Issue 2, 2011

Paradox

THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY

MUMS

PRESIDENT:	Sam Chow cme_csamc@hotmail.com
VICE-PRESIDENT:	Muhammad Adib Surani m.surani@ugrad.unimelb.edu.au
TREASURER:	Julia Wang julia.r.wang@gmail.com
SECRETARY:	Mark Kowarsky mark@kowarsky.id.au
EDUCATION OFFICER:	Richard Hughes mobyseven@gmail.com
PUBLICITY OFFICER:	Brendan Duong b.duong@ugrad.unimelb.edu.au
EDITOR OF *Paradox*:	Stephen Muirhead s_muirhead22@hotmail.com
UNDERGRAD REP:	Mel Chen m.chen11@ugrad.unimelb.edu.au
UNDERGRAD REP:	Narthana Epa narthana.epa@gmail.com
UNDERGRAD REP:	Ahmad Issa a.issa2@ugrad.unimelb.edu.au
UNDERGRAD REP:	Dana Ma iwantakudamon@hotmail.com
UNDERGRAD REP:	David Schlesinger d.schlesinger@ugrad.unimelb.edu.au
UNDERGRAD REP:	Trithang Tran t.tran17@ugrad.unimelb.edu.au
POSTGRADUATE REP:	Jeff Bailes j.bailes@pgrad.unimelb.edu.au
POSTGRADUATE REP:	Yi Huang y.huang16@pgrad.unimelb.edu.au

WEB PAGE:	www.mums.org.au
MUMS EMAIL:	mums@ms.unimelb.edu.au
PHONE:	(03) 8344 4021

In this edition of Paradox

Regulars

Special Features

Articles

EDITOR:	Stephen Muirhead
SUB-EDITORS:	Mel Chen, Richard Hughes, Kristijan Jovanoski, Jiaying Zhang
WEB PAGE:	www.ms.unimelb.edu.au/∼paradox
E-MAIL:	paradox@ms.unimelb.edu.au
PRINTED:	9 May, 2011
COVER:	**Tokyo, Japan:** I heart MUMS? For more takes on the MUMS logo, see page 10.

Words from the Editor

Regular readers will know that over the last year or so, Paradox has been keenly following the plight of Julian Assange, not least because he is a former vice-president of MUMS (see Issue 2, 2010). So it was with some interest that Paradox saw the following in The Monthly magazine (Feb, 2011):

> *By 2004 Assange had reached the elevated position of vice-president of the students' Mathematics and Statistics Society and chief organiser of their Puzzle Hunt. ... Organising a puzzle hunt was a somewhat less engrossing ambition than planning world revolution.*

Having done the former but not the latter, it is hard to pass comment. Though, with the AGM coming up on Friday 13 May, you might like to nominate to join Julian in reaching such an 'elevated position'.

On a different note, as a follow up to the piece in the previous edition on *Letters and Numbers*, we'd also like to point the reader in the direction of an episode of the comedy series *The IT Crowd* called 'The Final Countdown' (Season 4, episode 2), whose plot revolves around Moss's (a lead character) appearance on the TV show *Countdown* (on which *Letters and Numbers* is based). After Moss breaks records on the show, he is invited to join a secret society of past *Countdown* contestants. However, when another member of the society gets jealous, he is challenged to a bout of 'street' Countdown, in order to settle the feud. Overall, it is a hilarious episode, one well worth checking out.

In the current edition of Paradox you'll find articles on a diverse range of mathematical topics, from infinity to the gamma function and even evolutionary biology. This edition also gives some indication of the global readership of Paradox – one of the articles was contributed by a reader from Puerto Rico!

Finally, it is with a tinge of sadness that I announce that this will be the last edition of Paradox that I edit. Which means Paradox is in need of a new editor! If you are interested, either contact me in person or via email, or else come along to the MUMS AGM (Friday 13 May, 2:15 pm, Russell Love Theatre) to nominate for the position. I can assure you, it is a thoroughly rewarding experience! Paradox will also need new sub-editors, so if you are interested in a smaller role, you can contact me as well.

Thanks to all those that have contributed to Paradox over the past two years, and I wish the new editorial team the best of luck.

— Stephen Muirhead

Words from the President

Well, well, well. Looks like it's time for some pizza, and a new committee! The Annual General Meeting will take place on Friday 13 May at 2.15pm in the Russell Love Theatre. Positions available are: President, Vice President, Paradox Editor, Secretary, Treasurer, Education Officer, Publicity Officer, six undergraduate representatives and two postgraduate representatives.

In other news, twenty supervillains decided to raid the University of Melbourne one week earlier this semester (see the wrap of the 2011 Puzzle Hunt on page 6 of this edition). Fortunately, *Wild Goldfish: Pleasant Quilt* rescued Glyn Davis, and with him defeated the villains. The local heroes came from behind to beat some highly fancied professional puzzle-hunting teams to the chase, first digging up a random Santa hat before finally finding Glyn Davis, who turned out to be Spiderman.

Do check out our seminars, which are aimed at undergraduate students, and cover a variety of interesting topics. These happen on Fridays at 1pm, Lowe Theatre, Redmond Barry Building. Refreshments in the MUMS room follow these seminars, and we usually play some games afterwards. We'll continue to have games nights, where we teach you all of the board and card games we play. Feel free to just come into the MUMS room at any time though: we have a new game, *Dominion*.

Finally, keep your eyes peeled for our popular trivia afternoon, which is expected to take place on the last afternoon/evening of semester. This typically involves a lot of eating.

Good luck to the new committee!

— Sam Chow

Maths in the News 1:
There can be few fields of human endevour more opposed than boxing and mathematics. Yet that hasn't stopped young British boxer Nathan Cleverly, who remains undefeated and will contest the World Light-Heavyweight Title on May 21, 2011 (The Daily Express, 8 May 2011). At the same time as Cleverly's ascent to the summit of world boxing, he has been earning a BSc in maths from Cardiff University, graduating in 2010.

A Wrap of the 2011 Puzzle Hunt

Unless you've been living under a rock for the past seven years, chances are you would have heard of the annual MUMS Puzzle Hunt – or better yet, participated in one. For the uninitiated, the Puzzle Hunt is an event in which teams from all over the world dedicate an entire week to solving puzzles and fighting for the top prize; and is arguably MUMS's largest event. But these are not your ordinary sudokus or crosswords – indeed, many of these puzzles will have at least one lateral step which requires you think outside the box.

What follows is an insight into the inner workings of the Puzzle Hunt. The 2011 MUMS Puzzle Hunt took place from 11–15 April 2011. The number of participants continue to grow each year, and we saw close to 900 participants this year! We note though, that people who participate in our Hunts often complain of withdrawal symptoms and post-puzzlehunt depression whenever the Hunt ends. You can visit the Puzzle Hunt website at `puzzlehunt.mums.org.au` for more information, and to look at past puzzles.

But back to the story at hand. I first organised the Puzzle Hunt in 2009. I was a second-year student then, and I was very eager to create the best Puzzle Hunt the world had ever seen. Having participated in every Hunt since 2005, I had a rough idea of what makes certain puzzles more enjoyable and memorable than others. I wrote seven puzzles that year, most of which were derived from ideas I'd built up over the years. As one might expect, planning for the Hunt takes place many months before the actual Hunt. The meta has to be set up as early as possible so that we can write puzzles that conform to it.

Wild Goldfish: Pleasant Quilt, winners of the 2011 MUMS Puzzle Hunt.

I organised the Puzzle Hunt again in 2010. I was a third-year student then, and I was eager to create the best Puzzle Hunt we'd had so far. In other words, I was planning to outdo my performance from the previous year. So that was what I did. I didn't write as many puzzles as I did in 2009, but I was more involved with test-solving and checking puzzles for any possible errors. That didn't go quite so well, as I recall we had quite a lot of errata that year. Regardless, it was an overall enjoyable experience for most teams.

For the third year running, I organised the Puzzle Hunt in 2011. By this time, I pretty much understood every aspect of how the Puzzle Hunt ran. And like the previous two years, we introduced even more changes. Some of the biggest changes we made with this year's Puzzle Hunt were to partially remove the story, and replace them with villain biographies instead. Many teams found this more fun. We also shortened it to 4 days, a move which was not very popular with many teams. Overseas teams in particular thought this was a huge disadvantage, as well as people who work office-hours and really only get free time during the weekend. The other change was that we made the meta slightly easier, which was a reasonable thing to do given that teams have taken a really long time to solve the meta in the past. An overwhelming majority of teams highlighted that they would prefer the hunt not to end when the hidden item is found, but instead let the Hunt continue until all the hints have been given out (much like CiSRA and SUMS Puzzle Hunts). While it is a tradition to end the Hunt in this way, we might be looking at alternatives that can make everyone happy in the future.

I thought I would write about some specific puzzles in this article. If you've never done participated in any Puzzle Hunts and want to look at a few puzzles to get an idea of the beauty that surrounds them, then these are the puzzles you would want to look at. To that end, I decided to review the five most popular puzzles this year[1], which were:

1. **Puzzle 3.1 - The Drover, by Stephen Muirhead**
   Large thanks to Kim Ramchen, Sam Chow and Corey Plover as well for programming and creating graphics for it. I can easily think of a few reasons why people enjoyed this one: It looks fun, it looks approachable, and there's a good *Aha!* to be found when you get the first string which tells you to TURN SHEEP INWARDS.

[1] As obtained from the feedback we received.

2. **Puzzle 4.2 - Kangaroo Jack, by Corey Plover**
This was the very first puzzle drafted for the 2011 Puzzle Hunt, and as such I have fairly fond memories of it. I recall trying out different types of *joeys*, such as the Aussie U17 football team, before finally hitting upon the correct class of words known as *joey words*. I am extremely impressed with the way the Corey has managed to make the shaded joeys spell something when it would seem extremely restricted to most people.

3. **Puzzle 2.2 - Heckyl and Jeckyl, by Corey Plover**
This was easily the puzzle that received the most feedback in the Hunt. I would also go so far as to say that this is personally my favourite puzzle this year. It was really interesting how there were so many pairs of words, differing by one letter, that had almost the same definition! Again, another masterpiece by Corey. The many guesses such as MAINELUI MVORTYUBK seem to imply that many teams got stuck with 'single letter sets' and were unsure as to what to do with the letters.

4. **Puzzle 4.1 - Mr. Game & Watch, by Muhammad Adib Surani**
This was one of the few puzzles that was created backwards: from an answer phrase into a puzzle. I picked SIMISEAR because STARMIE was a very nice counter, and then it just sort of merged itself with some clocks. I blame Professor Layton for this one, having just completed *Professor Layton and the Unwound Future*[2] earlier this year. They had one too many puzzles which invovled clocks and angles, and I thought about how I could use it to give letters. It turns out that you can give five letters just by measuring the angle between the hour and minute hands to five decimal places.

5. **Puzzle 4.5 - The Contortionist, by Muhammad Adib Surani**
This was by far the puzzle that took me the longest time to design. This is an Oskar's cube, and solving it gives the moves which turns a solved Rubik's cube into the SUPERFLIP position. A number of people have asked me how I constructed this, and I don't think I've ever given a satisfactory answer. But it does involve a lot of programming and trial and error. I essentially used a depth-first search lots of times to ensure that the sequence always produces a tree for the main maze component. Anytime it creates a cyclic component, I would add 4 consecutive moves in the same direction (which changes the maze but not the cube) to try to force it into a tree.

[2] A puzzle video game on the Nintendo DS.

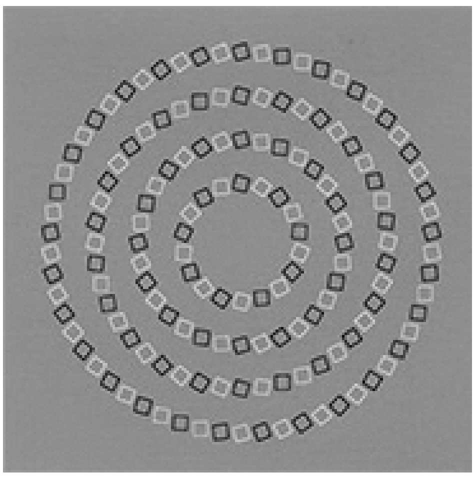
From the feedback, the puzzles that people enjoy most tend to follow a similar pattern: they have an obvious (but not necessarily easy) first step, followed by a lateral next step. And a good lateral step is usually one in which you can obtain an *Aha!* in finite time. As the list above implies, the Day 4 puzzles were some of the most enjoyable, and many teams found it a shame that the Hunt ended early before they managed to attempt most of them.

