevin Andrews, Minister for Employment and Workplace Relations, and CC'd to *the Age*:

Dear Mr. Andrews,

A vital but often over looked cultural value of our great country is that floor numbering in Australian buildings follows the scheme: $\{\text{ground}, 1, 2, \ldots\}$

and not

$\{1, 2, 3, \ldots\}$

or

$\{\text{RC}, 1, 2, \ldots\}$

as used by non-Australian cultures.

This idiom is not only part of the Australian way of life, but has important fire-safety implications. For the last 7 years approximately 500 Australians per year died from fire related injuries, and of these 78% died in building fires. On the face of it, it seems likely that some portion of these deaths can be attributed to a desperate confusion as to which floor leads to safety.

In addition, the Australian idiom, which starts 'counting from zero' (ground), closely matches the method used in the sciences and all modern computer programming languages. The Australian manner of floor counting has been credited as the single most important factor in the rapid and intuitive adoption by Australian science students of the counting from zero technique relative to students of other countries and likely plays a factor in the high performance of Australians in the prestigious International Mathematical Olympiad (IMO) and ACM programming competitions. The technique is present in nearly every computer program and the multibillion dollar Australian ICT and biomedical industries could not function without it.

We seek a commitment that a Coalition government by the Australian people in November 24 will demonstrate that it cares for Australian lives, Australian students, Australian traditions & Australian industry by embodying the Australian floor-counting idiom in the Australian Cultural Values Test.

Sincerely, Julian Assange Vice President emeritus of the Melbourne University Mathematics and Statistics Society.

— Stephen Muirhead

SET Theory

The real-time card game known as SET is deceptively simple, yet terribly addictive. The objective is to find SETs of 3 cards known as SETs (hereafter capitalised to distinguish it from the mathematical use of 'set'). SET cards look just like the following, except that the colour (R, G or P) will be labelled on the bottom left hand corner of each card since this will be printed grayscale.

So what does a SET deck look like? Each card has 4 attributes:

1. A number: one, two or three.
2. A shading: solid, open or striped.
3. A colour: red, green or purple.
4. A symbol: oval[1], squiggle or diamond.

There is exactly one card in the deck for each possible set of attributes, thus a total of $3^4 = 81$ cards in the deck.

Definition: Three cards are called a SET if, for each of the four attributes, the cards are either all the same or all different.

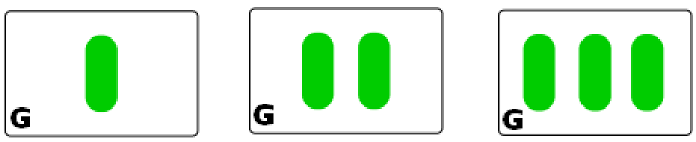


Figure 1: A really simple SET.
Colour = all the same. Shape = all the same.
Shading = all the same. Number = all different.

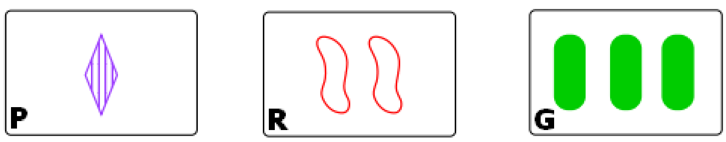


Figure 2: A slightly harder SET.
Colour = all different. Shape = all different.
Shading = all different. Number = all different.

[1]Though the game literature refers to one of the shapes as an oval, none of them truly are. The shape in question, with semicircular ends and parallel sides, is properly referred to as a stadium.

Alternative formulation: If some attribute can be grouped as "two of (something) and one of (something else)" then it is not a SET.

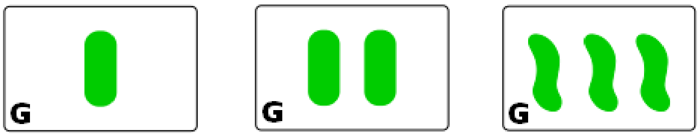


Figure 3: Not a SET.
Fails because the shapes can be grouped as "two ovals and one squiggle".

In the canonical way of playing SET, the dealer lays out 12 cards on the table. When someone sees a SET they call 'SET!' and takes the cards that forms the SET, and the dealer lays out three more cards to replace those. In the event that there is no SET among the twelve cards, the dealer deals out three more cards to make fifteen, or even more if necessary. This goes on until there are no more SETs on the table, and whoever collected the most SETs wins.

As you can see, it can be a fairly stressful game. If you'd like to play it, you're always welcome to come down to the MUMS room where we always have a SET deck handy. But for the purposes of this article, we're going to discuss the mathematics behind this wonderful game, for there are many beautiful mathematics to be discovered from it.

