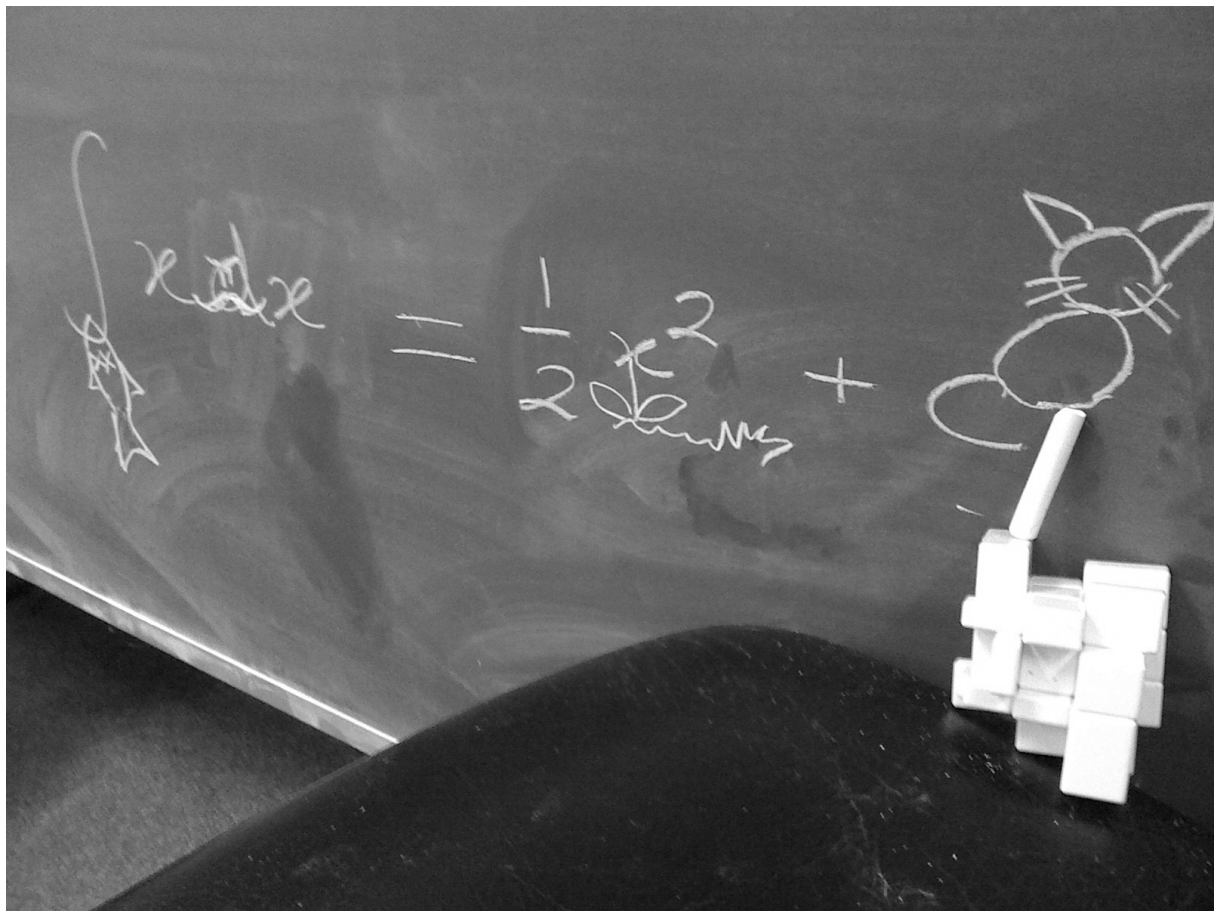

Paradox

Issue 4, 2011

THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY



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COVER:	Rubik's Turtle, defender of all Rubik- skind from the forces of evil! See page 6 for more!

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The trouble with integers is that we have examined only the very small ones. Maybe all the exciting stuff happens at really big numbers, ones we can't even begin to think about in any very definite way. Our brains have evolved to get us out of the rain, find where the berries are, and keep us from getting killed. Our brains did not evolve to help us grasp really large numbers or to look at things in a hundred thousand dimensions.
— Ronald L. Graham

Words from the Editor

Welcome to this year's fourth and final issue of Paradox, the magazine produced by the Melbourne University Mathematics and Statistics Society (MUMS). You will be pleased to find that problems and solutions have returned to Paradox but they return with a twist! There are also some relatively straightforward problems no-one has been able to solve yet in case you find the Paradox ones to be far too easy...

In this issue, you will learn more about what life as a mathematician is like from the next MUMS alumnus to be interviewed by Paradox, how a famous mathematician did not show any working for his proofs and got away with it, and how the seemingly random number on a taxi in London inspired a whole new generalisation of numbers.

Joining our now regular features of interviews and biographies is *The Adventures of Rubik's Turtle*, a photo comic strip which shall span multiple issues of Paradox. Read about our new hero's humble beginnings in this issue and stay tuned to see if he makes it through his perilous adventures in one piece!