I have had a really great time organising Puzzle Hunt for the last 3 years. Unfortunately, I may no longer be studying at Melbourne University next year, and will have to pass the torch down to our bright young juniors. If you're interested in running the Puzzle Hunt, or joining the committee (be it writing puzzles, test-solving puzzles, drawing posters, etc.), feel free to come down to the MUMS room and discuss with our friendly team. We're always more than happy to have new members in the committee. It's definitely been one of the most rewarding experiences I've ever had. I may not be organising it in the future, but I'll surely like to help out with future hunts and keep on writing puzzles.

Last but not least, I would like to congratulate the team *Wild Goldfish: Pleasant Quilt* for their outstanding win (see the photo above). Teams *plugh* and *The Elite* (second and third place winners respectively) had also sent their men (and women) down to look *ON TOP OF ALL HELP PHONES*, but *Wild Goldfish* completely outraced them to the act of retrieving the final hidden item, which was a Spiderman with the Vice-Chancellor's head pasted on. Congratulations once again to *Wild Goldfish: Pleasant Quilt*!

— Muhammed Adib Surani

An optical illusion:



Are the circles concentric or overlapping?

Mathematical Miscellany

The international adventures of MUMS

A quick tour of the world demonstrates the lengths people have gone to to rip-off MUMS's name and logo (a Möbius band in the shape of infinity):

Christchurch, New Zealand: We can only hope this little restaurant survived the earthquake.

Singapore: Playground equipment, branded 'Infinity'.

London, UK: Virgin Media with yet another take on the logo.

Not to be outdone, Paradox has also has its share of copycats:

Amsterdam, The Netherlands: Given its location, there's a fair chance this 'coffee shop' deals in substances of an entirely stronger nature than caffeine.

Jokes and anecdotes

One day while lecturing, G H Hardy was in the middle of a long, complicated proof, when he said: 'Now, it is obvious that ... '. Hardy suddenly stopped and stared at the blackboard, confused. To the students' suprise, Hardy put down his chalk and walked out of the room. Ten minutes later, Hardy returned, picked up his chalk, and continued: 'Yes, it *is* obvious that ... '!

∞

The Three Laws of Thermodynamics (paraphrased):

1st Law: You can't get anything without working for it.
2nd Law: The most you can accomplish by work is to break even.
3rd Law: You can't break even.

∞

Q: What do you get if you cross an elephant with a zebra.
A: Elephant zebra sin theta.
Q: What do you get if you cross an elephant with a mountain climber.
A: You can't. A mountain climber is a scalar.

∞

Some dubious methods of proof:[3]

Proof by intmidation – 'It is, of course, trivial to show that X is true.'

Proof by vigorous handwaving –'If you think about it enough, and stare at these cloud-shaped diagrams, it will be clear that X is true.'

Proof by reduction to the wrong problem – 'X is equivalent to this easy problem Y, so X is true.'

Proof by importance – 'A large body of useful consequences flow from X, so X is true.'

Proof by astonishment – 'Surely, you're not implying that X isn't true?'

Proof by haste – 'X is true, but we don't have time to go into the details.'

Proof by mercy – 'X is true, but I'll spare you the details.'

Proof by unclaimed prize – 'No one has claimed our $10,000 prize for a counterexample, so X is true.'

Proof by assuming a contradiction – 'Assuming Y and not Y are true, then X follows.'

[3]This is just a selection from a much larger list, that gets circulated among mathematicians from time to time. See `school.maths.uwa.edu.au/~berwin/humour/invalid.proofs.html` for a more complete version.

A poem in tribute to J H Michell (1863 – 1940)

The wave resistance of a ship (near
a wall) can be calculated. The drag force of a body
moving with constant velocity (thru
water) is zero. A standing wave (in a musical
instrument) *bounces* back 'n' forth. (Another way to think of
a curve is to look at its tangents). How high a wave rises
depends on its wave length. The curvature of
a straight line, is identically to zero. As the wave grows (in
height) the wind pushes it along *further*. The curvature of
a circle, of radius R is constant. One often speaks of
a journey, from A to Z. (Paths and loops
are central to the study of topology). Ocean currents
allow for the larvae of fish, to be carried along
great distances. When the wind blows across the water
the water's surface *changes*. The incredibly mass of
a material 'plopping' into the ocean, creates
major ripples, generating waves travelling at speeds of
up to 300 kph. When the waves reach the beach
they wash inland. The rougher the water
the easier it is, for the wind to transfer its energy.

As the wave enters shallow water, it slows down. . .
The first Australian mathematician (of
World renown) was born in Maldon. In 1854, a Miner
and his wife (Grace né Rowse) migrated
from Devon A Wrangler is a student who has completed
a Mathematics course with first-class honours. (Insects
are endowed with extraordinary power).

He lectured in Mathematics
at Melbourne Uni. for 30 years (and was appointed Professor
in 23). (:The Drawer is the one signing
the cheques). (He never married, and
after his Cambridge days, never travelled again. Stress is
the measure of the force (per unit area) over
a *deformable* body. His researches were into hydro-
dynamics, and elasticity. A beam deforms, and stresses
develop inside. A natural phenomena includes
a volcanic eruption. His only publication (after 1902) was
a textbook; 20^{th} century rigour with 18^{th} century clarity.

He hated the company of those who
seemed *insincere*. (There are numerous nerves that lead
to the facial muscles). He was punctual; found
relaxation in classical music; A sidecar

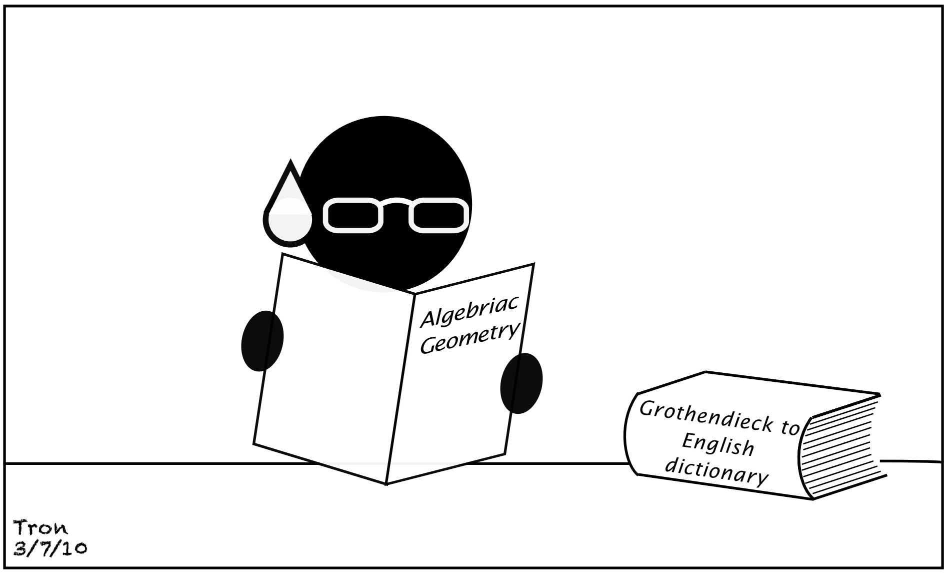
on a motor cycle; a bathtub. He loved playing the organ
and gardening. A connoisseur and lover of Australian natives.
So why did he stop, publishing?
Pneumonia has no "one" cause. Increase the tension on
a string, results in a higher frequency. Standing
in-line, is *not* pushing yourself forward. E. coli (and other
bacteria) swim by the action of rotary motion;
in groups of two to six. The *problem* is simply: to work out
the *drag-force* on a steadily moving ship. The only
solution for boundary-value problems *then* were
for special shapes like circles, and spheres. He goes on
to prove explicitly, that as the resistance vanishes
at high speeds $R \to 0$ as $U \to \infty$.
Draw the shades down on the next 25 years.
What did he expect? Instant salvation for a wonderful
formula? Barely a ripple))))))).
Thomas Havelock made explicit use of his integral (in
1923). No one had noticed it before. The integrals scale with
the square of the ship's beam. Anxiety is followed
by a "for", "about", or concerning". A thing is important if
its relevant, and of a great concern to somebody.
There is no single cause of aging. Ammonia smells.
He lived in a back water, called Australia.
The *integrals* must've frightened 'em, like the giant tentacles
of an octopus in the Pacific Ocean or something.
He collapsed into obscurity. (To *deprive* of confidence is
to discourage). For simplicity sake, the water is taken
to be *infinitely* deep. But, physical strength at 70 is
only that at 20. For a string of finite stiffness the harmonic
frequency, departs progressively from its harmonics.
The distinctive sound of the violin
is a product of all its parts. The "length" of the violin
determines the wavelength, of the sound. A twisted wire can
add style, dimension, and elegance, flash and/or fun
to jewellery. (And almost (as an after
thought) he included in his paper (on its last two pages)
a study of the "shallow-water" case) (ie of the limit
of the depth of water, as it vanished
into nothing). He's buried in the Boroondara cemetery.

— π.O

π.O is a Melbourne poet. His work can be found in selected bookstores.

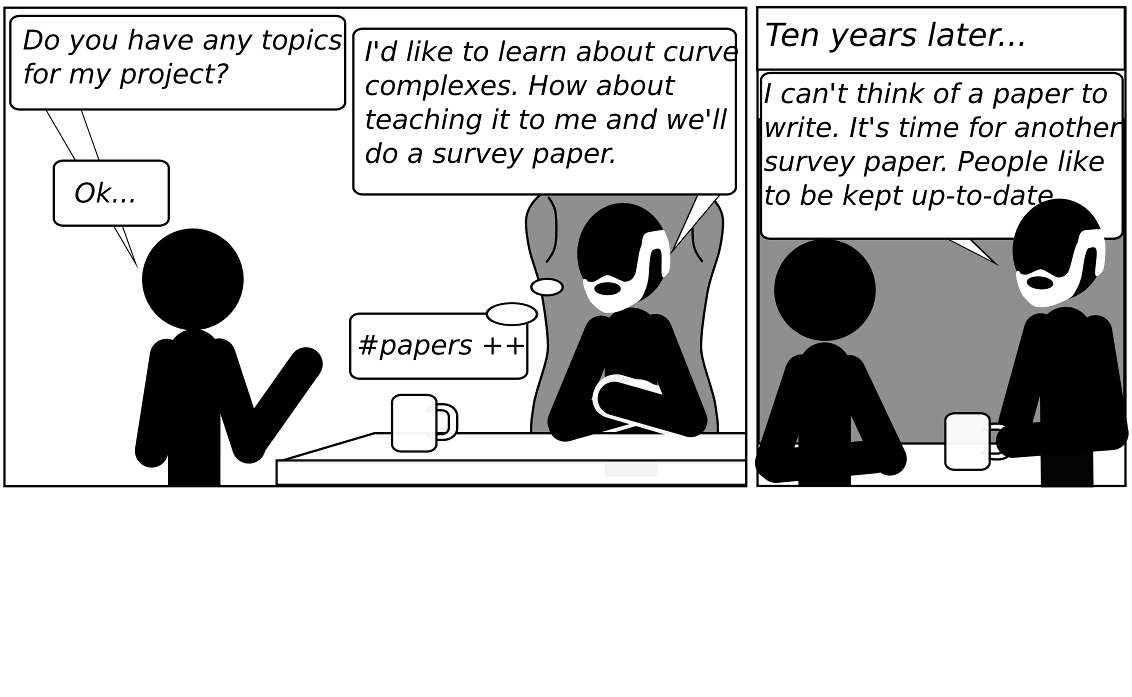
Paradox **Comics**

Dummy's Guide to Algebriac Geometry

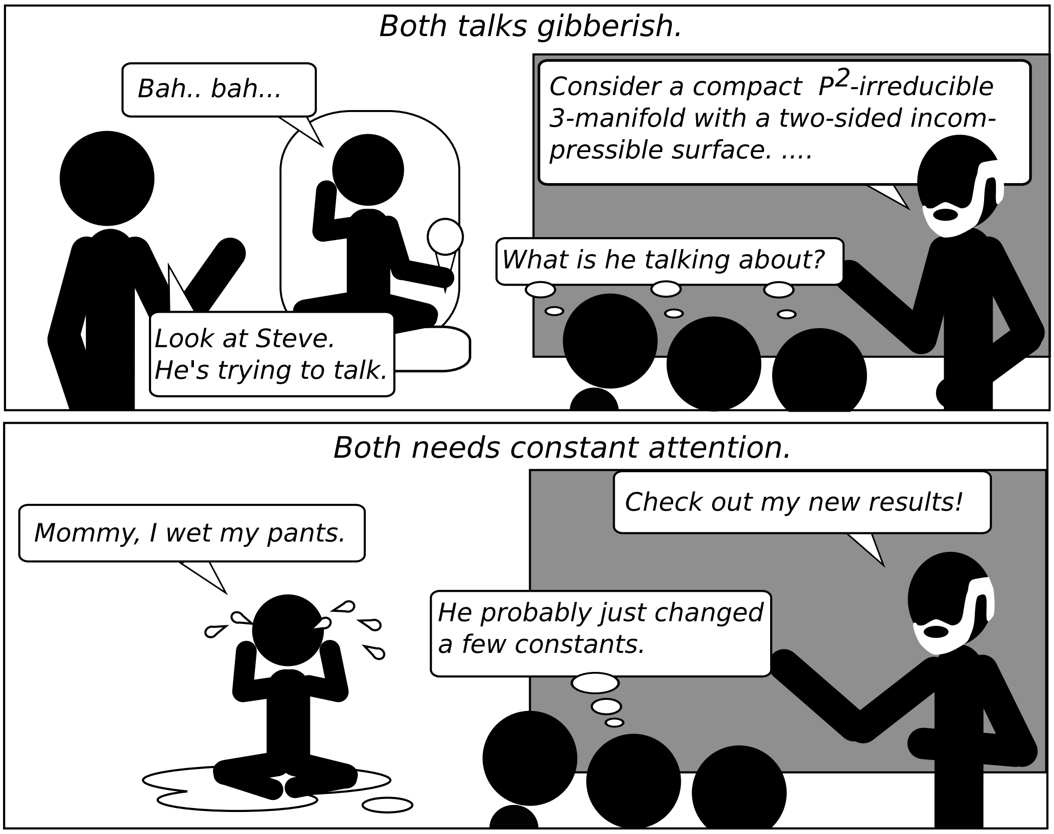


Tribute to Abstruse Goose

SURVEY - a way to increase your number of publications



MATHEMATICIANS ARE GROWN-UP BABIES - PART 1



To be continued ....