'Although children often beat adults, the game has a rich mathematical structure linking it to the combinatorics of finite affine and projective spaces and the theory of error-correcting codes,' Diane Maclagan and Benjamin Lent Davis remarked in a paper published in the September 2003 Mathematical Intelligencer. In 2002, 'an unexpected connection to Fourier analysis was used to settle a basic question directly related to the game of SET, and many related questions remain open.'

We're not going to jump into any complicated mathematics, but a lot of nice things can be proven just by using fairly simple mathematics. Of course, we start with the most important statement of all:

Fundamental theorem of SET: Given any two cards, there exists exactly one card (called the *third card*) which forms a SET with those two cards.

For example, consider the colour attribute. If the two cards are both red, then the third card must also be red. If the two cards are red and green, then the third card must be purple. In this sense, the attributes of the third card are all uniquely determined from the other two cards.

Corollary: The probability of producing a SET from 3 randomly drawn cards in 1/79.

Finding the number of unique SETs is also easy from the fundamental theorem, picking any two cards will produce a unique third card that gives a SET. But this SET can be rearranged in $3! = 6$ different ways, thus a total of $\frac{81\times 80}{6} = 1080$ unique SETs.

There exists a really nice geometric interpretation of SET, which is a 4-dimensional "wrap around" version of noughts and crosses. Or it can be equivalently formulated as follows: Let $\mathbb{Z}_3$ be the cyclic field with 3 elements. This means that numbers wrap around in that $1 + 2 = 0$ and $2 + 2 = 1$ etc.

Consider the vector space $\mathbb{Z}_3^4$ in which points are 4-tuples of the form (x_1, x_2, x_3, x_4) where each coordinate represents one of the the attributes, each taking 3 possible values. We use the same order as given before, so for example the point (0, 1, 2, 0) would be "One open purple oval" whereas the point (2, 1, 0, 2) would be "Three open red diamonds."

Using this system, notice that three cards form a SET if and only if the three vectors add to 0. To see this, consider that for each attribute, they are either all the same or all different. So the 4 possibilities are $0 + 0 + 0 = 0, 1 + 1 + 1 = 0, 2 + 2 + 2 = 0, 0 + 1 + 2 = 0$, i.e. if they form a SET, they will equal to zero. Furthermore, these are the only possible ways of making zero, and so they are equivalent. We state this more explicitly:

The Affine Collinearity Rule: Three cards $a, b, c \in \mathbb{Z}_3^4$ form a SET if and only if $a + b + c = 0$.

Theorem: If we have found 26 SETs from the standard deck, then the remaining 3 cards must also form a SET.

Proof: We assert without proof that the sum of all cards = 0 (fairly intuitive). The sum of the first 26 SETs will then have to be $26 \times 0 = 0$, and thus the sum of the last 3 cards is zero, so they must form a SET.

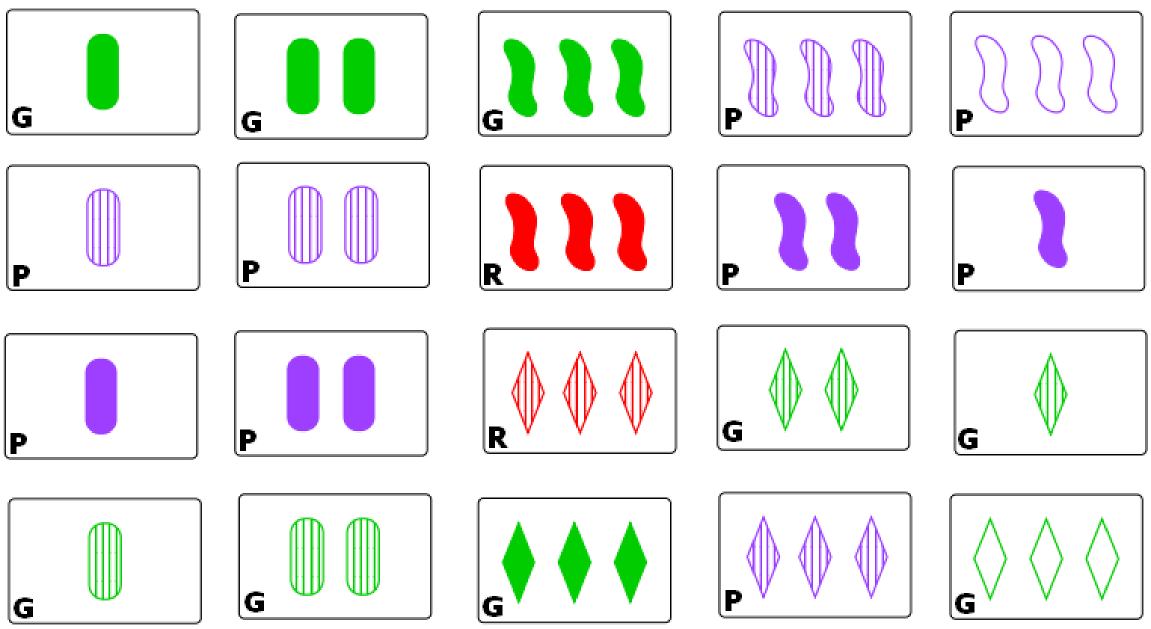
One question that usually comes to mind when playing is: How many cards will I need on the table to guarantee that there will be a SET? Anyone who has played SET more than a few times will know that 12 cards are often not enough to find a SET. So how high can we actually go without having any