As always, we rely on contributions from people like you to keep Paradox awesome. Thanks to all who have made submissions so far — if your article has not been included in this issue, then you can be sure that it will appear in Issue 1 next year. If you have anything new that you would like to submit to Paradox, then feel free to just drop by the MUMS room and ask any one of our friendly MUMSians about it or you can always send an email to paradox.editor@gmail.com.

Finally, whether you have been diligently preparing for or are now desperately scrambling to prepare for your upcoming wave of end-of-semester exams or assignments, I wish you all the best with your assessments — just remember that it will all be over very soon with a long and hopefully relaxing summer awaiting you. In the meantime, enjoy the end of semester with a beloved MUMS Trivia Night — I know I will!

— Kristijan Jovanoski

Words from the President

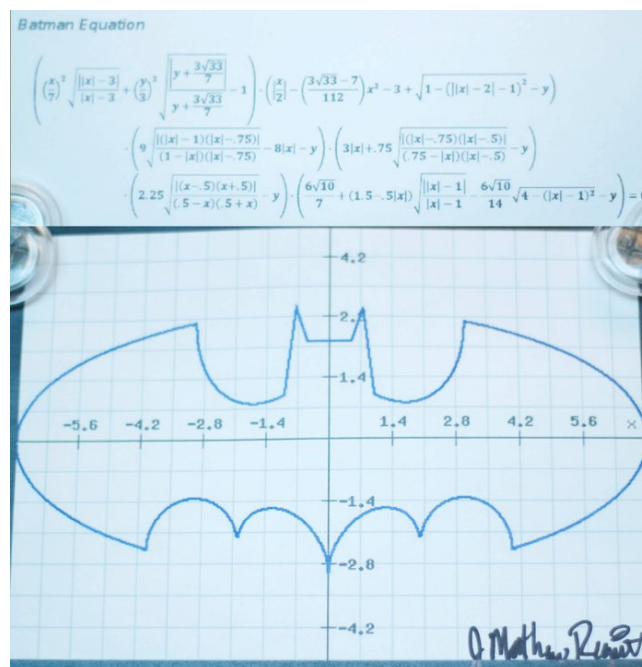
Hi everyone and welcome back to the final issue of Paradox for the year. In the time between the previous issue and this current one, MUMS has successfully run both the School Maths Olympics and the University Maths Olympics. This was my first time being involved so closely with the organisation of these two events.

I just want to say that it has been a lot of fun and I would like to thank everyone who helped out with the smooth running of both events. A special thanks goes to all of the participants for making both competitions so successful.

As the year winds down, one last event awaits us — the MUMS Trivia Night to be held on Friday 28th October. If you are reading this before the 28th then remember to compete — it's always lots of fun!

Finally, I wish you all the best for exams and hope you enjoy your summer holidays. I hope to see all of you again next year.

— TriThang Tran



The Batman equation.

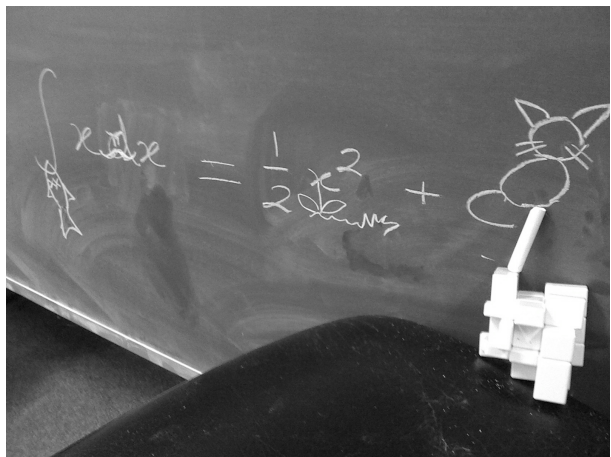
The Adventures of Rubik's Turtle

— Dougal Davis & Jinghan Xia

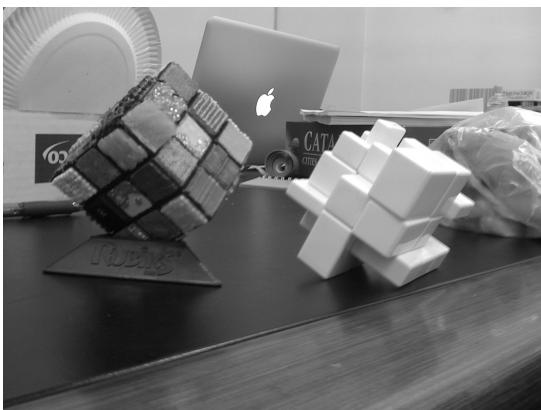
Episode 1: A hero is born



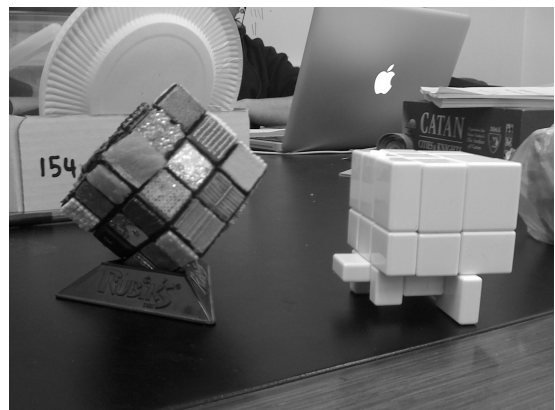
A poor little orphan Rubik's cube is shunned by the other cubes because he is different.