— Tharatorn Supasiti

Maths in the News 2:
Is it time to rethink the the leap year? According to British mathematician Adam Goucher (The Guardian, 11 March 2011), the current rule (years divisible by 4 are leap, with the exception of years divisible by 100 but not 400) results in our calendar shifing by an average of 26 seconds a year, which will put it one day out by the year 4000 (significantly better than the old rule – all years divisible by 4 – which put us a day out every 150 years). Instead, Goucher proposes a new, clearer, rule: years divisible by 4, but not divible by 128, are leap. Using this rule, it would take half a million years for the calander to be a day out.

A Few Words on the Gamma Function

Introduction

If you wanted to calculate binomial coefficients, the probability that no one at a party has the same birthday, or the number of ways of dealing 4 hands of 13 cards from a standard deck, you would most likely use the factorial function. The factorial of n is defined as

$$n! = n \times (n-1) \times \cdots \times 1$$

for all natural numbers $n \geq 1$, and 1 when $n = 0$. Another common definition is recursive. Set $0! = 1$ and let

$$n! = n \times (n-1)!$$

Notice that the factorial is not defined for any old real number, just the non-negative integers. This raises an interesting question about interpolation: can we construct a *nice* function for positive real numbers which is equal to the factorial function where the factorial is defined? *Nice* means that the function has a reasonably simple definition and nice properties like differentiability.

The gamma function, $\Gamma(x)$, is a positive answer to this question.

Definition

For $x > 0$, we define $\Gamma(x)$ by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Readers with a little analysis under their belts may be concerned that the integral does not converge. To see that it does, split the integral up:

$$\Gamma(x) = \int_0^1 t^{x-1} e^{-t}\, dt + \int_1^\infty t^{x-1} e^{-t}\, dt.$$

The second integral converges by the integral test[1], as the sum

$$\sum_{n=1}^{\infty} n^{x-1} e^{-n}$$

[1] A test that bounds the integral above by an infinite sum, which is then tested for convergence.

is finite for all positive x. If $x > 1$, the first integral is proper[2] and therefore finite. If $x < 1$, we compare it to

$$\int_0^1 t^{x-1}\,dt.$$

Properties

Since e^{-t} gets very small as t approaches ∞, we have the following 'fixed point' for the gamma function:

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1.$$

We can now show that the gamma function more or less solves our interpolation problem. Since the expression

$$\frac{t^{-x}}{x}e^{-t}$$

also vanishes as $t \to \infty$, we can integrate by parts to obtain

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}\,dt \\ &= \frac{t^x}{x}e^{-t}\Big|_0^\infty - \frac{1}{x}\int_0^\infty t^x(-e^{-t})\,dt \\ &= \frac{1}{x}\Gamma(x+1)\end{aligned}$$

or

$$\Gamma(x+1) = x\Gamma(x).$$

This is called the *functional equation* for $\Gamma(x)$. The factorial satisfies a similar equation. Indeed, a simple induction shows that

$$\Gamma(n+1) = n!$$

so the gamma function (or rather, the function $\Gamma(x-1)$ really does generalise the factorial.

[2] A *proper* integral is an integral computed over a finite domain and where the integrand is also finite on this domain.

The last property we mention is called *log convexity*. A real function f is said to be convex if the line joining any two points on the graph of the function lies above the graph (apart from the endpoints). A useful equivalent formulation is as follows: if $1/p + 1/q = 1$ and $p, q > 1$, then

$$f\left(\frac{x}{p} + \frac{y}{q}\right) \leq \frac{f(x)}{p} + \frac{f(y)}{q}.$$

It turns out that the logarithm of the gamma function is convex. Hence, it is log convex. To show this, we can use a result from real analysis called *Hölder's inequality*. If f, g are integrable on the interval (a, b), with a, b possibly infinite, then

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \left\{\int_a^b |f(x)|^p\,dx\right\}^{1/p} \left\{\int_a^b |g(x)|^q\,dx\right\}^{1/q}$$

with p, q as above. As $\Gamma(x)$ is the integral of a nonnegative function, it is nonnegative and the absolute value signs may be omitted:

$$\begin{aligned}\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) &= \int_0^\infty t^{\frac{x}{p}+\frac{y}{q}-1} e^{-t}\,dt \\ &= \int_0^\infty \left(t^{\frac{x-1}{p}} e^{-\frac{t}{p}}\right)\left(t^{\frac{y-1}{q}} e^{-\frac{t}{q}}\right)\,dt \\ &\leq \left\{\int_0^\infty t^{x-1}e^{-t}\,dt\right\}^{1/p} \left\{\int_0^\infty t^{y-1}e^{-t}\,dt\right\}^{1/q} = \Gamma(x)^{1/p}\Gamma(y)^{1/q}.\end{aligned}$$

Since $\log$ is increasing, $\log(xy) = \log(x) + \log(y)$ and $\log(x^p) = p\log(x)$,

$$\begin{aligned}\log\left\{\Gamma\left(\frac{x}{p} + \frac{y}{q}\right)\right\} &\leq \log\left(\Gamma(x)^{1/p}\Gamma(y)^{1/q}\right) \\ &= \frac{1}{p}\log\left(\Gamma(x)\right) + \frac{1}{q}\log\left(\Gamma(y)\right).\end{aligned}$$

Hence, $\log\left(\Gamma(x)\right)$ is convex. What is so interesting about these particular properties of the gamma function? Remarkably, the Danish mathematicians Bohr and Mollerup proved that if $f(x)$ is a log convex function on the positive reals such that $f(1) = 1$, and which satisfies the 'functional equation', then $f(x) = \Gamma(x)$. In other words, these three properties determine the gamma function.

The Bohr-Mollerup theorem

The proof of Bohr and Mollerup's theorem yields a beautiful expression for Γ. First, we use *another* formulation of convexity. (You might like to show that the characterisations are equivalent.) It turns out that a function f is convex if and only if, for any point x and $b > a > 0$,

$$\frac{f(x+a) - f(x)}{a} \leq \frac{f(x+b) - f(x)}{b}.$$

Because $\Gamma(x)$ is log convex, it follows that for any natural number $n \geq 2$ and $0 < x \leq 1$,

$$\begin{aligned}\log(\Gamma(n)) - \log(\Gamma(n-1)) &\leq \frac{\log(\Gamma(n+x)) - \log(\Gamma(n))}{x} \\ &\leq \log(\Gamma(n+1)) - \log(\Gamma(n)).\end{aligned}$$

Some elementary manipulation using the properties of Γ and $\log$ yields

$$(n-1)^x(n-1)! \leq \Gamma(n+x) \leq n^x(n-1)!$$

The functional equation may be repeatedly applied to get

$$\Gamma(x+n) = x(x+1)\cdots(x+n-1)\Gamma(x).$$

We combine these two results, replacing n with $n+1$ on the left since the first inequality holds for all $n \geq 2$:

$$\frac{n^x n!}{x(x+1)\cdots(x+n)} \leq \Gamma(x) \leq \frac{n^x n!}{x(x+1)\cdots(x+n)} \frac{x+n}{n}.$$

As a result,

$$\Gamma(x)\frac{n}{x+n} \leq \frac{n^x n!}{x(x+1)\cdots(x+n)} \leq \Gamma(x).$$

If we let $n \to \infty$, the expression on the left approaches $\Gamma(x)$. Hence,

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)}$$

for $0 < x \leq 1$. We then use the functional equation to show that $\Gamma(x)$ is defined by this expression for all $x > 0$.

Proving the Bohr-Mollerup theorem is now straightforward. To obtain the limit expression above, we have *only* used three properties of Γ: it is log convex, satisfies the 'functional equation', and has a fixed point at 1. Therefore, *any* function with these properties is equal to

$$\lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)},$$

and must be identically equal to the gamma function for $x > 0$.

The beta function and the Gaussian integral

Bohr and Mollerup's result is very useful for proving identities involving the gamma function. For example, take the *beta function*, defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

where $x > 0$ and $y > 0$. Some routine manipulation shows that the function $\psi(x)$, given by

$$\psi(x) = \frac{\Gamma(x+y)}{\Gamma(y)} B(x,y),$$

has the three required properties. (Interested readers can consult Walter Rudin's *Principles of Mathematical Analysis*, p. 193–94, for details.) This means $\psi(x) = \Gamma(x)$, or

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

This relation may be applied to the Gaussian integral from probability[3] The change of variables $t = \sin^2\theta$ in the definition of $B(x,y)$, along with the trigonometric identity $1 - \sin^2\theta = \cos^2\theta$, yields

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta.$$

[3]The *Gaussian integral* is $\int_{-\infty}^{\infty} e^{-s^2}\,ds$ which can be shown to have value $\sqrt{\pi}$. The integral is crucial in probability theory because the ubiquitous normal distribution has density function proportional to e^{-s^2}.

Setting $x = y = 1/2$ and using the value $\Gamma(1) = 1$ then gives

$$\Gamma(\tfrac{1}{2})^2 = 2\int_0^{\pi/2} d\theta = \pi.$$

As a result, we have the most famous non-integer value of the gamma function, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. But

$$\Gamma(\tfrac{1}{2}) = \int_0^{\infty} t^{-1/2}e^{-t}\,dt.$$

Making the substitution $t = s^2$, we get $dt = 2s\,ds$ and

$$\Gamma(\tfrac{1}{2}) = 2\int_0^{\infty} e^{-s^2}\,ds = \int_{-\infty}^{\infty} e^{-s^2}\,ds$$

since the integrand is an even function. We conclude that

$$\int_{-\infty}^{\infty} e^{-s^2}\,ds = \sqrt{\pi}.$$

Conclusion

Mathematicians have a penchant for generalisation. We started with the problem of extending the factorial to positive reals, but the process can be taken further. In fact, Γ can be defined for all *complex* numbers except the non-positive integers.

But I hope I have shown that the theory of the real gamma function is rich and interesting in its own right. On that note, here are some elegant identities readers might like to try proving using the methods of this article or otherwise:

- $\Gamma(x) = \dfrac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\dfrac{x}{2}\right)\Gamma\left(\dfrac{x+1}{2}\right)$
- $\Gamma\left(n + \dfrac{1}{2}\right) = \dfrac{(2n)!}{4^n n!}\sqrt{\pi}$
- $\Gamma(x) = \dfrac{1}{x}\displaystyle\prod_{n=1}^{\infty}\dfrac{(1+\frac{1}{n})^x}{(1+\frac{x}{n})}.$

— David Wakeham

More Great Lives

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

—John von Neumann [1]

In the previous issue of Paradox,[2] I discussed five great mathematicians in terms of their lives outside their mathematical careers, particularly concerning their achievements in other fields as well as their amusing or tragic idiosyncrasies. This time I will discuss three very well-known and prominent mathematicians with a greater focus on how mathematics played a role in their wider lives. Herein you will read about, among many other things, what Leibniz was supposed to be doing for his wealthy patrons while he was involved in the infamous calculus dispute with Newton, why the Prussian Emperor would refer to Euler pejoratively as 'Cyclops', and why Gauss forbade his children from going into mathematics or science.

Gottfried Leibniz (1646 – 1716)

A German philosopher and mathematician who primarily wrote in Latin, French, and German, Gottfried Leibniz is best known for developing the infinitesimal calculus independently of Isaac Newton. He did so with superior notation that has been widely used ever since. In philosophy, Leibniz argued optimistically that the Universe is the best possible one a God could have created and that all the evil elements were balanced by good in an optimal manner.[3] He is widely considered second only to the Roman Emperor

[1] Keynote speech at the first national meeting of the Association for Computing Machinery in 1947.