SETs? Here's a relatively high bound that's not too difficult to prove:

Theorem: Every set of 47 cards must contain a SET within it.

Proof: By contradiction. Suppose we had a set of 47 cards with no SET in it. Then for any two cards in this set, its corresponding 'third card' is not in this set, and so every pair of two cards in this set produces a unique set. However, this then gives us a total of $\binom{47}{2} = \frac{47\times 46}{2} = 1081$ SETs. But we earlier showed that there are only 1080 SETs in the entire deck, which is a contradiction.

In actual fact, you only need 21 cards to guarantee that a SET exists. Unfortunately, the proof is much beyond the scope of this introductory article. However, I will end off this article with a possible set of 20 cards that does not contain any SETs (try it!):



— Muhammad Adib Surani

It's a Quantum Thing...

You'd be hard-pressed to find a more abused scientific discipline than quantum mechanics. It's been co-opted by theologians, new age proponents, quacks, cranks, and charlatans to promote everything from universal consciousness to crystal healing. Peddler of dubious medical treatments Deepak Chopra[1] has personally inflicted cruel and unusual punishments upon quantum mechanics so often he should be brought up for human rights violations.

Due in no small part to all of the misinformation about quantum mechanics in popular culture, the topic remains widely misunderstood. It prompts the question, therefore: what is quantum mechanics all about? And what does one actually learn in a course on quantum mechanics? The answer, probably to the dismay of many (but not, presumably, to your average Paradox reader), is maths. Maths with a physical interpretation and application, to be sure, but rather a lot of maths nevertheless. To understand why this should be the case, it is important to digest the following two facts:

1. Light is not the only object that behaves like a wave. All matter has a corresponding wavelength – it's just that for larger objects the wavelength becomes insignificant. This is because the relation is given by $\lambda = \frac{h}{p}$, where h is an extremely small physical constant called Planck's constant, and p is the momentum (i.e. mass times velocity) of the object.

2. Quantum objects can be described in terms of 'wavefunctions', typically denoted $\Psi(\mathbf{x}, t)$ when the function depends on position and time, and $\psi(\mathbf{x})$ when it only depends on position. This wavefunction is related to the probability that a particle can be found in a particular region of space at a particular time.

Of course, these observations aren't of much use to us unless we can actually do something with these wavefunctions; imagine if, in classical mechanics, all we could do is specify where and how fast an object was travelling, but not what it would do next! In classical mechanics we can use Newton's laws and Lagrangian mechanics to determine how a system evolves with time – analogously, in quantum mechanics we have the Schrödinger equation to tell

[1] Deepak Chopra is an Indian/American 'expert' on alternative medicine, who frequently invokes links between quantum mechanics and medicine.

us how a wavefunction evolves with time. For the one-dimensional case, this is:

$$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+V(x,t)\Psi(x,t)=i\hbar\frac{\partial}{\partial t}\Psi(x,t).$$

Don't be put off if everything in that equation looks unfamiliar to you – the basics of it are quite easy to grasp. All this equation is really doing is stating a relationship between: (i) how the wavefunction changes in space if we ignore time (the bit on the left); (ii) how the wavefunction changes in time if we ignore space (the bit on the right); and (iii) how this evolution is affected by a potential, just like in the Lagrangian formulation of classical mechanics (the bit in the middle).

There's other stuff too, mind you – no course in quantum mechanics would be complete without looking at the Heisenberg Uncertainty Principle, specifically using a statistical treatment of uncertainty and expectation values analogous to those of the standard deviation and mean. A closer look at these topics would require a briefing in operators and observables, a fascinating topic, but not the sort of thing one can cover in a single article.