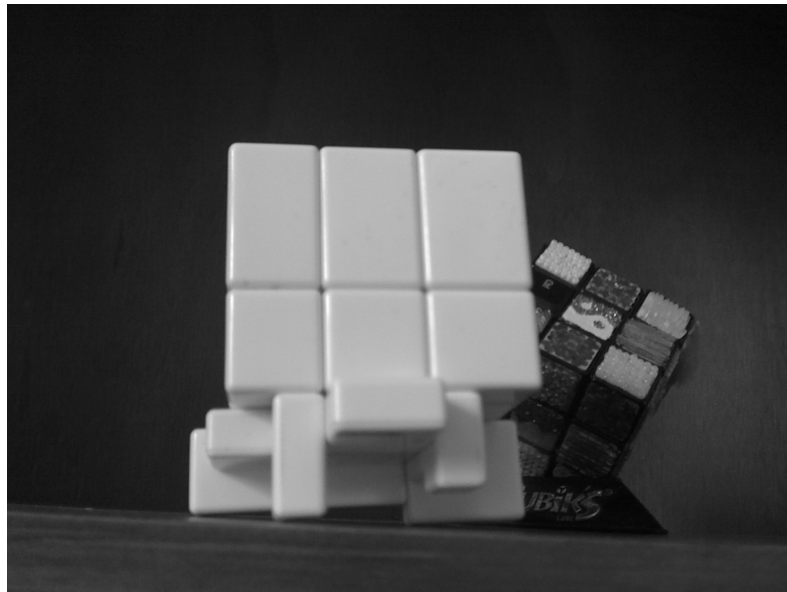
With no parents, no friends, and no future, he takes to vandalism and graffiti.



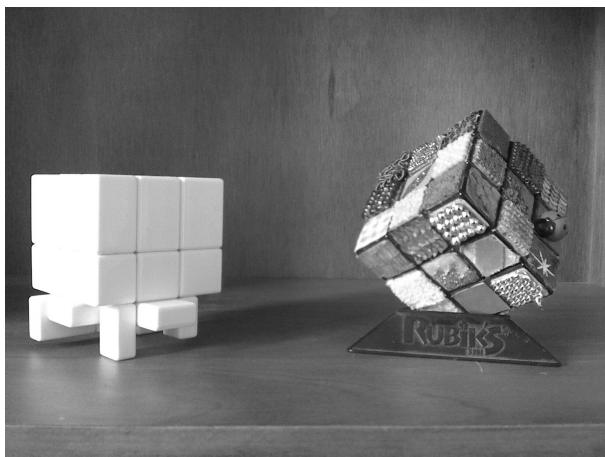
Until by chance he meets the mysterious Mother-of-all-Rubiks, who sees his true potential and takes him under her wing...



And through her kindness and wisdom, teaches him to become Rubik's Turtle!



Rubik's Turtle, defender of all Rubikskind from the forces of evil! He vows to protect the innocent, and fight to ensure that no Rubik's cube, no matter how small and insignificant, will be left prey to corruption and despair!



Mother-of-all-Rubiks gives the young hero his first mission...



To rescue a poor defenceless Rubik's cube from the evil Cyril and his Circus of Circuits!

Will Rubik's Turtle prevail in this perilous quest? Will he be able to wrest the innocent cube from the villainous Cyril and his treacherous band of discarded computer hardware? Find out next issue, in Episode 2 of *The Adventures of Rubik's Turtle!*

Unsolved Problems in Mathematics

If you think there's nothing more to do when you've finished your weekly tutorial's homework, then you're sorely mistaken! Here are some problems you can try your hand at in your spare time:

1. **Goldbach's Conjecture:** This problem looks really easy: every even number greater than 2 is the sum of two primes, e.g. $6 = 3 + 3$, $8 = 3 + 5$, etc. But can you prove that it's true for *all* even numbers? You may need to use immensely huge numbers, but they shouldn't be that different from small ones, right? Since Goldbach posed this conjecture to Leonhard Euler in 1742, no one has been able to prove it, so you can always have a shot whenever you feel like it!
2. **The Riemann Hypothesis:** Apparently, this is the most important unsolved problem in pure maths according to some experts. But what would they know? Surely the distribution of zeroes in relation to Bernhard Riemann's 'Riemann-zeta-function', an analytic number theory that can produce a never-ending sequence of numbers, is a relatively straightforward extension problem for you? Well, maybe it's a bit harder than it sounds. After all, the world's brightest mathematicians haven't solved it yet and computer calculations have counted at least 100 billion zeroes so far... a few more zeroes couldn't hurt!
3. **P vs. NP:** If the first two problems seem a little too dull and uninteresting, then the Clay Mathematics Institute has an example of a P vs. NP problem for you! Suppose that you are organising housing accommodations for a group of 400 university students. Space is limited and only 100 of the students will receive places. To complicate matters, the Dean has provided you with a list of pairs of incompatible students, and requested that no pair from this list appear in your final choice. Not too hard to generate a list of 100 such students from scratch, right? Just remember that the total number of ways in which you could pair up 100 students from 400 exceeds the number of atoms in the known universe... maybe some computer programming skills could help!