[2] See *Great Lives*, Issue 1, 2011.

[3] For more on Leibniz's solution to the theodicy, the problem of evil, see http://web.maths.unsw.edu.au/~jim/think.pdf

Marcus Aurelius[4] as a philosopher with the most experience with practical affairs of state.

It was his good fortune to have had a father who was a Professor of Moral Philosophy at the University of Leipzig. It was not such good fortune to lose his father to death when he was just six years old, yet he was given free access to his father's personal library from the age of seven onwards, allowing him to study a wide variety of advanced works that he would not have otherwise been permitted to read until much later. Since it was mostly written in Latin, Leibniz became so proficient in Latin that he was apparently able to compose three hundred hexameters of Latin verse in a single morning for a special school event when he was just thirteen.

Having completed his formal education with numerous degrees by the age of twenty-one, he found a wealthy noble patron to support his endeavours instead of beginning a career as a university professor. While his first patrons showed some genuine interest in intellectual matters and allowed him considerable freedom in his endeavours, the Dukes of Hanover, whom he eventually ended up serving, were not as interested in scholarly discussions and implored Leibniz to write on the family's history. Leibniz compiled this genealogy for the remaining three decades of life and used the project as an excuse to travel a lot around Europe. He never completed it.[5]

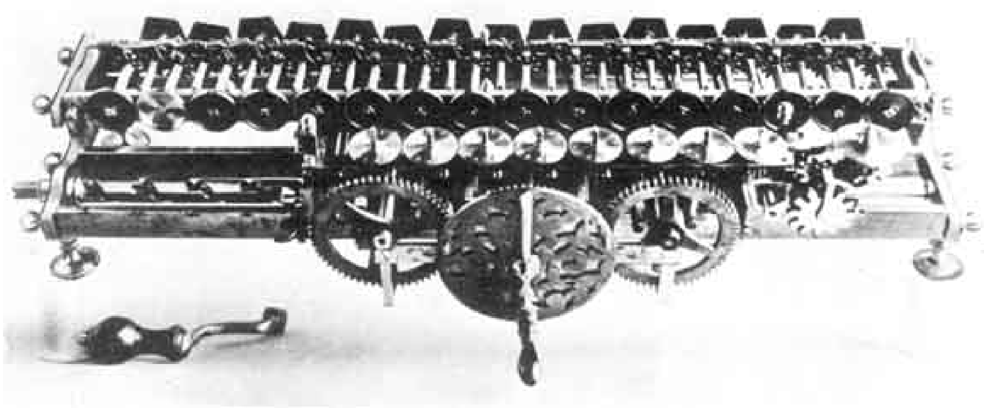
It was while making grand tours of Europe that Leibniz maintained correspondence with up to as many as six hundred collaborators while also doing much legal and diplomatic work for his patrons. He visited mines and unsuccessfully tried to negotiate export contracts for lead from these mines. He had embarked upon this particular project so that he could fund an academy in which a small group of capable individuals would be able to compile an encyclopaedia compilation of a universal artificial mathematical language in which all knowledge could be expressed. Despite his optimism, the academy was never established.

Leibniz was also one of the most prolific inventors in the field of mechanic calculators. He bettered Pascal's calculator that could add and subtract by adding automatic multiplication and division to it. He also incorporated into this improvement a stepped drum later named the Leibniz wheel which was more or less used for three centuries in mechanical calculators until the devel-

[4] Who is as well-known for his philosophical work *Meditations* as he is for his achievements as Roman Emperor.

[5] Martin Davis, *Engines of Logic* (2000), pp. 13-15.

opment of electronic calculators in the mid 1970s. The wheel consisted of a cylinder with a set of teeth of incremental length coupled to a counting wheel.



Leibniz's calculator.

He also refined the binary number system, thus beginning to lay the foundations for digital computing. He was fascinated by how the Chinese *I Ching* hexagrams correspond to the binary numbers from 0 to 111111, concluding that the Chinese had made major accomplishments in the pursuit of a universal language. He went so far as to consider the possibility that the Chinese characters were an unwitting form of this universal language although he did not pursue this idea much further.

Leibniz anticipated many ideas well ahead of his time. He called for a European confederation whose members would represent entire nations governed by a council or senate very much like today's European Union. He argued against his great rival Isaac Newton among other things (such as the infamous calculus dispute) that space, time, and motion are relative and not absolute, an idea that would only really be deeply explored in the time of Albert Einstein. He also believed that Europe would one day adopt a uniform religion, an idea to which he devoted considerable intellectual and diplomatic effort in seeking to reconcile the many divisions of Christianity with limited success. What he most likely did not anticipate was that one day a popular brand of biscuits in Germany, *Leibniz-Keks*, would be named after him.[6]

[6]These biscuits honour Leibniz because he was a resident of Hanover, where the company is based.

Leonhard Euler (1707 – 1783)

One of the most prolific mathematicians ever, Euler[7] was a Swiss mathematician and physicist. He is widely considered to be the pre-eminent mathematician of the 18th century and one of the greatest of all time. The French mathematician and astronomer Pierre-Simon Laplace would later exhort:

> *Read Euler, read Euler, he is the master of us all.*

Euler was also a devout Christian and a believer in biblical inerrancy who wrote apologetics and argued against prominent atheists throughout his life.[8]

Euler was born in Basel, Switzerland to a pastor who was a friend of the Bernoulli family. He received Saturday afternoon lessons from Johann Bernoulli,[9] who discovered Euler's mathematical talent and convinced his father that he should be studying mathematics rather than theology, Greek, and Hebrew.

Euler spent most of his adult life in St. Petersburg, Russia and in Berlin, Prussia. He first worked at the Imperial Russian Academy of Sciences in St. Petersburg for fourteen years, then at the Berlin Academy for twenty-five, and then back at the St. Petersburg academy for the rest of his life.

When he worked in Berlin, he was asked to tutor the Princess of Anhalt-Dessau, niece of the Prussian King Frederick. He wrote over two hundred letters to her which were later compiled into a book that became more widely read than any of his mathematical works. While these letters were mostly concerned with various subjects relating to physics and mathematics, they also re-

[7] Pronunciation 'Oiler'.

[8] For this he is commemorated by the Lutheran Church on their Calendar of Saints on 24 May.

[9] Johann Bernoulli was also the tutor of the French mathematician Guillaume de l'Hôpital. It is believed that the former actually discovered l'Hôpital's rule, although he had signed a contract with l'Hôpital giving the latter the right to use Bernoulli's discoveries as he pleased. As a result, l'Hôpital published the first textbook on infinitesimal calculus which mainly consisted of Bernoulli's work including what is now known as l'Hôpital's rule.

flected Euler's personality and religious beliefs. However, he was eventually forced to leave Berlin because Frederick had come to regard Euler as unsophisticated, especially compared to the French philosopher Voltaire whom he held in very high esteem. This is because Euler's conventional religious beliefs and limited training in rhetoric together with his tendency to debate matters he knew little about made him a frequent target of Voltaire's wit.

During Euler's second period at the St. Petersburg academy, he was asked by the Russian Empress Catherine the Great to confront the visiting French philosopher Denis Diderot concerning his arguments for atheism on the basis that they would otherwise corrupt members of her court. Diderot was thus informed that a mathematician had proved the existence of God to which he agreed to view the proof in Catherine's court. Euler advanced towards Diderot and announced:

Sir $\frac{a+b^n}{n} = x$ *hence, God exists—reply!*

According to the story, Diderot stood dumbstruck as laughter erupted in the court. Embarrassed, he asked to leave Russia, to which the Empress graciously complied.[10]

Euler's eyesight worsened throughout his life, so much so that Frederick began referring to him as 'Cyclops' while he stayed in Germany. He became nearly blind in his right eye after a fatal fever when he first stayed in St. Petersburg, but instead blamed it on his painstaking work in cartography at that time. He later suffered a cataract in his good left eye, leaving him almost totally blind. But he compensated for the loss of his sight with his mental calculation prowess and photographic memory.

Among Euler's feats included repeating Virgil's *Aeneid* from beginning to end without hesitation and he could even indicate which lines were first and last on every page of his copy. He was amazingly still able to produce on average one mathematical paper every week almost ten years after he became almost totally blind.

[10] This story is unfortunately busted by B. H. Brown in the American Mathematical Monthly in 1942. He provides evidence that Diderot was a more than capable mathematician and claims that he did not reply to Euler for other reasons. Brown's short paper can be found at `http://www.fen.bilkent.edu.tr/~franz/M300/bell3.pdf`

Carl Friedrich Gauss (1777 – 1855)

Often referred to as the Prince of Mathematicians, Gauss was a German mathematician and scientist who made significant contributions to many fields of science ranging from number theory to astronomy and optics. Nevertheless, he considered mathematics to be 'the queen of sciences'. He was born to poor working-class parents who had never recorded the date of his birth, remembering only that he had been born on a Wednesday eight days before the Feast of the Ascension, which itself occurs forty days after Easter. Gauss would one day solve this puzzle while also deriving methods to compute the date of Easter in both the past and the future.

He was an industrious perfectionist who refused to publish work that he did not consider to be complete and above criticism. He was not a prolific writer and his personal motto was *pauca sed matura*.[11] While he was a child prodigy whose talent had been recognized, his breakthrough came at the age of nineteen when he demonstrated that any regular polygon with a number of sides which is a Fermat prime[12] can be constructed by compass and straightedge.[13] The discovery inspired Gauss to choose mathematics instead of philology[14] as a career.

There are many stories of Gauss' prodigious abilities, most of them apocryphal. According to one, his gifts were noticed at age three when he corrected, mentally, and without fault, an error his father had made on paper while calculating finances. Another involved his teacher punishing Gauss by asking him to add all of the integers from 1 to 100 which he reputedly did

[11] Latin for 'Few, but ripe.'

[12] A Fermat number, named after Pierre de Fermat who first studied them, is a positive integer of the form: $F_n = 2^{2^n} + 1$ where n is a non-negative integer. In addition, every prime of the form $2^n + 1$ is a Fermat number, and such primes are called Fermat primes.

[13] Compass and straightedge is the construction of lengths, angles, and other geometric figures using only an idealized ruler and compass.

[14] The study of languages in written historical sources; a combination of literary studies, history, and linguistics.

correctly within seconds. Later in life, he was interrupted in the middle of a problem to be told his wife was dying, to which he apparently replied:

> *Tell her to wait a moment until I'm done.*

When he was once asked how he had been able to predict the trajectory of the dwarf planet Ceres with such accuracy, Gauss claimed that he used mentally calculated logarithms without using tables. Gauss also usually did not like to show the intuition behind his often very elegant proofs, instead preferring to erase all traces of how he discovered them. He justified this on the grounds that he believed all analysis must be as concise as possible.

While he did have some students, Gauss disliked teaching. He also only attended a single scientific conference in Berlin in 1928. Nevertheless, many of his students became prominent mathematicians, such as Richard Dedekind,[15] Bernhard Riemann,[16] and Friedrich Bessel.[17] He also corresponded with and had a profound influence on Sophie Germain, who made the first substantial contributions towards solving Fermat's last theorem. However, they never met despite their friendship and correspondence ceased when Gauss eventually lost interest in number theory.

Gauss plunged into a depression from which he never fully recovered following the death of his first wife and one of his sons soon afterwards. He clashed with his other sons, forbidding them from entering science or mathematics for 'fear of sullying the family name'. One of them, Eugene, wanted to study languages, but Gauss wanted him to be a lawyer instead. Unable to resolve their differences, Eugene left in anger and emigrated to the United States.[18] In contrast, Gauss' youngest daughter, Therese, looked after her father's household all the way until his death when she was thirty-nine; she married soon afterwards.

— Kristijan Jovanoski

[15] Dedekind made significant contributions to ring theory and the foundations of real numbers.

[16] Whom we have to thank for the unresolved Riemann hypothesis.

[17] He generalized what are now known as Bessel functions.

[18] While Eugene eventually became quite successful, it took him many years to counter the bad reputation he had among Gauss' friends and colleagues.