Really though, the bulk of a second year quantum mechanics course is related to solving the Schrödinger equation for different situations (read: potentials) and looking at various applications of the results. Along the way you encounter all of the paradoxical and weird aspects of quantum mechanics that tend to get twisted in popular culture, but you encounter them with a solid enough physical and mathematical backing to make sense of at least some of it, and not fall into popular traps for the rest. If nothing else you gain a whole new appreciation for the truth of that old crank rebuttal: *'If there's no maths, it's not really quantum mechanics.'*

— Richard Hughes

Solution to the puzzle on page 5:
Neither 'triangle' has a hypotenuse that is a straight line; one is concave and the other is convex.

Cutting a square into triangles

Not long ago, I attended an interesting student seminar that was part of a series held every Friday afternoon. What particularly struck me was the method used in the seminar to prove a surprising fact:

> Given a square, you cannot subdivide it into an odd number of triangles of equal area.

On the other hand, it is known that you can, in fact, subdivide a square into an even number of triangles of equal area.

The proof that I provide here closely follows an article by Paul Monsky.[1] It has two parts: the first is combinatoric in nature, whereas the second is a result from number theory. These two mathematical worlds collide to form a beautiful proof.

Combinatorial observations

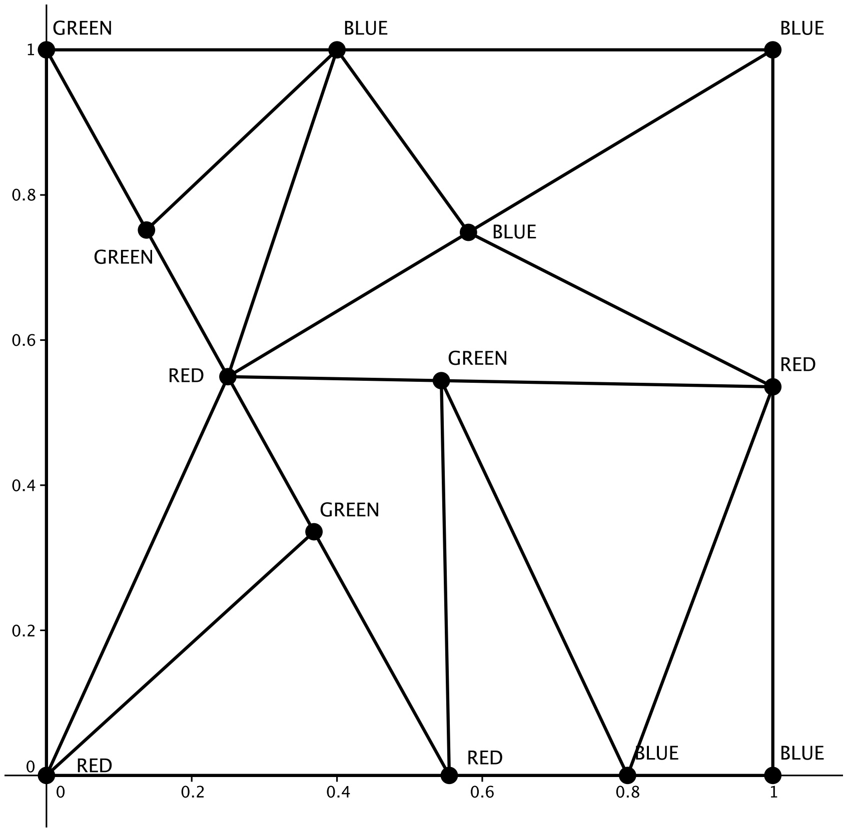
Here we will make a combinatorial observation about 3-colourings. First, we need to establish some terminology so that we may more easily discuss the topic.

For convenience, let $\Box$ denote a square in the Euclidean plane (think of normal Cartesian coordinates) with vertices at $(0,0), (0,1), (1,0)$ and $(1,1)$. Suppose that we divide the square $\Box$ into m triangles. We shall call this subdivision $\mathcal{S}$. We can observe that this subdivision $\mathcal{S}$ forms a pattern called a *graph* in the square $\Box$ as shown below:

This graph consists of a set of vertices and edges linking two vertices. So, a *vertex* is exactly a vertex of some triangle in the subdivision $\mathcal{S}$ and an *edge* is a line segment that contains exactly two vertices. Given a triangle T in the subdivision $\mathcal{S}$, each of the three sides is called a *face*. A face will generally contain more than one edge.

By 3-colouring, we mean: (1) fixing a set of three colours, say red (R), blue (B) and green (G); (2) associating each vertex to any of these three colours.

[1] Paul Monsky, 'On Dividing a Square into Triangles', *American Mathematical Monthly*, 77:161-164, 1970.