— Kristijan Jovanoski

Interview with a MUMS Alumnus

Background

Norm Do has been an active member of MUMS since 1998. He graduated in 2002 with Honours in both Science (majoring in mathematics) and Engineering (electrical engineering stream), and completed his PhD in mathematics at the University of Melbourne in 2009. Soon after, he commenced a post-doctoral fellowship at McGill University in Montréal before returning to the University of Melbourne as an Australian Research Council (ARC) Postdoctoral Fellow in 2010.

MUMS

Why did you join MUMS?

I always liked maths and pretty much knew I wanted to study it at uni. I was lucky enough to know the MUMS committee through school friends and thought they were cool (albeit in an uber-geeky way)! After I joined, it turned out to be pretty fun! It's important to remember that MUMS offered, and still offers, more than just events. It provided an environment for mathsy activities (like deciphering lectures) and was an opportunity to socialise with people who had similar interests, not necessarily maths-related.

How have you been involved in MUMS?

Previously, I've been the First Year Rep, Second Year Rep, Postgrad Rep, Education Officer, and Paradox Editor (and possibly even more)! I also helped out with the SMO (Schools Maths Olympics) and the UMO (University Maths Olympics). Since I've been back in Melbourne, I've given two MUMS seminars. Oh, and I always attend the other MUMS seminars for the free food (and sometimes, the educational value too)!

You've been involved in MUMS for over *ten* years! Have there been many changes over this time?

Well, MUMS keeps adding to their repertoire of activities. For example, back in the day, Puzzle Hunt didn't even exist on the MUMS calendar!

I think the MUMS Committee has also become more diverse in terms of gender and background. In my first year with MUMS, the committee seemed to be made up of people from only two schools!

Oh, and the new room is definitely better... it's more open and just a little less stinky than the old one. The only downside is that when I enter the MUMS room now, I'm almost certainly the oldest person there! Back in the day, our room was a dark, windowless corridor. It was so long and narrow that we used to play two square and cricket in there. These days, it's been snazzified by BHP and they're probably using it for more dignified purposes!

And how has Paradox changed?

You know what, Paradox has not changed at all! I'm surprised at how similar Paradox has been over the years... it's been consistently awesome! The Editor still seems to be using the same template we used all those years ago.

Actually, now that I think of it, there is one thing that's changed and it was introduced by me — staples! I was the first ever official Paradox staple boy... which means that every issue of Paradox was stapled by me. In fact, the stapler in the MUMS room belongs to me.¹ I got it as a present the first year I was anointed staple boy. I think readers have me to thank for their Paradoxes not falling apart in their hands!

What is your favourite MUMS event?

Probably the UMO, because I always get something out of it — first prize [laughs]! My team won the UMO last year, and I hope to win again this year if my team, the Contravariant Funksters, perform up to scratch!² I've participated every year since 1995, apart from a few times when I was overseas or helped write the questions.

Have you really won first prize every year?

Ok, no, that's not true... but it's close enough! Look, I'm getting old, you can't expect me to keep up with all those spry youngsters.

¹The current Paradox Editor became so sick of struggling with this stapler that he ended up decommissioning it and bought another stapler that looks almost identical to the original one, except this one is much, much better!

²Ed note: They didn't finish in the top three this year.

Career

Why did you choose to become a mathematician?

I've always been drawn to maths, even as a child. In grade four, my primary school teacher would give me a tougher textbook and tell me to sit at the back of the class and read it... and for me that was a treat! I feel like my mind has always been drawn to the structure and beauty of maths. I've always been driven by curiosity, be it in maths or other areas. But for most of my life I didn't know what a mathematician actually does.

Now that you *are* a mathematician, can you explain what they actually do?

I'm still working that out [laughs]! Well, to give you an idea of what I've been doing this semester: I spend a quarter of my time teaching undergraduates [Calculus 2], a quarter of my time going to graduate lectures and seminars, a quarter of my time checking my email [laughs], and a quarter of my time actually sitting down with pen and paper trying to solve real original research maths problems.

What problems are you currently working on?

I like to work on problems from combinatorics, geometry, or physics, but preferably all three! The problem I'm currently working on is this: someone gives you a bunch of polygons, and you want to glue them together to make a surface. For example, four triangles can be glued together to make a tetrahedron, which is a very simple example of a surface. I'd like to know how many ways there are to do this in a general setting. It turns out the numbers that arise turn up in other areas of mathematics, as well as theoretical physics.

What is it like being a mathematical researcher?

As an academic doing maths, the lifestyle is very good. By that, I mean that the environment is great. Also, being a mathematician is not an evil job. You're generally working through self-motivation rather than having a boss telling you what to do. It's pretty liberating to be able to do whatever you like - you make up your own hours, and no one tells you to stay back until 7 pm. But I usually end up staying back that late anyway if I'm really keen on a problem or really excited about a paper I'm reading! There are many opportunities for travel, and in this line of work you come across really

interesting people, as you can imagine! They obviously love maths but are usually keen on other activities as well.