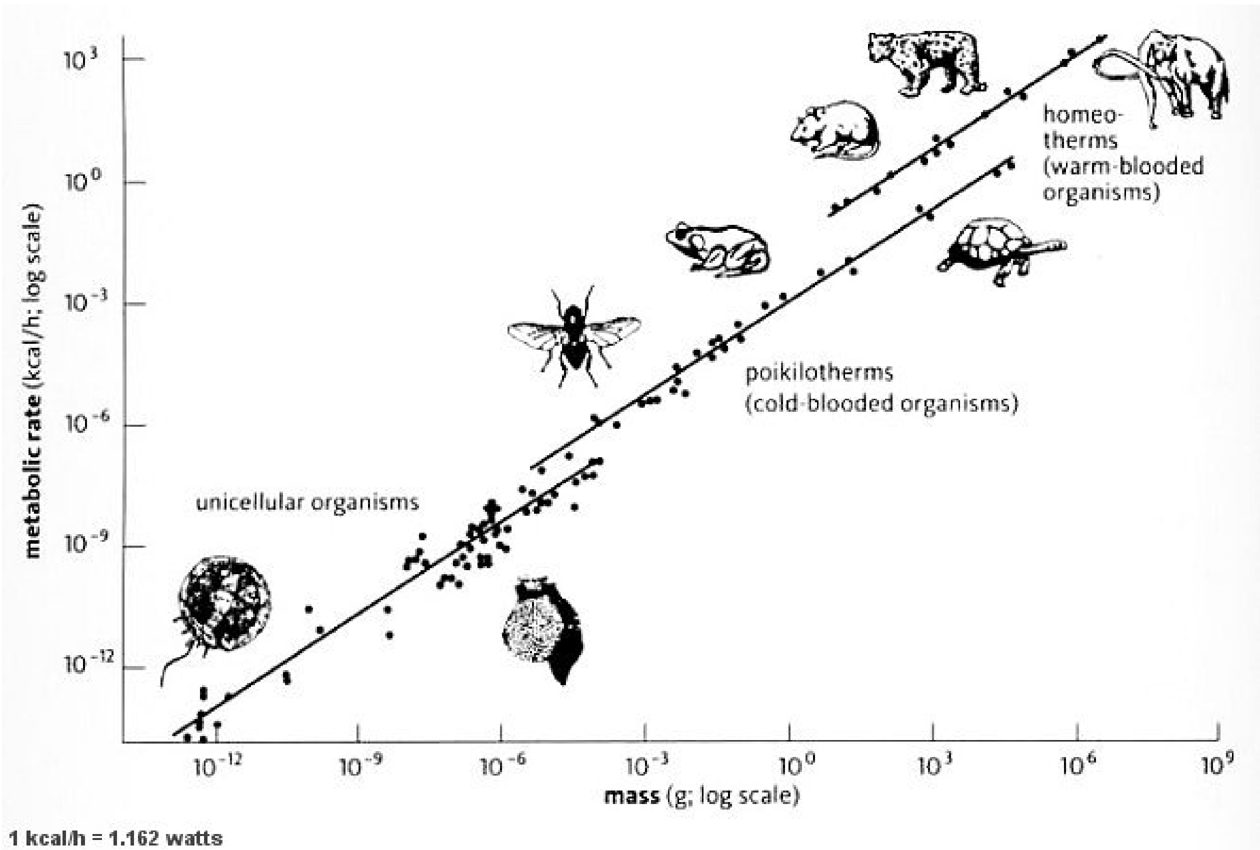
Fractal Geometry in Nature

I recently came across a good book by Richard Dawkins, who is well-known for his fundamentalist atheism. And no, the book is not *The God Delusion*, but *The Ancestor's Tale*. It describes a journey back to the origin of life itself.

Several of the discussions in the book are fascinating, particularly the relationship between metabolic rate and the body mass of a species: the metabolic rate is expected to increase with increasing mass. The figure below is a log-log graph with metabolic rate on the y-axis and mass on the x-axis, with its slope telling us about the relationship between the two. Suppose that M is the body mass and k is the metabolic rate. In slope-intercept form, the graph tells us that $\log k = s \log M + c$, so:

$$10^{\log k} = 10^{s \log M + e} \quad \Rightarrow \quad k = EM^s.$$

The question of interest here is: *what value of s can we expect*?



Log-log graph plotting the metabolic rate of cetain species against their body mass.

First, we might like to estimate upper and lower bounds for the slope. For the upper bound, it is reasonable to expect that the metabolic rate increases linearly with respect to its body mass. So we can expect that at most, $k \propto M$. That is, the slope (or s) should not exceed 1.

For the lower bound, we need to first assume that the metabolic rate depends on the surface area. For example, bacteria require nutrients from their environment for survival. In this case, nutrients must permeate their membranes, meaning that the rate at which the nutrients enter the cell is proportional to its area.

Since Darwin's theory of evolution, natural selection has been the most successful theory yet to describe the current diversity of life, and it is also reasonable to assume that natural selection will favour organisms with a greater surface area. From our assumption above, we know that the total area is proportional to a square of length ($A \propto L^2$), since the dimension of a surface is 2. Assuming that density is uniform, we may conclude that body mass is proportional to its volume V, which in turn is proportional to L^3. Thus,

$$k \propto A \propto L^2 = (L^3)^{\frac{2}{3}} \propto V^{\frac{2}{3}} \propto M^{\frac{2}{3}} \tag{1}$$

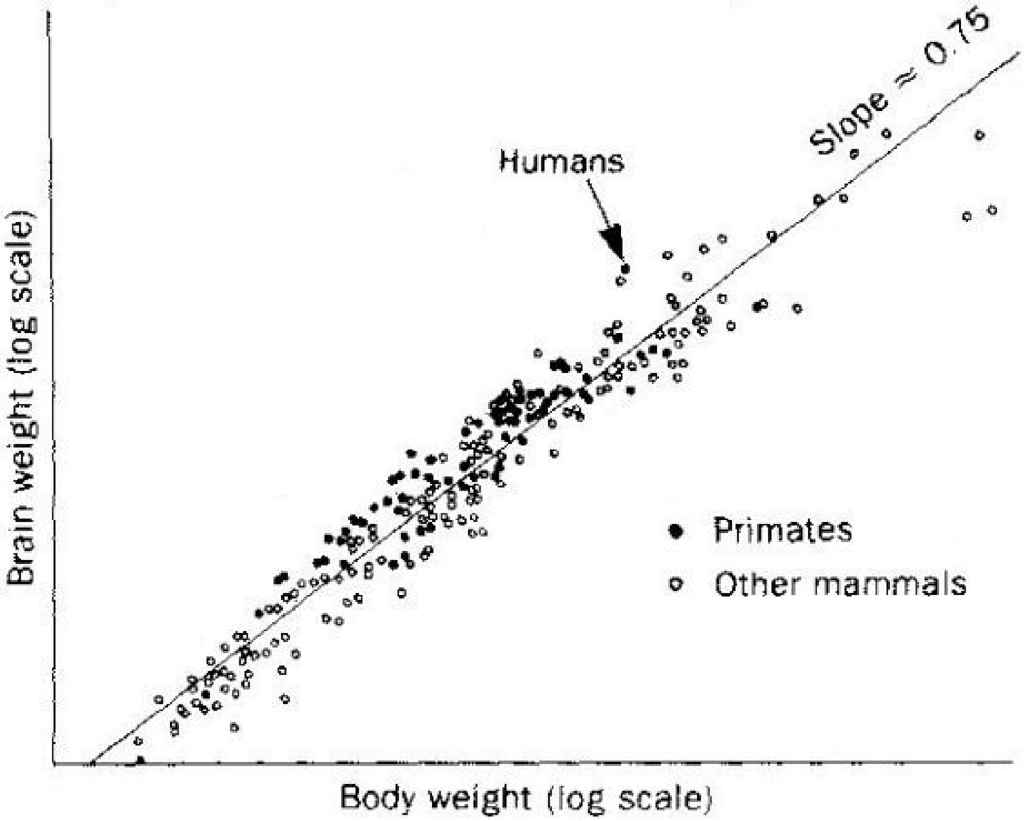
and so the slope should be at least ${}^2/_3$.

Indeed, from the above graph, we can see that the slope is approximately ${}^3/_4$. Note that this slope fits nicely in between ${}^2/_3$ and 1, our lower and upper bounds.

What is more amazing is that many aspects of living things including brain size (see the below graph) follow this ${}^3/_4$-rule (more generally, the slope is a multiple of ${}^1/_4$). This is known as Kleiber's Law. We shall try to understand this phenomenon mathematically. This is based on the work by Geoffrey West, James Brown and Brain Enquist.[1] Before we proceed further, I would like to point out that this law covers 20 orders of magnitude. There are not many laws that are satisfied empircally on this scale.

As aforementioned, metabolic rate is expected to be proportional to the rate at which nutrients enter the body. Using this, West et al reduced this problem to a transportation problem. When the organism is small, this is proportional to its surface area, since every cell is close to the external environment. Hence, we should expect that the rate is proportional to the area of the outer wall.

[1] West, G, Brown, J H and Enquist, B J, 'The fourth dimension of life: fractal geometry and allometric scaling of organisms', *Science* (1999) 284 (5420), 1677–9.



Graph plotting the brain mass of certain species against their body mass.

However, in a larger organism, most of the cells are far away from the external environment. Thus, the organism needs a circulation system to deliver food to individual cells. Insects develop a highly-branched pipework for circulating the air around their bodies, whereas in humans this pipework is restricted to our lungs, which are in turn connected to blood vessels. So, what does this mean mathematically?

First, we would like to understand how the $^2/_3$-rule may arise. The total surface area A can be parametrised as a function of various length scales $L_1, L_2, \ldots$ We may think of each L_i as the average length of arms or legs for a particular species. So, we may write the total surface area as $A = A(L_1, L_2, \ldots)$. Precise formulation is unnecessary for this calculation. This can be expressed as

$$A = L_1^2 \Phi\left(\frac{L_2}{L_1}, \frac{L_3}{L_1}, \ldots\right), \tag{2}$$

where Φ is a dimensionless function. The factor of L_1^2 is there because of dimensionality. Suppose that we scale the organism by a factor of α. Then the resultant area A' is exactly

$$A(\alpha L_1, \alpha L_2, \ldots) = \alpha^2 L_1^2 \Phi\left(\frac{L_2}{L_1}, \frac{L_3}{L_1}, \ldots\right) = \alpha^2 A. \tag{3}$$

Similarly, the volume can be expressed as $V = V(L_1, L_2, \ldots)$ and

$$V = L_1^3 \Psi\left(\frac{L_2}{L_1}, \frac{L_3}{L_1}, \ldots\right).$$

After the scaling of factor α, we get a similar expression, $V' = \alpha^3 V$ and hence:

$$\frac{A'}{V'^{\frac{2}{3}}} = \frac{\alpha^2 A}{(\alpha^3)^{\frac{2}{3}} V^{\frac{2}{3}}} = \frac{A}{V^{\frac{2}{3}}}$$

We find that we end up with equation (1). But inversely, we also see that $L_i \propto V^{\frac{1}{3}}$ and thus we may write $V = AL$, where L is some length function on L_i.

There are problems with this approach. I will outline them here:

1. It ignores the fact that the surface area through which nutrients can enter is not the same as the surface area A. This means the metabolic rate cannot be proportional to the surface area. The area we need to focus on instead is the total area of exchange surface area, such as that of the lung for air or the total surface area of blood capillaries.
2. The supply system occupies spatial volume inside the body. If one doubles the number of cells, one will need to increase the volume of supply system too. For example, an elephant would require more blood than a human would.

With these new insights, let a be the total effective exchange area from (1) and let v be the total volume of active material from (2). Now, we need to assume the following:

1. The transportation system is hierarchical fractal-like. For example, in human lungs, incoming air is separated by the two primary bronchi. These are further subdivided multiple times until reaching the terminal bronchioles. The upshot is to show that the respiratory system has hierarchy. One can represent it as a rooted tree with many branches coming out at each node.
2. We wish to express the area a in a similar fashion to A. However, there is an invariant length factor, denoted by l_0, which we will need to consider.

The motivation for this factor is that there is a physical limit to how small a certain biological object can be. For example, a blood capillary cannot be smaller than the size of red blood cells. Otherwise, it cannot transport the blood. As in this example, the limitation generally occurs near the ends of the system.

3. Independent of the organism, the supply system would have most likely minimised inefficiency (or waste) in the system via evolution as a result of natural selection. It turns out that in general such a system occupies roughly the same fractional space inside the body. In mammals, the volume of blood occupies about six to seven per cent of the body. This means that $v \propto V$ or that the total volume of active material is proportional to the body volume.

From the second assumption, we may write the effective exchange area a as a function of the length scale l_i and an invariant length l_0. That is, $a = a(l_0, l_1, l_2, \ldots)$. Proceeding as usual, we have

$$a = l_1^2 \phi \left(\frac{l_0}{l_1}, \frac{l_2}{l_1}, \frac{l_3}{l_1}, \ldots \right)$$

where ϕ is a dimensionless function of l_i/l_1. The l_1^2 factor is justified by considering scaling l_i by a factor $1+\epsilon$, where ϵ is infinitesimally small. Taking ϵ to zero, this should approach the analog of equation (2).