Alternatively, we may think of it as dividing a set $\mathcal{V}$ of vertices into three disjoint sets $\mathfrak{R}, \mathfrak{B}$ and $\mathfrak{G}$. Finally, we say that a face is of type $\mathfrak{RB}$ if it has an endpoint in red and the other in blue. Now, we are ready to state a fact:

Fact 1: Suppose that there is no face that contains vertices of all three colours and the square $\square$ contains an odd number of faces of type $\mathfrak{RB}$. Then there is a triangle with each vertex of different colour.

First, if the face F is of type $\mathfrak{RB}$, then by assumption, it can only have red or blue vertices. So, the face F can only contain an odd number of edges of type $\mathfrak{RB}$. To see this, let $N_{\mathfrak{RR}}$ and $N_{\mathfrak{RB}}$ be the number of edges of type $\mathfrak{RR}$ and $\mathfrak{RB}$ respectively. Similarly, let $N_{\mathfrak{R}}$ be the number of vertices of type $\mathfrak{R}$. By counting the number of vertices of red colour, we have the following equation:

$$2N_{\mathfrak{RR}} + N_{\mathfrak{RB}} = 2N_{\mathfrak{R}} - 1.$$

This implies that $N_{\mathfrak{RB}}$ must be odd. Also, faces of other types will have no edges of type $\mathfrak{RB}$.

We shall prove this by contradiction. Suppose there is no triangle with vertices of all three colours. This means that for each triangle there is either 0 or 2 faces of type $\mathfrak{RB}$. So each triangle T contains an even number of edges of type $\mathfrak{RB}$. Let N_T be a number of edges of type $\mathfrak{RB}$ in triangle T. Then,

$$\sum_T N_T = 2N_i + N_o,$$

where T ranges over all triangles in the subdivision $\mathcal{S}$ and N_i (N_o) is the number of interior (exterior) edges of type $\mathfrak{RB}$. Since the left hand side is even, while the right hand side is odd, we get a contradiction.

p-adic valuation

Let's start with a motivation for p-adic valuation. One idea comes from studying how we can construct the reals $\mathbb{R}$ from the rationals $\mathbb{Q}$. Given a real number, say π, we may think of it as

$$\pi = 3.1415\ldots$$

But what does "…" really mean? One interpretation uses the notation of limits. That is, we should write π as a sequence of rationals that converges to it. One sequence is

$$3, 3.1, 3.14, 3.141\ldots.$$

But this gives rise to the following question: what does it mean for a sequence to converge to a point? The technical answer is the following: the sequence $(x_1, x_2, \ldots)$ of rational numbers *converges* to a number L (L may not be rational) if for every $0 < \epsilon \in \mathbb{Q}$, there is a natural number $N > 0$ such that for all $n > N$, $|L - x_n| < \epsilon$. In a nutshell, what this means is that after N, all x_i's are within ϵ of L. But notice that the definition relies on the notion of an *absolute value* $|\cdot|$. So, what if we change this definition?

To do so, we need to observe some of the properties of the absolute value. Essentially, $|\cdot| : \mathbb{Q} \to \mathbb{Q}^+$ is a map from the rationals to the positive rationals (remember that at the moment, we don't know what a real number is). It satisfies the following:

$$|xy| = |x| \cdot |y|, \tag{1}$$
$$|x + y| \leq |x| + |y| \tag{2}$$
$$|x| = 0 \iff x = 0. \tag{3}$$

This is called the *Archimedean* absolute value. However, if one imposes a stronger condition by replacing the second property with $|x+y| \leq \max\{|x|, |y|\}$, we get a *non-Archimedean absolute value* or *ultrametric*. It turns out that there is only one such ultrametric.

Fix a prime number $p > 0$. Note that for our purposes we will later concentrate on the case $p = 2$.

Definition 1: The *p-adic valuation* $|\cdot|_p$ is a map from $\mathbb{Q} \to \mathbb{Q}^+$ defined in the following way: since every rational can be written uniquely as $p^n \frac{a}{b}$, where p does not divide either a or b, let

$$\left|p^n \frac{a}{b}\right|_p = \frac{1}{p^n}.$$

This valuation satisfies all the properties we demand of a non-Archimedean absolute value. I said before that this was the only such ultrametric. In fact, there is another: the *trivial absolute value* $|\cdot|_0$, where $|x|_0 = 1$ for all $x \neq 0$, and $|x|_0 = 0$ if $x = 0$. But Ostrowski showed that any non-trivial absolute value on the rationals is equivalent to the normal absolute value or the p-adic absolute value. It is amazing that by imposing three simple conditions that an absolute value must satisfy, we are left with only two possible types.