What is an 'evil job'?

In general, for most jobs, it's hard to say what effect your job has on other people since the effect is often so long-range. On the other hand, in mathematics, your interactions with other people are very natural and pleasant. When you teach, you're helping others; when you collaborate, you're working together on a problem. It's a career that's really driven by curiosity. Your impact on other people is quite immediate and generally very positive — although some of my calculus students might beg to differ!

Who are the interesting people you have met on your travels?

I've come across various Fields Medallists... one example is Shing-Tung Yau [1982 Fields Medallist]. I also know mathematicians who are circus performers — they work on the weekends busking on the streets. I know others who are in heavy metal bands... actually that's not that uncommon really. A colleague of mine paid people \$20 to turn up to his maths lectures dressed as the 'mathematician of the day'. Sometimes, it seems to me that I'm the only normal mathematician in the world!

Is this what you imagined yourself doing?

When I was a student I was pretty short-sighted — I didn't think about where I would end up. I'm glad about that, because if I had thought harder I might have gone into a more vocational degree. I like being an academic, but I know that if I want to move into the real world, there are millions of jobs waiting for someone numerate like me!

It sounds like job prospects for someone studying maths are quite good?

Well, I recently made an appearance as the token academic (out of seven speakers) at a maths careers evening held by the [MU Maths] Department. One thing was very apparent to me from listening to the other speakers. Mathematics, particularly the ability to think logically as well as creatively, is very much appreciated in government, industry, and finance... and if you're willing to look, it's easy enough to get a job.

I imagine that some would be worried about becoming a mathematician because it may not pay as well as a job in industry. Your thoughts?

I seem to be doing all right... although I'm also lucky enough to be married to a doctor! To be honest, I think I get paid more than I deserve... but don't tell that to the ARC though [laughs]! In Australia, academics generally get paid reasonably well. It isn't comparable to working in finance... but maybe per hour it might be.

What are your plans for the future? Do you envisage yourself in academia for a long time?

My current post lasts for two more years, and then I'm back on the job market, which is one of the worst aspects of being an early-career mathematician. I may have to move to another country temporarily, but that's also pretty exciting from my perspective. My current plan is to ride the academic wave for as long as possible. I can't think of anything else I'd rather do at the moment!

Random and Interesting

I've been stalking you a bit and noticed that you are involved in many maths activities and camps?

That's a bit creepy... but yes, I've been kind of busy! I used to teach motivated high school students; I teach at NMSS (National Maths Summer School); I've been involved in Australia's Maths Olympiad Program since I was a student; and hopefully I'll be starting up a mathematical radio segment with the ABC (Australian Broadcasting Company) soon!

These activities aren't a part of my job, but my job gives me the flexibility to participate in all sorts of extracurricular activities. Teaching students is one of the things I love, and what makes it more exciting is if they're keen and talented.

Have there been any students you've taught that really stick out in your memory?

Julian Assange? He's the only one who's gained notoriety — there's no one else really who can compare to that. I tutored him while he was in

first year. . . he was obviously very intelligent and thoughtful. Usually I don't interact with students a lot after class, but I got to know him through MUMS. We spent countless hours talking in the MUMS room about maths, education, and lots of other things. Of course, conversations about politics did come up as well! I certainly considered Julian a friend but I don't know whether I'll see him again any time soon. Some old MUMS people got together and sent him a birthday video for his last birthday (obviously I didn't see him in person) and I heard that he appreciated it!

What was the video about?

I can't tell you that — that's classified information. Just joking. . . it was just a few of us wishing him happy birthday and singing the song very very badly!

Now, I heard that you met your wife at NMSS?

Yes, I met my wife at NMSS, or 'nerd camp' as it's often known, while we were both still teenagers. I have to say, I wasn't expecting to go to nerd camp and come back with a future wife, but there you have it! That's what happens when you put a bunch of like-minded teenagers in the same room! It's weird how things work out — who meets their partner at nerd camp?! Actually, it happens a lot more often than you probably think!

Then you pursued a long-distance relationship for seven years! That is quite amazing. . .

Around the time I met my wife-to-be, I also became acquainted with the internet. . . so I spent far too much time talking to her on ICQ (if you know that that is) [an instant messaging program popular before the days of MSN]! I also managed to clock up a fair few frequent flyer points over those years. Most of my friends thought she was made up; even after they met her they told me I didn't have a chance with her (or anyone else for that matter)! And now we've been married for nearly three years. . . but it feels like twenty. Just kidding!

— Lu Li

If you are interested in being interviewed for Paradox, please send an email to paradox.editor@gmail.com. Include your name, occupation, and relation to or interest in MUMS.