If the organism is scaled by a factor of α, then:

$$a' = \alpha^2 l_1^2 \phi \left(\frac{l_0}{\alpha l_1}, \frac{\alpha l_2}{\alpha l_1}, \frac{\alpha l_3}{\alpha l_1}, \ldots \right) = l_1^2 \phi \left(\frac{l_0}{\alpha l_1}, \frac{l_2}{l_1}, \frac{l_3}{l_1}, \ldots \right).$$

Note that l_0 is invariant and hence unaffected by scaling. Unlike equation (3), the function ϕ does not remain invariant. By fixing l_i, it is a function of α. Although ϕ depends on α, we may parametrise it as a power law:

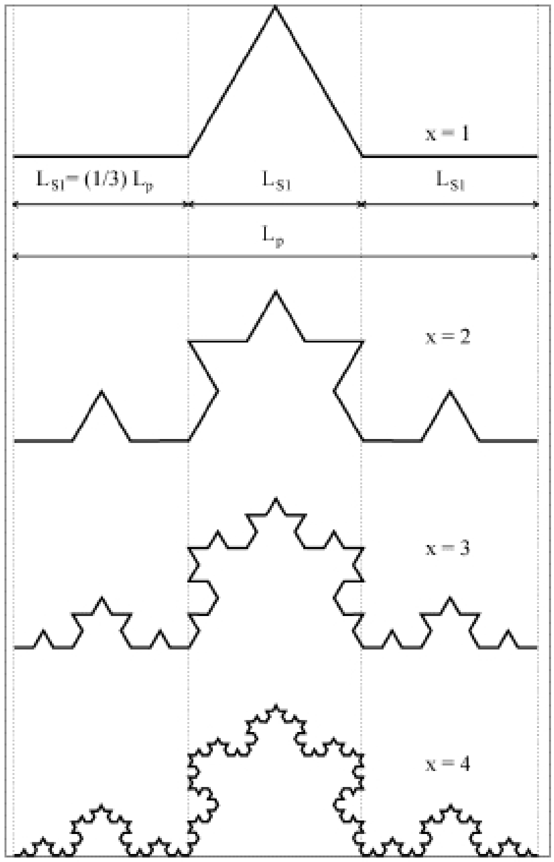
$$\phi \left(\frac{l_0}{\alpha l_1}, \frac{l_2}{l_1}, \frac{l_3}{l_1}, \ldots \right) = \alpha^{\epsilon_a} \phi \left(\frac{l_0}{l_1}, \frac{l_2}{l_1}, \frac{l_3}{l_1}, \ldots \right), \tag{4}$$

where ϵ_α is an arbitrary exponent. So, we may conclude that $a'/a = \alpha^{2+\epsilon_a}$.

The $2+\epsilon_a$ can be thought of as a fractal dimension of an idealised self-similar fractal, which we will now focus on. As a aside, although the physical system is not actually fractal, fractal language can give quite a good approximation.

Fractal dimension

We may think of fractals as a sequence of approximations towards the object of interest. The example to the right is called Koch's curve. Note that any points on Koch's line are infinitely far apart. So, in one sense, its 'dimension' should be greater than one. On the other hand, it is not a surface, so its dimension should be smaller than two. This fractal dimension is a way to encapsulate this behaviour. Note that there are many definitions, but we will focus on the easiest one (in my opinion).



Koch's curve.

In the aforementioned approach to fractals, for each iteration, there is a repeated unit throughout the structure that we can simply count. Let $N(n)$ be the number of repeated units in the object after n iterations and let $l(n)$ be the 'length' of the repeated unit. Then we can define the *fractal dimension* D as

$$D = \lim_{n\to\infty} -\frac{\log N(n)}{\log l(n)}.$$

For Koch's curve, $N(n) = 4^n$ and assuming that $l(1) =^1 /_3$, then $l(n) =^1 /_{3^n}$. Therefore, the fractal dimension of Koch's curve is

$$D = \lim_{n\to\infty} -\frac{\log N(n)}{\log l(n)} = \lim_{n\to\infty} -\frac{\log 4^n}{-\log 3^n} = \frac{\log 4}{\log 3}.$$

As an example, we could check that this definition agrees with our normal sense of dimension for a plane. In practice, it is not that easy to calculate fractal dimensions. Generally, a line that has a fractal dimension close to two means that the line is highly fractal. One example that comes to mind is a space-filling curve.

Putting it all together

The effective exchange area a has a fractal dimension of $2+\epsilon_a$. This means that $0 \leq \epsilon_a \leq 1$, since we know that the area must be at least dimension 2. Given that it is embedded in a three dimensional space, it cannot exceed 3.

Now, we may calculate the total volume v of active material. In a similar way, we get $v = l_1^3 \psi(l_0/l_1, l_2/l_1, l_3/l_1, \ldots)$. Under scaling by a factor of α, the volume $v' = \alpha^{3+\epsilon_v} v$, where $0 \leq \epsilon_d \leq 1$ is an arbitrary exponent. Thus, the effective exchage area $a \propto v^{\frac{2+\epsilon_a}{3+\epsilon_v}}$.

Similarly, we can apply the same idea to the length characteristic l to conclude that the length characteristic after scaling is $l' = \alpha^{1+\epsilon_l} l$. Expressing $v = al$, we can conclude that $\epsilon_v = \epsilon_a + \epsilon_l$ and that

$$a \propto v^{\frac{2+\epsilon_a}{3+\epsilon_a+\epsilon_l}} \propto M^{\frac{2+\epsilon_a}{3+\epsilon_a+\epsilon_l}}$$

via our assumption. Since ϵ_a and ϵ_l are independent and a typical organism would have maximised their scaling of a as a consequence of natural selection, we must have $\epsilon_a = 1$ and $\epsilon_l = 0$. To see this, fix any value of ϵ_a. The maximum occurs at $\epsilon_l = 0$. Optimising $(2+\epsilon_a)/(3+\epsilon_a)$ gives the desired result. Therefore, the metabolic rate is:

$$k \propto a \propto M^{\frac{3}{4}}.$$

Some interesting consequences

1. Since ϵ_l is zero, the fractal dimension of the internal length is one. This implies that the characteristic length l is not itself fractal, which agrees with our intuition that the distance from one cell to the other should be minimised.
2. As ϵ_a is one, this means that the fractal dimension of the effective exchange area a is three. So, the exchange surface structure is a volume-filling one. That is, organisms have evolved to maximise the constraints in three-dimensional space by exploiting a ‘fourth’ spatial dimension.
3. This idea can be generalised in other dimensions. For a roundworm, one may approximate it to a one-dimensional object. From this we can conclude that $a \propto M^{D/(D+1)}$, where D is the spatial dimension.

Final remarks

This model is not entirely accurate, but it gives a good rough reason as to why nature follows the $^3/_4$-rule (which is not the only rule that has been proposed). However, it does not explain why nature adopts fractal arrangements. There have been claims recently that this reasoning may not be entirely correct and that a fifth dimension may be needed.[2] Adding to the confusion, the debate on metabolic rates in plants is ongoing, as it seems that plants follow a 1-rule rather than the $^3/_4$-rule.

— Tharatorn Supasiti

Where does infinity begin?

> It was at this point that the stranger said, 'Look at the illustration closely. You'll never see it again'.
>
> — Jorge Luis Borges
> *The Book of Sand*

The last of the numbers. . . how might we find it? Surely what comes *after* is where infinity begins? Perhaps the largest number is not as far away as we imagine, and at the same time, it may be closer than we think.

The concept of infinity is deceptive; the approaches taken in literature differ greatly from those in mathematics. This article compares Georg Cantor's rigorous-infinite with Jorge Luis Borges's random-infinite, starting with the assertion that the infinite starts at 1.

The natural numbers

To begin, we must remember that in mathematics it is always necessary to define concepts properly to avoid people having similar but not necessarily identical ideas as to the nature of the discussion. Let us start with natural numbers:

[2]See, eg, Etienne, R, Apol, M E and Olff, H, 'Demystifying the West, Brown & Enquist model of the allometry of metabolism', *Functional Ecology* (2006) 20, 394–99.

> **Definition:** *We call a* natural number *any of the elements of the series 1, 2, 3, 4,... etc; they are also called the* positive integers. *We denote the conglomerate of all natural numbers by the symbol* $\mathbb{N}$.

We use the natural numbers every day to count objects and quantities: a book has 100 pages, a square has 4 corners, etc.[1] As our ancestors began counting their cattle, their sheep, their children, their friends, and their enemies, they began with the idea of the unity: the number 1, the first natural number. One is the smallest number that we can count: there is one (1) Sun in our solar system and one Moon orbiting our planet.

The number two (2) is for plurality; it appears when the number one takes the lead in counting. For that reason, the number two is called the *successor* of the number 1; similarly, 3 is the successor of 2, etc. Note that if 3 is the successor of 2, which is the successor of 1, then 3 is the successor of the successor of 1. This property of *successiveness* can be projected indefinitely because if the number 1 has it, and if the number 2 has it, then there is no reason to stop at a certain point.

We can use the symbol ($'$) to denote the successor of a natural number; we can say that $2 = 1'$ (ie. 2 is the successor of 1), $3 = 2' = 1''$, etc. Therefore, $\mathbb{N}$ can be written:

$$\{1,\ 1',\ 1'',\ 1''',\dots\}$$

What is interesting about this notation is that we do not need the Indo-Arabic numerals (the 10 symbols 1, 2, 3, ..., 9, 0) to represent the natural numbers. But eventually, this notation ($'$, $''$, $'''$, $\dots$) will become unmanageable, and so we must again return to the decimal notation system that we always use.

Is there a natural number without a natural successor?

No matter what notation we adhere to, the property of successiveness will always be present because we intuitively imagine it to be the case. Note that it is this property that makes the number: the concept of number arises from the succession of the number 1. The number 1 alone by itself is not a number

[1] For the purpose of this article, since the zero number is not used for counting, we will not classify it as a natural number.

but instead a unity. If plurality were nonexistent, that is, if of everything were to be a unity, then we would not have invented the counting system and the science of mathematics.

Suppose there is a number k so big that it has no successor. Then its predecessor should be $k-1$. Now the addition k to its predecessor: $k+(k-1)=2k-1$. But this number is greater than k. This contradiction arises because we assumed that there could exist a natural number k so big that no other number can be bigger than it. Hence, k has a successor and it cannot be the largest number.[2]

Once we intuitively accept the notion of succession, then we start to accept the existence of infinity because infinity is the unstoppable, the unlimited. It should be emphasized that we are not identifying the unreachable with the infinity, because infinity is not something that is never reached, it is an 'activity' that never stops.

Dictionaries define 'infinite' as: 1. extending indefinitely, endless; 2. immeasurably or inconceivably great or extensive, inexhaustible; 3. subject to no limitation or external determination.

But what if an entity, instead of having no end, had no beginning? What if it had no beginning and no end? Are they different kind of infinites? We'll later see some examples of these cases.

Where does infinity begin?

The series of the natural numbers cannot end because there will always be a number that will be the successor to any we might imagine to be the greatest of them. Thus, our quest for the greatest natural number is always doomed to fail: there will never be such thing as the number that is larger than all others.

When we talk about the endless sequence of natural numbers, we say: '$1, 2, 3, \ldots$ to infinity', thinking of the infinite as if it were a place, or a limit which we cannot go beyond, or the representation of the greatest natural number we can imagine. Yet the infinite is much more than the definition that

[2] This simple argument uses the concept of successorship. Archimedes demonstrated the infinite of primes, a proof that can also be used to argue in favor of the infinitude of the natural numbers.

dictionaries give us. It is not an 'inconceivably big' natural number or an 'inexhaustible quantity' because infinity is not a 'value' or a 'number'.

In light of the above, we will formulate a very simple definition of infinity:

Definition: Infinity *is the set of all successors of unity, and unity itself.*

That is, infinity is a direct consequence of the successorship property of the natural numbers.

With this definition we are not assigning any number to the infinite. Nevertheless, in order to do mathematical operations with the infinite we need to assign some measure of the 'quantity' of elements that sets have, especially the 'quantity' of elements of the set $\mathbb{N}$. This is the concept of the cardinality of infinite sets:

Definition: *The* cardinality *of* $\mathbb{N}$ *is denoted by* $\aleph_0$ *(aleph-null).*

If the infinite is the sequence itself, then it is necessarily unbounded. From that point of view, we can assert:

The infinite starts at the number 1.

This is because the number 1 is not the successor of another natural number.