We may observe that $|1|_p = |-1|_p = 1$, and that $|x+y|_p = \max\{|x|_p, |y|_p\}$ unless $|x|_p = |y|_p$. Now, as one can extend the normal absolute value to the reals, one can also extend the p-adic value to the reals as well.

Fact 2: Given a p-adic value on the rationals $\mathbb{Q}$, there is an absolute value $|\cdot|_p^{\mathbb{R}} : \mathbb{R} \to \mathbb{R}^+$ defined on the reals such that for any rational $q \in \mathbb{Q}$, $|q|_p^{\mathbb{R}} = |q|_p$. In fact, we may also extend this valuation for any field extension containing $\mathbb{Q}$.

For simplicity, we shall drop the $\mathbb{R}$. Before we proceed, please note that $|2|_2 < 1$. Originally, I intended to give an explicit expression for the valuation, but I was told that the existence was not proven by the construction of one.

Here comes the beautiful part. Colour the points of the plane in the following way:

$$\begin{array}{llll} (x,y) & \text{is red (in } \mathfrak{R}) & \text{if } |x|_2 < 1 & \text{and} \quad |y|_2 < 1, \\ (x,y) & \text{is blue (in } \mathfrak{B}) & \text{if } |x|_2 \geq 1 & \text{and} \quad |x|_2 \geq |y|_2, \\ (x,y) & \text{is green (in } \mathfrak{G}) & \text{if } |y|_2 \geq 1 & \text{and} \quad |y|_2 > |x|_2. \end{array}$$

Now, we shall demonstrate that this colouring allows us to use Fact 1.

Fact 3: Let $P = (x, y)$ and $P' = (x', y')$. Suppose that $|x' - x|_2 < 1$ and $|y' - y|_2 < 1$. That is, P' is a translation of P by a point of type $\mathfrak{R}$. Then P' is of the same type as P.

We shall basically do a case bash:

1. If P' is of type $\mathfrak{R}$, then $|x'|_2, |y'|_2 < 1$. Since $|x|_2 = |x' + (x - x')|_2 \leq \max\{|x'|_2, |x - x'|_2\} < 1$, and similarly for y. So, $P \in \mathfrak{R}$.

2. If P' is of type $\mathfrak{B}$, then $|x'|_2 \neq |x' - x|_2$, so $|x|_2 = \max\{|x'|_2, |x' - x|_2\} = |x'|_2 \geq 1$. Note that

$$\begin{aligned} |y|_2 &\leq \max\{|y'|_2, |y' - y|_2\} \\ &\leq \max\{|y'|_2, 1\} && \text{since } |y' - y|_2 < 1 \\ &\leq |x'|_2 && \text{since } |x'|_2 \geq 1 \text{ and } |x'|_2 \geq |y'|_2 \\ &= |x|_2 \end{aligned}$$

   So, P is of type $\mathfrak{B}$.

3. If P' is of type $\mathfrak{G}$, we may apply a similar argument.

Fact 4: If L is a line in $\mathbb{R}^2$, then L cannot contain a vertex of some colour. That is, it is 2-colourable. This implies that no face in the subdivision $\mathcal{S}$ contains vertices of three colours.

We shall show this by contradiction. Suppose there is a line L that contains vertices of all three colours. We may use Fact 3 to translate the line so that the origin $(0, 0)$ is contained in the line L. Let $P = (x, y)$ and $P' = (x', y')$ be blue and green vertices on the line L respectively. Since they are both on the same line that passes through the origin, we get $x'y = y'x$ and thus $|x'y|_2 = |y'x|_2$.

But that is absurd since

$$\begin{aligned} |xy'|_2 = |x|_2|y'|_2 &> |x|_2|x'|_2 && \text{since } P' \in \mathfrak{G} \\ &\geq |y|_2|x'|_2 = |yx'|_2 && \text{since } P \in \mathfrak{B} \end{aligned}$$

Fact 5: If a triangle T has vertices of all three colours. Then $|\text{ Area } T|_2 > 1$.

Again, we may translate (without changing colouring and area) the triangle so that the origin $(0,0)$ is one of the three vertices. Let $P = (x, y)$ and $P' = (x', y')$ be blue and green vertices of the triangle T. Recall vector calculus. The area of a triangle is exactly half the determinant of the 2×2-matrix formed by the two column vectors P and P'. Thus, the area of the triangle T is $\frac{1}{2}(xy' - yx')$.

$$|\text{Area } T|_2 = \left|\frac{1}{2}\right|_2 |xy' - yx'|_2 = 2 \times \max\{|xy'|_2, |yx'|_2\} = 2|x|_2|y'|_2 > 1$$

since $|xy'|_2 > |yx'|_2$, $|x|_2 \geq 1$ and $|y'|_2 \geq 1$.