Biography: Srinivasa Ramanujan (1877–1920)

With almost no formal training in pure mathematics, the Indian autodidact Srinivasa Ramanujan made substantial contributions to mathematical analysis, number theory, infinite series and continued fractions. His talent was said by the prominent English mathematician G.H. Hardy¹ to be in the same league as legendary mathematicians such as Euler, Gauss, Newton, and Archimedes. According to Hardy's personal ratings of mathematicians on the basis of pure talent on a scale from 0 to 100, he gave himself a score of 25, J. E. Littlewood (a long-time collaborator with Hardy) 30, David Hilbert 80, and Ramanujan 100.



He was a person who was somewhat shy and quiet, yet dignified and with pleasant manners. He spent much of his life in India and only came to Cambridge University after much coaxing from Hardy. He credited his abilities to his family Goddess, Namagiri of Namakkal, claiming that he dreamed of blood drops symbolising her male consort, Narasimha, after which he would receive visions of scrolls of complex mathematical content unfolding before his eyes.

Despite Ramanujan's Brahmin Hindu upbringing, Hardy claims that he believed all religions seemed equally true to him and that his religiousness has been romanticised and overstated in terms of belief. However, Ramanujan was a very strict vegetarian to the point of religious fervor. This contrasts with his abilities, which Hardy believed were surprisingly inconsistent:

The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations and theorems... to orders unheard of, whose mastery of continued fractions was... beyond that of any mathematician in the world, who had found for himself the functional equation of the zeta function and the dominant terms of many of the most famous problems in the analytic theory of numbers; and yet he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was...

¹See *Great Lives* in Paradox Issue 1, 2011 for more about Hardy's life.

When asked about the methods Ramanujan used to arrive at his solutions, Hardy said that they were:

Arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account.²

While he was still in India, Ramanujan recorded the bulk of his results in four notebooks of loose leaf paper. These results were mostly written up without any derivations, leading to the misperception that Ramanujan simply thought up the final result directly and could not actually prove them. However, it is more likely that since paper was very expensive then, Ramanujan would do most of his work and possibly his proofs on slate,³ after which he would then transfer just the results to paper. It is also possible that Ramanujan considered his workings to be of personal interest only.

Unfortunately, Ramanujan was plagued by health problems throughout his life and living in England took its toll. His health eventually took a harsh turn for the worse, possibly due to his stress and obsession with mathematics, or possibly because vegetarian food was scarce during the First World War which was raging at the time. Suffering from tuberculosis and a severe vitamin deficiency, he was eventually confined to a sanatorium. He returned to India in 1919 and died soon afterwards at the young age of 32. In contrast, his wife died at age 94.⁴

Ramanujan also recorded the discoveries of the last year of his life in a notebook that was only rediscovered in 1976 by the American mathematician George Andrews. Andrews and Bruce Berndt have published several textbooks providing proofs for the formulas included in the notebook and according to the latter:

The discovery of this 'Lost Notebook' caused as much excitement in the mathematical world as the discovery of Beethoven's Tenth Symphony would cause in the musical world.

— Kristijan Jovanoski

²This reason does not excuse failing to provide working for a solution unless you are Ramanujan reincarnated.

³Using a slate was common for mathematics students in India during Ramanujan's time.

⁴When Ramanujan was 21, he was married to a nine-year old bride. In the branch of Hinduism to which he belonged, marriage was consummated only after the bride turned 17 or 18.

Taxicab Numbers

G. H. Hardy once arrived at Srinivasa's Ramanujan's¹ residence in a cab numbered 1729. He commented that the number seemed to be rather dull and hoped that it was not an unfavourable omen, to which Ramanujan immediately replied:



No, it is a very interesting number; it is the smallest number expressible as the sum of two [positive] cubes in two different ways.

Generalisations gave rise to the notion of the n th 'taxicab number', typically denoted $Ta(n)$ or $Taxicab(n)$. It is defined as the smallest number that can be expressed as the sum of two positive algebraic cubes in n distinct ways. $Ta(1)$ is rather simple:

$$Ta(1) = 2 = 1^3 + 1^3$$

However, $Ta(2)$ is much larger; in fact, it is 1729, the number that has now been immortalised as the Hardy-Ramanujan number:

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

First discovered in 1657 by Bernard Frénicle de Bessy, $Ta(2)$'s famous story featuring Ramanujan brought the concept of taxicab numbers to prominence and in 1954, G. H. Hardy and E. M. Wright proved that such numbers exist for all positive integers n . While their proof is easily converted into a program to generate such numbers, it does not claim whether the numbers generated are actually the *smallest* and hence cannot be used to find the actual value of $Ta(n)$.

¹See this Paradox Issue's biography for more about the relationship between Hardy and Ramanujan.

Why must the algebraic cubes be positive? Because if negative numbers are allowed, then we can form more (and smaller) instances of numbers that can be expressed as the sums of two cubes in n distinct ways. Such numbers are known as *cabtaxi* numbers, but for the sake of brevity, they shall not be discussed any further.

The most famous taxicab number has made a few appearances in popular culture. 1729 and its interesting property is mentioned by the somewhat-insane mathematician Robert played by Anthony Hopkins in the 2005 film *Proof*, while in the animated television series *Futurama*, the spaceship is designated as Nimbus BP-1729 in Season 2 and the robot character Bender's serial number is revealed to be none other than 1729 in a Christmas card he receives in the episode *Xmas Story*.