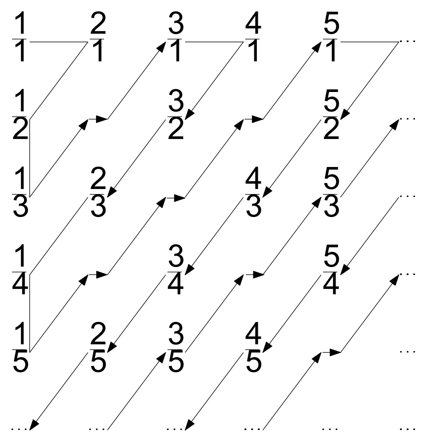
If the sequence of natural numbers is the infinite, then there is no way to show by mathematical proof that the natural numbers are infinite, simply because *we cannot demonstrate by any proof what we accept as true by prior definition*. Any other mathematical proof that deals with infinity is implicitly assuming that the infinite is the sequence of all successors of the number 1, regardless of whether that proof is related with $\mathbb{N}$ or not. Hence, $\mathbb{N}$ is the infinite series *de facto*, and $\aleph_0$ is the cardinality of the first infinity.

So what about the fractions? Are they the next 'aleph'? It certainly appears that there are more fractions than cardinal numbers.

Georg Cantor (1845-1918)—a Russian-German mathematician—took up the task of showing that the fractions (the rational numbers) are numerable, that is, that there are as many natural numbers as fractions. He established a one-to-one correspondence between the set of all fractions and $\mathbb{N}$. It had to be done this way because there is no other way of demonstrating that a series or a set of elements is infinite without establishing a correspondence between this set and $\mathbb{N}$.

In the figure below on the left, all the possible positive fractions are aligned in a square matrix. This matrix holds every proper and improper fraction (like $2/3$, $173/91$), and every natural number in all of its possible representations such as $1 = 2/2 = 22/22$, or $5 = 25/5 = 125/25$, etc. Note that this representation highlights that the set of rational numbers consists of infinite series of infinite elements.

1/1	2/1	3/1	4/1	5/1	...
1/2	2/2	3/2	4/2	5/2	...
1/3	2/3	3/3	4/3	5/3	...
1/4	2/4	3/4	4/4	5/4	...
...	...	...	...	...	



However counterintuitive, there are no more fractions than natural numbers. When the fractions are arranged or displayed as in the figure on the left, we can count all of them by following the path given by the arrows in the figure on the right. In this way, we can enumerate all the fractions as if they were 'the first fraction, the second fraction, the third fraction' etc. Of course, there is no such thing as 'a first fraction', but what the above figures illustrate is that among the many possible arrangements of all the fractions, this one provides enough to establish a counting process for them.

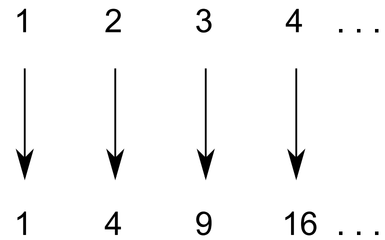
To go deeper into the assertion that 'there is no such thing as a first fraction', we can try to find the smallest positive fraction next to unity, that is, the first fraction next to 1. Let us denote that fraction as p/q. But there is always a fraction between 1 and p/q given by $(1 + p/q)/2$. Hence, even when we can talk about the natural number next to 1, we cannot use the same concept with the fractions. Since the fractions are infinitely numerable, then they are of the same kind of infinity as $\mathbb{N}$. As such, we can say that the rational numbers are *countable*.

The question that remains is: how can these two sets be equipollent (equinumerous) if the natural numbers enjoy the property of natural succession but the fractions do not? That is, each natural number always follows one, and only one, other natural number, but given any fraction we cannot say exactly which fraction follows.

On the other hand, between any two fractions there is always a fraction that is 'in the middle' of the two (just find the arithmetic average of them), while between two integers there is not always another integer 'in the middle'. From this, we can infer that when infinite sets are involved, the properties of both series are not always necessarily shared.

Galileo Galilei (1564-1642) foresaw that when we refer to infinity, we cannot think of it with the same attributes that are used in the finite cases. For example, he reasoned that there is the same amount of square numbers as natural numbers. This is counterintuitive because when we walk along the number line, the natural numbers are evenly spaced, while the squares are more and more spaced apart.

So, the concepts 'greater than' and 'less than' do not apply when we handle infinite 'quantities'. Note that as shown in the figure, the square of the numbers are becoming more and more scarce, that is, they are increasingly separated from one another while the natural numbers are always separated by a unit distance. This strange behavior is known as 'the Galileo paradox.' What it implies is that we cannot say that there are more natural numbers than squares, but can only assert that there is a one-on-one correspondence between them.



The Galileo paradox: though the square numbers are a subset of the integers, there are the same number of squares as integers.

Returning to Cantor, he was also able to demonstrate that the real numbers—the set of all possible decimals—are infinitely innumerable, that is, there is no way to establish a one-to-one correspondence between $\mathbb{N}$ and the set of decimals. This is a strange discovery because if every fraction can be converted to a decimal, and fractions are infinitely countable, then it seems intuitive that the decimals are also infinitely countable. The reason behind this is that not every decimal can be converted to a fraction; in fact, the great discovery lies in the fact that the 'decimal fractions' (as they are called) form an infinitely small quantity compared to the real numbers that cannot be expressed as fractions.

The symbol of infinity

How can we symbolize something that we cannot even imagine? We symbolize the unit with the numeral 1 and the plurality with 2, 3, etc. But infinity is not a number and for this reason it has no numeral assigned. But nothing prohibits us from using a symbol as an abbreviation when we refer to infinity.

The symbol of infinity has always been used on Tarot cards in the image of the Magician to symbolize the implementation of the impossible and the unlimited powers of the magic that transcend natural laws. In early versions of these cards, specifically, in the Tarot of Marseilles around the year 1500, the Magician Arcane (also known as the Juggler) appears with a mysterious wizard's hat that is very similar to the now widely-used infinity symbol ∞.

In mathematics, John Wallis (1616-1703) is usually credited with introducing ∞ as a symbol for infinity in 1655 in *De Sectionibus Conicis*. Loosely speaking, infinity is often informally symbolized by ∞, but in the theory of sets that Cantor developed, the formal symbol for infinity is $\aleph_0$ when we talk about the 'smaller' of the infinites, which is the infinitude of the natural numbers. The symbol $\aleph_0$ is usually read as 'aleph sub 0' (or aleph-null) to distinguish it from other infinitudes that 'follow' this first infinity.

The infinite is… infinite

From the work of Georg Cantor, the founder of set theory, we know that there are actually several types of infinites. What we have seen so far is only the 'smallest' and 'easiest' of all of them! Cantor established 'categories' of infinity to differentiate them. The first infinity is denoted by $\aleph_0$, the next one (the next higher) is denoted by $\aleph_1$ etc.

If the infinity of the natural numbers is $\aleph_0$, then what is the infinity $\aleph_1$? To get near what Cantor speculated to be the 'next' infinity, we need to invoke the idea of subsets of sets. Intuitively, we know that a set is any collection or grouping of objects, be they concrete or imaginary. For example, we can imagine the set A of all numbers ending in 3: $A = \{3, 13, 23, 33, \ldots\}$, or the set of all the houses that have one door painted white.

In contrast, a subset is formed by any set of some or all the elements of the given set. For example if we have the set of the first 4 natural numbers: $B = \{1, 2, 3, 4\}$, some of the subsets of this set are: {2, 3}, {1, 3}, {3, 4}, {2, 3, 4}. The number of subsets that can be obtained from a set of n elements is 2^n. If there is a set such that $A = \{a, b\}$, then it has 2^2 subsets {a}, {b}, ∅ and {a, b}. The set ∅—the empty set—is considered to be a subset of any set, and in turn, every set is considered to be a subset of itself. The set of all subsets of any set is called the *power set*.

It can be shown that the amount of subsets that can be obtained from a set is always greater than the number of elements in the set in question. It was this property that allowed Cantor to show that the quantity of real numbers in the real number line is 'greater' than the quantity of natural numbers in the same line. You might think that this is logical, because between two integers there is always an infinite number of points. But when we go into detail, the same can be said of fractions in that there is always an infinite number of fractions between two integers. However, it is easy to show that the fractions are numbered (we saw this earlier), while the real numbers are not.

Reviewing:

$$\aleph_0 = \{1, 2, 3, \ldots\} \text{, and}$$

$$\aleph_1 = \{\ldots, 0.2675, \ldots, \pi, \ldots, 3.6719887\ldots, 5^3 \ldots, \tfrac{73694}{37}, \ldots, 41^{235.999311}, \ldots\}$$ [3]

Notice how easy it is to conceptualize the 'first' infinite ($\aleph_0$), and how difficult it is to conceptualize the 'next' infinite ($\aleph_1$), because it contains all possible integers *and* all possible decimals. The infinites that follow the infinity $\aleph_0$ are given a special name: they are called *transfinite numbers*. Hence, $\aleph_1$ is the first transfinite.

According to Cantor,[4] the series of the infinites is:

[3] Note that here, as elsewhere in the article, we are assuming the continuum hypothesis. This hypothesis has an interesting story of its own that is well worth investigating, but one that we won't go into here.

[4] As well as the generalised continuum hypothesis.

$$\aleph_0, \aleph_1, \aleph_2, \text{etc.}$$

The implications of the transfinite arithmetic include the following:

$$\aleph_0 + 1 = \aleph_0, \quad \aleph_0 + \aleph_0 = \aleph_0$$

$$\aleph_1 + 1 = \aleph_1, \quad \aleph_1 + \aleph_1 = \aleph_1$$

$$2^{\aleph_0} = \aleph_1.$$

We must internalize these expressions as *concepts* rather than as equations because if we literally understand the equation $\aleph_0 + 1 = \aleph_0$ and subtract $\aleph_0$ from both sides, then we will come the conclusion that $1 = 0$ which is clearly false.[5]

Have we seen it all?

The concept of infinity is very elusive to our finite minds; while we can cope with the set of all numbers under 100 in our minds, we cannot so easily cope with the set of all numbers greater than 100. No matter how much we try, we will always have a limited view of everything that is unlimited, but that will not stop us from being bold enough to continue scrutinizing the boundaries of the infinite.

Infinity has been explored by several authors for many centuries, each contributing a little to our understanding of this field. Apart from the Cantorian mathematical rigor used for dissecting the infinite, there is an author who, in my personal estimation, has faced the vertigo of infinity like no other fictional author and essayist: the Argentine writer Jorge Luis Borges (1899-1986).

The infinite *Book of Sand*

Jorge Luis Borges is the author of the weird and fantastical short story titled *The Book of Sand*. It is about a book he buys, but that can never be opened on

[5] This is how some of fallacies in mathematics are built.

the same page, because the book is infinite. He can never find the first page, never the last page, never the same figures. There is never a way of opening the book on the page where it was bookmarked the previous day. The few short passages of this short story that follow highlight why this story is so fascinating:

> I live alone in a fourth-floor apartment on Belgrano street, in Buenos Aires. Late one evening, a few months back, I heard a knock at my door. I opened it and a stranger stood there. He was a tall man, with non-descript features—or perhaps it was my myopia that made them seem that way.
>
> I saw at once that he a foreigner.
>
> "I sell Bibles," he said.
>
> "I don't only sell Bibles. I can show you a holy book...It may interest you".
>
> I opened it at random. The script was strange to me. The pages, which were worn and typographically poor, were laid out in double columns, as in the Bible. The text was closely printed and ordered into versicles. In the upper corners of the pages were Arabic numbers. I noticed that one left-page bore the number 40,514 and the facing right-page bore the number 999. I turned the leaf; it was numbered with eight digits. It also bore a small illustration, like the kind used in dictionaries–an anchor drawn with pen and ink, as if by a schoolboy's clumsy hand.

Here is where Borges starts playing with the infinite. When one opens a book the pages are sequential, but that's not what happened to him. Upon first inspection he saw page 40,514 facing page 999. When he turned the page the number was an eight-digit number such as 56,783,452.

> It seems to be a version of Scriptures in some Indian language, is it not?
>
> "No," he replied. Then, as if confiding a secret, he lowered his voice. "I acquired the book in a town out on the plain in exchange for a handful of rupees and a Bible. Its owner did not know how to read. I suspect that he saw the Book of Books as a talisman. He was of the lowest caste; nobody but the other untouchables could tread his shadow without contamination. He told me that his book

> was called the Book of Sand because neither the book nor the sand has any beginning or end."

We have a similar experience every time we go to a beach or to a river bank: there is no way to have on hand the same grains twice with absolute certainty.

> The stranger asked me to find the first page.
>
> I laid my left hand on the cover and, trying to put my thumb on the flyleaf, I opened the book. It was useless. Every time I tried a number of pages came between the cover and my thumb.
>
> "Now find the last page. "
>
> Again I failed. In a voice that was not mine, I barely managed to stammer, "This can't be."
>
> Still speaking in a low voice, the stranger said "It can't be, but it *is*. The number of pages in this book is no more or less than infinite. None is the first page, none the last. I don't know why they're numbered in this arbitrary way. Perhaps to suggest that the terms of an infinite series admit any number."