Bringing them all together

Suppose that we can subdivide the square $\square$ into m triangles of equal area. This implies that the area of each triangle is $^1/_m$. According to the colouring scheme, $(0,0)$ is red, $(0,1)$ is green, and both $(1,0)$ and $(1,1)$ are blue. So, the square $\square$ has an odd number of faces of type $\mathfrak{RB}$. Fact 4 tells us that we may apply Fact 1 to show that there is a triangle consisting of vertices of all three colours. By combining Fact 5 and the hypothesis that there are m triangles, we get $|^1/_m|_2 > 1$. This implies that 2 divides m.

Final remarks

In modern mathematics, this idea of solving a problem using a technique in a different area of mathematics is almost a central theme. Another instance of this is the classification of 3-manifolds using group theory (Mostow Rigidity Theorem), which is proved using methods from measure theory. However, if you want to move up to 4-manifolds, you need recursive theory to show that this cannot be done.

— Tharatorn Supasiti

Solutions to Problems from Last Edition

We had a number of correct solutions to the problems from last issue. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

Ying Wan Yap solved problem 1 and may collect 2 dollars.
Raj Dahya solved problem 3 and may collect 3 dollars.
Adrian Khoo solved problems 1 and 5 and may collect 6 dollars.
Kevin Fray solved problems 3, 6 and 7 and may collect 13 dollars.

1. Prove that the sum of the 2009th powers of the first 2009 positive integers is divisible by 2009.

   Solution: First of all, it is easily shown that $a + b | a^n + b^n$ for all integers a and b and for all odd n. Then, letting n run from 0 to 1004, and setting $a = n$ and $b = 2009 - n$, we have that $2009 | 0^{2009} + 2009^{2009}$, $2009 | 1^{2009} + 2008^{2009}$, $\ldots$, $2009 | 1004^{2009} + 1005^{2009}$. Hence 2009 also divides the sum of these expressions.

2. Two cylists, Sam and Steve, simultaneously set off from one end of a road, cycle back and forth along the road (turning instantaneously at the end-points), and stop when they arrive simultaneously at the end opposite from where they started. At this point Sam has travelled the length of the road nine times, and Stephen 13. How many times did they pass each other going in opposite directions?

   Solution: There are three states that can describe Sam and Steve's status: (1) Sam and Steve are travelling towards each other, (2) Sam and Steve are travelling away from each other, (3) Sam and Steve are travelling in the same direction. There are exactly three ways to change states: (1) to (2), occurring when they pass each other going in opposite directions; (2) to (3), occurring when they are travelling in opposite directions and one 'turns' at the end of the road; (3) to (1), occurring when they are travelling in the same direction and one 'turns' at the end of the road. Thus, the sequence of 'passes' and 'turns' proceeds: TPTTPTTP…TTPT. Since Sam turns eight times and Stephen 12 times, they pass each other $\frac{8+12}{2} = 10$ times. A slight complication arises if Sam and Steve turn simultaneously, as in this case the state jumps from (2) to (1) with two 'turns' completed. However, this does not affect the resulting sequence of Ts and Ps, and hence the answer.

3. Prove that a set of size n has no more that $n!$ partitions into disjoint subsets.

   Solution: The result holds for $n = 1$. Suppose it holds for $n = 1, 2, \ldots, k-1$, and we'll show that it holds for $n = k$. Consider a partition of $1, 2, \ldots, k$ into disjoint subsets in which the element k occurs in a subset of size t. There are $\binom{k-1}{t-1}$ possible such subsets, and for each, there are at most $(k-t)!$ valid partitions of the remaining $k-t$ elements, by our inductive hypothesis. Hence, for a given t, there are $\binom{k-1}{t-1\times(k-t)!=\frac{(k-1)!}{(t-1)!}}$ partitions. Each summand is at most $(k-1)!$, and there are k summands $(t = 1, 2, \ldots, k)$, so the total is at most $k!$.

4. If you place twenty-one 3x1 blocks on a chessboard so that there is one square not covered, what are the possible positions for this square?

   Solution: Colour the chess board red, green and blue as follows:

R	G	B	R	G	B	R	G
G	B	R	G	B	R	G	B
B	R	G	B	R	G	B	R
R	G	B	R	G	B	R	G
G	B	R	G	B	R	G	B
B	R	G	B	R	G	B	R
R	G	B	R	G	B	R	G
G	B	R	G	B	R	G	B

   There are exactly 21 red, 22 green and 21 blue squares. Now, since each 3x1 block occupies exactly one R, one B and one G square, the square not covered must be green. This argument equally holds if you rotate the colouring 90 degrees in any direction. Thus the missing square must be a green square that remains green under 90 degree rotations. The only candidates are the four squares in positions $(3,3)$, $(6,3)$, $(3,6)$ and $(6,6)$. Constructing an example of a covering for each of these square is simple enough.