What about the other taxicab numbers? Only the first six are known to date, and the ones after $Ta(2)$ were found with the help of computers. John Leech found $Ta(3)$ in 1957:

$$\begin{aligned} 87,539,319 &= 167^3 + 436^3 \\ &= 228^3 + 423^3 \\ &= 255^3 + 414^3 \end{aligned}$$



It was a while until $Ta(4)$ was found to be a whopping 6,963,472,309,248 in 1991 and $Ta(5)$ was found to be 48,988,659,276,962,496 in 1999. Only three years ago, $Ta(6)$ was recently announced to be an incredibly huge 24,153,319,581,254,312,065,344! When will $Ta(7)$ be discovered?

But it can be even more challenging: what if a taxicab number is restricted such that it is not divisible by any cube other than 1^3 ? If we describe it as a *cubefree* taxicab number T and write it as $T = x^3 + y^3$, then the numbers x and y must be relatively prime for all pairs (x, y) .

Among the taxicab numbers $Ta(n)$, only $Ta(1)$ and $Ta(2)$ are cubefree taxicab numbers. But $T(3)$, the smallest cubefree taxicab number with three representations, was discovered by Paul Vojta (unpublished) in 1981 while he was a graduate student and it was not $Ta(3)$:

$$\begin{aligned}T(3) &= 15,170,835,645 \\ &= 517^3 + 2468^3 \\ &= 709^3 + 2456^3 \\ &= 1733^3 + 2152^3\end{aligned}$$

Have any more of these numbers been discovered? In 2003, $T(4)$ was found to be a colossal 1,801,049,058,342,701,083 — taxis may be small but taxicab numbers and their cubefree companions are certainly not!

— Kristijan Jovanoski

Pure mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere — Archimedes did it in a bathtub.

— Richard J. Trudeau, *Dots and Lines*

Solutions to Problems from Issue 2, 2011

We had a number of correct solutions to the problems from last issue. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

David Wakeham solved problems 1, 2, and 6 and may collect \$9.

Steven Xu solved problems 1, 2, 3, 4, 5, and 6 and may collect \$18.

1. Three people take it in turns to toss a coin; the first to throw a head wins. What probability does each player have of winning the game?

Solution: Consider the first throw: if it is a head, Player 1 wins; if it is a tail, the game effectively resets with Player 2 as the new Player 1. Hence, Player 2 has a half chance of replacing Player 1 as first player, and so has exactly half as much chance of winning the game as Player 1. An identical argument holds for Player 3. So the probabilities for the three players are $(\frac{4}{7}, \frac{2}{7}, \frac{1}{7})$.

2. Find a closed form for the infinite sum $\frac{1}{1 \times 2} + \frac{1}{5 \times 6} + \frac{1}{9 \times 10} + \dots$

Solution (thanks to Andrew Elvey Price): $\frac{1}{1 \times 2} + \frac{1}{5 \times 6} + \frac{1}{9 \times 10} + \dots = (1 - \frac{1}{2}) + (\frac{1}{5} - \frac{1}{6}) + (\frac{1}{9} - \frac{1}{10}) + \dots = (1 - \frac{1}{4} + \frac{1}{5} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \dots) + (-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \dots) = (\frac{2}{2} - \frac{2}{8} + \frac{2}{10} - \frac{2}{16} + \frac{2}{18} - \frac{2}{20} + \dots) + (-\frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \dots) = \frac{1}{2} + \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{10} + \frac{1}{12} - \frac{1}{14} - \dots = (\frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \frac{1}{14} + \dots) + (\frac{1}{4} - \frac{1}{9} + \frac{1}{12} - \frac{1}{16} + \dots) = \frac{1}{2}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots) + \frac{1}{4}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots)$. The alternating series on the left is just the famous Leibniz formula for $\frac{\pi}{4}$ (try to prove this!), whereas the alternating series on the right is the Taylor expansion of $\log(2)$. Hence the sum is $\frac{\pi}{8} + \frac{\log(2)}{4}$.

3. How many ways can we choose three non-empty and non-intersecting subsets from $\{1, 2, \dots, 2011\}$?

Solution: If we drop the 'non-empty' condition, there are 4^{2011} ways to select three non-intersecting subsets (for each element of the set, we must choose between one of four options: either include it in one of the three subsets, or leave it out). Now apply the inclusion-exclusion principle to the non-empty condition. For each of the subsets, exactly 3^{2011} of the ways counted above have the subset as empty (instead of four options for each element, there are only three). Similarly, for each pair of subsets, there are 2^{2011} ways to have both of the subsets empty. Finally, there is $1^{2011} = 1$ way to have all three subsets empty. Moreover,

In the first, it is clear that the position marked (a) will force a three-in-a-row. In the second, after filling in the obvious forced moves, it becomes clear that the position marked (b) will force a three-in-a-row.

6. A calculator is broken so that the only keys that work are $\sin, \cos, \tan, \arcsin, \arccos$ and \arctan . The display originally shows 0. Prove that for any positive rational q , there is a finite sequences of keys that will return q (ignoring decimal place restrictions).