Here we have the mathematician inside Borges reasoning like this: An infinite series is infinite by its elements and not by where it begins. *It does not need to have a beginning nor does it need to have order*. Borges' infinity is unmanageable because there is no way to know which page is coming next. It is the concept of randomness extended to the realm of the infinitude.

> I went to bed and did not sleep. At three or four in the morning I turned on the light. I got the impossible book and leafed through its pages. On one of them I saw a mask engraved. The upper corner of the page carried a number, which I no longer recall, elevated to the ninth power.

Borges realizes that in an infinite book the page numbers will become soon so large that they will occupy the whole page. That is why he uses exponents. A figure to the ninth power could be something such as 22,676,1909. Of course, in some cases, a number on a page could be larger than the book itself!

> Summer came and went, and I realized that the book was monstrous. What good did it do me to think that I, who looked upon

> such a volume with my eyes, who held it in my hands, was any less monstrous? I felt that the book was a nightmarish object, an obscene thing that affronted and tainted reality itself.
>
> I thought of fire, but I feared that the burning of an infinite book might likewise prove infinite and suffocate the planet with smoke.

Borges implies that he is losing his mind, that it is a nightmare to have an infinite object at hand, and so decides to get rid of this 'monstrous' book.

> I recalled reading that the best place to hide a leaf is in a forest. Before retirement, I worked on Mexico Street at the Argentine National Library, which contains nine hundred thousand volumes. I knew that to the right of the entrance a curved staircase led down into the basement, where books and maps and periodicals are kept. One day I went there and, slipping past a member of the staff and trying not to notice at what height or distance from the door, I lost the Book of Sand on one of the basement's musty shelves.

What makes the *Book of Sand* so particular? What type of infinity lies behind this 'monstrous' book? Since the book had no beginning nor end, Borges is speaking about infinite series with no starting and no ending term such as the following:

$$\{\ldots, 48, 567, 67357^{561}, 894662, \ldots\}, \quad \{\ldots, 8888, 10000, 561, 7^{7^7}, \ldots\}.$$

Each time he opened a book, a new book reshuffling appeared. So, what this book incarnates is the set of all possible arrangements of the natural numbers. Since the book is infinite in two directions (without beginning or end), any numbering of the pages is similar to the set of the natural integers both negative and positive. Therefore, there are as many instances of the *Book of Sand* as re-orderings of the set of all integers:[6]

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

There are no negative page numbers in Borges' book: books always use non-zero positive integers for page numbers. But the set displayed above is

[6] The interested reader might like to show that this is $\aleph_1$.

equipollent with the set of all positive natural numbers so no generality is lost. To prove that any page of the *Book of Sand* can be made to correspond to some integer (positive or negative) follow this mapping: if a page n is even then map it to $\frac{-n}{2}$, but if the page is odd map it to $\frac{n-1}{2}$.

Rudy Rucker, author of the book *Infinity and the Mind* and many other mind-boggling books on mathematics says the following about Borges:

> Borges is the writer of fiction who has written most profoundly about the infinity. Most of the relevant stories are in this anthology [*Labyrinths* (New York: New Directions, 1962)]. Of particular interest is his essay 'Avatars of the Tortoise,' ... This essay begins with a paragraph that could serve as epigraph of *Infinity and the Mind*: "There is a concept that corrupts and upsets all others. I refer not to Evil, whose limited realm is that of ethics. I instead refer to the infinite."

Final thoughts

Georg Cantor discovered and explored for posterity many unexpected properties and characteristics of the mathematical infinity that even today are still little understood, debated, and require further investigation. Sadly, he paid for his many unprecedented results with his mental health.

Almost simultaneously, in the other hemisphere of the world, Jorge Luis Borges —with a graceful literary intuition—was also writing about the infinite, but from another point of view: the infinite as a non-static, indomitable, insurmountable entity.

Cantor stated and explored infinity in ironclad dense theorems while Borges challenged readers to recreate and imagine infinity in their minds. Both pursued infinity from different horizons; what a marvellous journey the infinite makes!

— Ernesto Pérez Acevedo, Puerto Rico

Solutions to Problems from Last Edition

We had a number of correct solutions to the problems from last issue. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

Shir Peled solved problems 1, 2 and 3 and may collect $8.
Alfred Liang solved problems 1, 2, 5 and 6 and may collect $12
Steven Xu solved problems 2, 3, 4, 5 and 6 and may collect $15.

1. Two players take turns placing coins of radius 1 onto an mxn table, such that no two touch. The first player unable to place a coin loses. For which m and n does the first player have a winning strategy?

   Solution: The first player always has a winning strategy: (1) On the first turn, place a coin in the center of the table; (2) On subsequent turns, place a coin 180 degrees around the center from the second player's previous placement. Since this place is always available, the first player can always move if the second player can. And since the game permits no draws, this is a winning strategy for the first player.

2. Prove that each 4x7 grid coloured black and white at random contains a rectangle such that each (distinct) corner square is of the same colour. Prove that this is not true of a 4x6 grid.

   Solution: There are two cases: (1) Every column has exactly two white and two black squares; (2) At least one column has three squares of the same colour. In the first case, there are only $\binom{4}{2} = 6$ possible column colourings, so two of the seven columns are identical (pigeon-hole principle), and so will contain such a rectangle. In the second case, without loss of generality we assume that the first column has white squares in the first three rows. Then in the first three squares of each of the remaining six columns there must be at least two black squares. But these can be arranged in only $\binom{3}{2} = 3$ ways, and so there must be two columns with identical arrangements, forming a rectangle.

   An example of a 4x6 grid that doesn't contain such a rectangle is:

1	1	1			
1			1	1	
	1		1		1
		1		1	1

3. 15 people, each holding a ball, stand in a field. Each passes their ball to the person nearest them; no two distances are equal. Prove that: (1) there is someone without a ball; and (2) no one has more than five balls.

   Solution: For (1), consider the two people closest to each other, who receive each other's balls. If either of them receive another, then someone ends up without. If neither receives another, ignore them and repeat the operation. Since the number of players is odd, this operation must terminate with a single player. This player must pass a ball, but cannot receive one, and so ends up without a ball.

   For (2), assume one player (O) ends up with six balls. Consider the six players that passed to him $(A_1, A_2, A_3, A_4, A_5, A_6)$. Now, one of the angles $A_iOA_j \leq 60°$. But then, by the cosine rule, the distance A_iA_j cannot be the longest in the triangle A_iOA_j. Hence one of A_iO or A_jO is longer than another distance in the triangle, and so either A_i or A_j did not pass their ball to O, contradiction.

4. A square piece of paper is progressively cut up into smaller pieces by, at each step, taking one piece and cutting it into two by a straight cut. What is the minimum number of cuts needed to get a hundred octagons?

   Solution (thanks to Steven Xu): Let $Z =$ (total number of edges among all the pieces of paper) $-3\times$(total number of pieces). There are three types of cut: (1) A cut passing from a corner to another corner, which converts a piece with $m + n + 2$ edges into a piece with $m + 2$ edges and another with $n + 2$ edges, subtracting one from Z; (2) A cut passing from a corner to an edge, which converts a piece with $m + n + 1$ edges into a piece with $m + 2$ edges and another with $n + 2$ edges, leaving Z unchanged; (3) A cut passing from an edge to another edge, which converts a piece with $m + n$ edges into a piece with $m + 2$ edges and another with $n + 2$ edges, adding one to Z. Since Z starts at 1 and must attain at least $800 - 300 = 500$, and since each cut adds at most 1 to Z, we require at least 499 cuts. The required pieces can be achieved with 499 cuts as follows: use 99 cuts to convert the square into 100 quadrilateral strips, then use another 4 cuts per square to snip off the four corners from the quadrilaterals.

5. For any convex polyhedron with an even number of edges, prove it is possible to attach an arrow to each edge such that each vertex of the polyhedron has an even number of arrows directed towards it.

   Solution (thanks to Alfred Liang): Label arrows at random. Note that the sum of the incoming degrees is the number of edges. Since the number of edges is even, the number of vertices with odd incoming degree is also even. Select two such vertices with odd incoming degree, 'connect' them by taking a path of edges and invert all the arrows on the path. Only the parity of the incoming degrees of the selected vertices will change. Thus the number of vertices of odd incoming degree has decreased by two. Since the number was initially even, repeating this operation will eventually reduce this number to zero.

6. Which positive integers cannot be represented as $\frac{a}{b} + \frac{a+1}{b+1}$, with a and b positive integers?

   Solution: All n such that $n - 2 = 2^k, k \in \mathbb{N}_{\geq 0} = \{3, 4, 6, 10, 18, \ldots\}$. To prove this, write $\frac{a}{b} + \frac{a+1}{b+1} = \frac{2ab+b+a}{b(b+1)} = n \in \mathbb{N}_{>0} \Rightarrow b|a \Rightarrow a = kb, k \in \mathbb{N}_{>0} \Rightarrow (b+1)|(2kb+k+1) \Rightarrow (b+1)|(k-1) \Rightarrow k = m(b+1)+1, m \in \mathbb{N}_{\geq 0} \Rightarrow n = (2b+1)m+2 \Rightarrow n = 2$ or $n-2$ contains an odd multiplier and so $n-2$ cannot be a power of 2.

   On the other hand, for all $n \neq 2^k+2, k \in \mathbb{N}_{\geq 0}$, we can reverse this process to determine a suitable a and b to give the required representation.

7. $2n$ people sit around a table with k chocolates distributed among them. A person may give a chocolate to their neighbour, but only after first eating one themselves. Nominating a *head* of the table, what is the minimum k such that, irrespective of the initial distribution of the lollies, there is a way for the *head* to get a chocolate? What is the minimum k such that *everyone* can get a chocolate?

   Solution: Consider giving chocolates to just the person sitting opposite the *head* (the *tail*). If they get less than 2^n, the *head* cannot get a chocolate. So $k \geq 2^n$. But 2^n is sufficient in every case. First, a lemma: if $\{x_1, \ldots, x_i\}$ are in a row and $|x_1| + \ldots |x_i| \geq 2^{i-1}$, where $|.|$ denotes the number of chocolates, then x_1 can get a chocolate. We prove this by induction. If $|x_1| \geq 1$, we are done. Assume it is true for k. For $k+1$, if $|x_1| + \ldots + |x_k| \geq 2^{k-1}$ then we are done by the assumption. But if it is not, then $|x_{k+1}| \geq 2^k - (|x_1| + \ldots + |x_k|) \geq 2(2^{k-1} - (|x_1| + \ldots + |x_k|))$ so we can give $2^{k-1} - (|x_1| + \ldots + |x_k|)$ to x_k, permitting x_1 to get a chocolate via the assumption. Now, if X and Y are the number of chocolates on the left and right arc from *head* to *tail* respectively, T the number held by *tail*, and assuming $X \geq Y$, then $X + Y + T = k \geq 2^n \Rightarrow T \geq 2^n - X - Y \geq 2(2^{n-1} - X)$ and so the *tail* may give $2^{n-1} - X$ chocolates to his left, so $X_{new} \geq 2^{n-1}$. Applying the lemma completes the proof.

   With this technique in mind, the second half of the problem will remain open for another issue.

Paradox **Problems**

Below are some puzzles and problems for which cash prizes are awarded. Anyone who submits a clear and elegant solution may claim the indicated amount (up to a maximum of four per person). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

1. ($2) Three people take it in turns to toss a coin; the first to throw a head wins. What probability does each player have of winning the game?
2. ($3) Find a closed form for the infinite sum $\frac{1}{1\times 2} + \frac{1}{5\times 6} + \frac{1}{9\times 10} \cdots$
3. ($3) How many ways can we choose 3 non-empty and non-intersecting subsets from $\{1, 2, \ldots, 2011\}$?
4. ($3) Starting with the word $aa \ldots ab$ (a appears 2011 times), we may exchange any a for bba (and back again) and any b for aba (and back again). Can we eventually form the word $baa \ldots a$ (a appears 2011 times)?
5. ($3) 3D Misère Tic-Tac-Toe is played like regular Tic-Tac-Toe on a 3x3x3 grid, except that a player *loses* if they place three tokens in a row/column/diagonal. Does either player have a winning strategy in 3D Misère Tic-Tac-Toe, and if so, what is it?
6. ($4) A calculator is broken so that the only keys that work are $sin, cos, tan, arcsin, arccos$ and $arctan$. The display originally shows 0. Prove that for any rational q, there is a finite sequences of keys that will return q (ignoring decimal place restrictions).
7. ($5) (From Issue 1, 2011) $2n$ people sit around a table with k chocolates distributed among them. A person may give a chocolate to their neighbour, but only after first eating one themselves. What is the minimum k such that *everyone* can get a chocolate?

Paradox would like to thank Ernesto Pérez Acevedo, Sam Chow, Tharatorn Supasiti, Muhammad Adib Surani, David Wakeham and π.O for their contributions to this issue.