5. If $n+1$ is a multiple of 24, show that the sum of divisors of n is also divisible by 24.

   Solution: Since n is not a square (as it is 2 mod 3), we can pair up the positive divisors into pairs $\{a, b\}$ such that $ab = n$. Then, considering modulo 24, $a + b \equiv a + \frac{n}{a} \equiv a - \frac{1}{a}$, so it suffices to show that $a \equiv \frac{1}{a}$, i.e. $a^2 \equiv 1$ modulo 24. Since a is both odd and not divisible by 3, $a^2 \equiv 1$ mod 3 and mod 8. Then the Chinese Remainder Theorem gives the result.

6. Prove that for every positive integer n, there is an integer x such that $x^2 - 17$ is divisible by 2^n.

   Solution: We prove this by induction. For $n = 1$, 2 or 3, $x = 1$ satisfies the required property. Now suppose $2^n | x^2 - 17$, for some n larger than 3. We need to find y such that $2^{n+1} | y^2 - 17$. Write $x^2 - 17$ as $k \times 2^n$. If k is even, then we can take y as x an we are done. If $k = 2m + 1$, then take y as $x + 2^{n-1}$. Now we have $y^2 - 17 = (x + 2^{n-1})^2 - 17 = x^2 - 17 + x.2^n + 2^{2n-2} = (2m+1).2^n + x.2^n + 2^{2n-2} = 2^n.(x+1) + 2^{n+1}(m + 2^{n-3})$. Since x must be odd and n is greater than 3, we have that $2^{n+1} | y^2 - 17$ as required.

7. For a 5x5 array of 1s and 0s, a *move* consists of choosing a square to change state (from 0 to 1 or 1 to 0), which causes each adjacent square in the same row or column to also change state. If the grid starts off containing 24 0s and a solitary 1, for which positions of this 1 can a combination of *moves* reduce the grid to all 0s?

   Solution (thanks to Kevin Fray): Colour the array in black and white as follows:

x	x		x	x
x	x		x	x
x	x		x	x

   This colouring has the property that any *move* affects an even number of black squares, and so the total number of black squares in state 1 cannot change parity. Thus the solitary 1 must lie on a white square. This argument is equally valid for a colouring which is a 90 degree rotation of the first colouring. Thus the solitary 1 must lie on a white square which remains a white square after a 90 degree rotation. The only candidates are the squares in positions $(2,2)$, $(4,2)$, $(2,4)$, $(4,4)$ and $(3,3)$. In fact, a solitary 1 in each of these positions can be removed by making moves as follows:

$(2,2)$ and rotations

	M		M	
M	M		M	M
			M	
M	M	M		
	M			

$(3,3)$

			M	M
		M		
	M	M		M
M				M
M		M	M	

Paradox **Problems**

Below are some puzzles and problems for which cash prizes are awarded. Anyone who submits a clear and elegant solution may claim the indicated amount (up to a maximum of four cash prizes per person). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

1. How many terms in the arithmetic sequence $8, 21, 34, \ldots$ consist solely of the digit 9? (2 dollars)

2. Prove that for any $n \geq 6$ an equilateral triangle can be dissected into n smaller equilateral triangles. (3 dollars)

3. Find all positive integers n such that $2^{200} + 2^4 + 2^n - 2^{103}$ is a perfect square. (3 dollars)

4. Does there exist an $n > 1$ such that the integers from 1 to n^2 can be arranged in an nxn grid so that the products of the integers in every row and column is constant? (3 dollars)

5. How many 'valid' position-pairs of the hour and minute hand on a 12-hour analogue clock (ie. positions that are possible given the motion of these hands over time) are equally 'valid' if the hands switch places? (3 dollars)

6. A 5x5 array contains all 1s or 0s. A *move* consists of toggling an nxn grouping of squares in the array (the grouping must lie fully within the array), where n can be 2, 3, 4 or 5. The array originally contains 24 0s and a solitary 1. For which starting positions of the 1 can the grid be modified to contain all 0s? (4 dollars)

7. Find all solution in positive integers to the following system of equations: (1) $a + b + c + d = 12$; (2) $abcd = 27 + ab + ac + ad + bc + bd + cd$. (5 dollars)

Paradox would like to thank π.O, Sam Chow, Narthana Epa, Tharatorn Supasiti, Muhammad Adib Surani, James Wan and Julia Wang for their contributions to this issue.