Solution: In fact, we can do better: we can return any number of the form \sqrt{q} where q is a positive rational (and so in particular $\sqrt{q^2}$). First, notice that since $\sin(\arctan(\sqrt{\frac{1}{x}})) = \sqrt{\frac{1}{x+1}}$, we can initially get any number of the form $\sqrt{\frac{1}{n}}$ for n a positive integer. Second, notice that for any number x we have displayed, we can perform the operation: (1) $\tan(\arccos(\sin(\arctan(x)))) = \tan(\frac{\pi}{2} - \arctan(x)) = \frac{1}{\tan(\arctan(x))} = \frac{1}{x}$. Third, notice that for any $\sqrt{\frac{1}{k}}$ we have displayed we can perform the operation: (2) $\cos(\arctan(\sqrt{\frac{1}{k}})) = \sqrt{\frac{k}{k+1}}$. Performing operations (2) and then (1) we get the operation: (3) $\sqrt{\frac{1}{k}} \rightarrow \sqrt{1 + \frac{1}{k}}$. Starting with our original $\sqrt{\frac{1}{n}}$, and performing successive operations (3) (perhaps multiple times) and (2), we find that we can write the square root of any positive finite continued fraction. Since any positive rational number can be expressed as a positive finite continued fraction, we are done.

7. (From Issue 1, 2011) $2n$ people sit around a table with k chocolates distributed among them. A person may give a chocolate to their neighbour, but only after first eating one themselves. What is the minimum k such that *everyone* can get a chocolate?

Solution: This question will remain open until a solution is provided!

— Stephen Muirhead

The mathematics are distinguished by a particular privilege, that is, in the course of ages, they may always advance and can never recede.

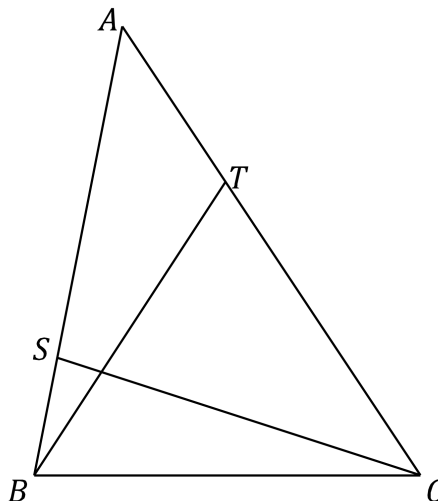
— Edward Gibbon, *Decline and Fall of the Roman Empire*

Paradox Problems

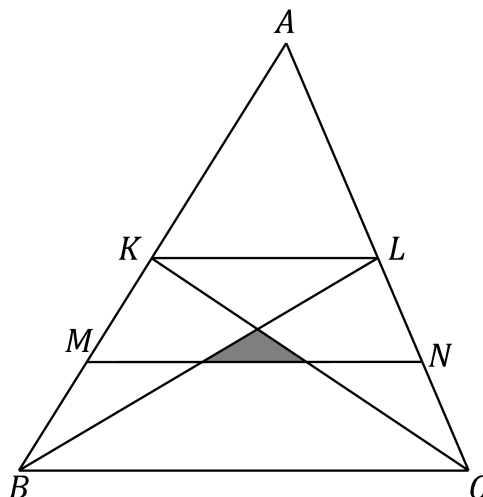
Below are some puzzles and problems for which cash prizes are awarded. Anyone who submits a clear and elegant solution may claim the indicated amount. Anyone who is able to do so for all seven questions will be awarded a \$50 voucher from the Melbourne University Bookshop!

Either email your solutions to the Editor (paradox.editor@gmail.com) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

1. (\$2) Evaluate $\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{2011^2}\right)$.
2. (\$3) A tromino is an L -shaped tile made of three connected unit squares. how many ways are there of tiling a $3 \times n$ chessboard with trominoes where n is a positive integer? (Every square must be covered and overlaps are forbidden).
3. (\$3) Let ABC be a triangle with $\angle ABC = 80^\circ$ and $\angle BAC = 40^\circ$. Let S and T be points on segments AB and AC respectively with $\angle BCS = 20^\circ$ and $BT = SA$. Find $\angle STA$.



4. (\$3) Let ABC be a triangle with area 1 and let K, L, M, N be the midpoints of AB, AC, KB, LC respectively. Find the area of the triangle formed by lines KC, LB and MN .



5. (\$4) There are 33 knights on a chess board. Prove that one of the knights is attacking at least two other knights.
6. (\$4) What is the smallest positive integer n such that $n \nmid 2^{2^{2^{2^2}}} - 2^{2^{2^2}}$?
7. (\$5) (From Issue 1, 2011) $2n$ people sit around a table with k chocolates distributed among them. A person may give a chocolate to their neighbour, but only after first eating one themselves. Nominating a *head* of the table, what is the minimum k such that, irrespective of the initial distribution of the lollies, there is a way for the *head* to get a chocolate? What is the minimum k such that *everyone* can get a chocolate?

— Andrew Elvey-Price

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