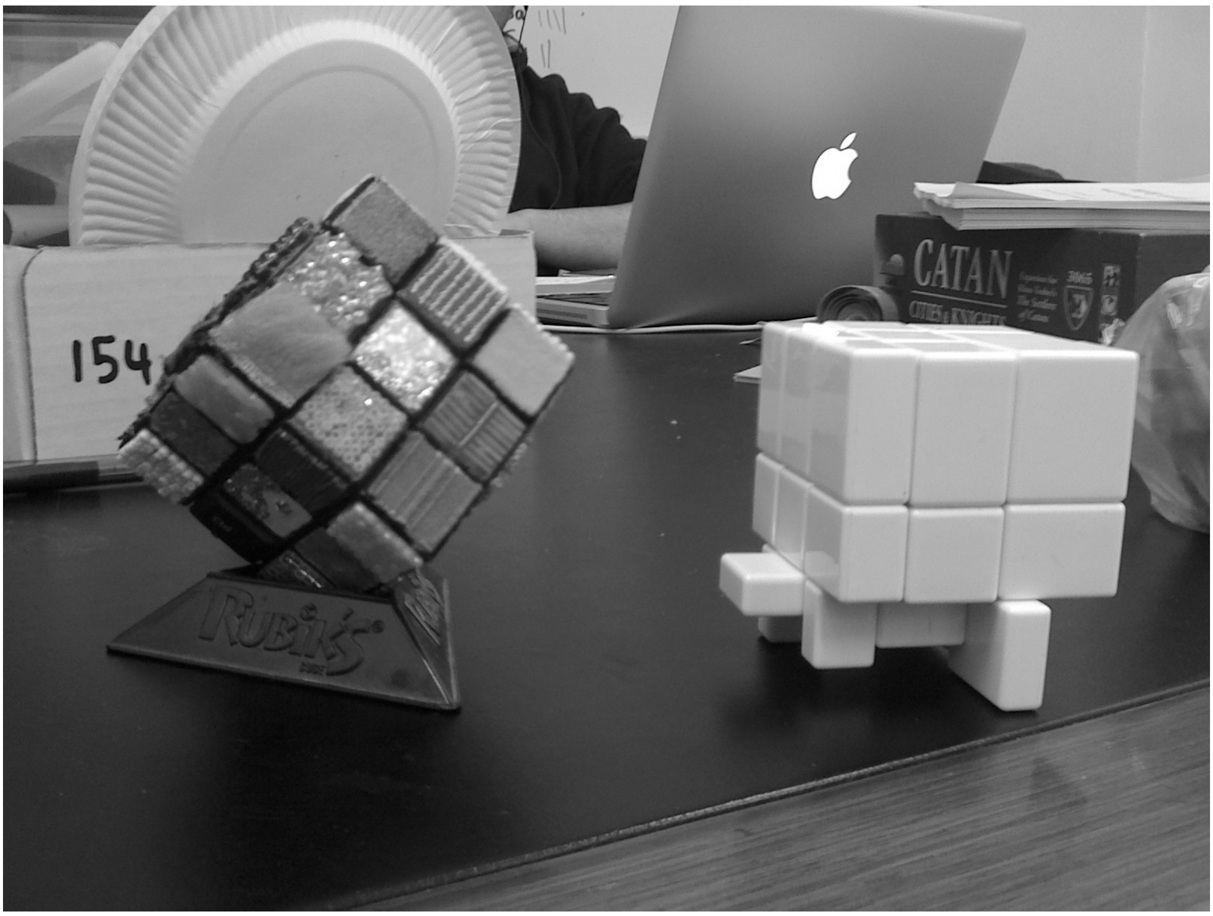
Issue 1, 2012

Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society



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Cover:	*Rubik's Turtle* embarks on his first mission against Cyril and his Circus of Circuits! (See page 6 for more.)

In This Edition of Paradox

Regular Features

Special Features

Puzzles

Two men lost in a hot air balloon drift through clouds for hours. Suddenly, the clouds part, and the two men see the top of a mountain with a man standing on it. "Hey! Can you tell us where we are?!"
The man doesn't reply. Minutes pass as the balloon drifts past the mountain. When the balloon is about to be swallowed again by the clouds, the man on the mountain shouts: "You're in a balloon!"
"That must have been a mathematician." "Why?" "He thought long and thoroughly about what to say. What he eventually said was irrefutably correct. And it was of no use whatsoever..."

Words from the Editor

Welcome to 2012's first issue of Paradox, the magazine produced by the Melbourne University Mathematics and Statistics Society (MUMS). Over the past few years, this publication has morphed from a problem-solving extravaganza to a forum for bad maths jokes, and then a recovery period coinciding with a preoccupation about a now world-famous past Vice-President.

Now you will find some bad maths jokes still, but they are scattered throughout this issue. The next MUMS alumnus to be interviewed by Paradox is quite a local celebrity, and the next biography tells the life of a famous mathematician who was so accomplished in all that he did that even some of his contemporaries began to doubt his ability to do truly anything!

One of the unsolved problems assigned in the last issue receives a thorough discussion as to why it still remains unsolved, and a particular profession's prestige is traced back to a series of difficult mathematics examinations once held at the University of Cambridge. Meanwhile, our new photo comic strip hero embarks on his very first mission in Episode 2 of *The Adventures of Rubik's Turtle*!

While Paradox has existed since 1981, another publication by MUMS in the past has recently been unearthed in the depths of the library archives. If you want to help us discover more about this publication, please send us an email to paradox.editor@gmail.com. If you are instead interested in learning more about Paradox in the past, there are copies of old issues in the library as well as on our website.

The problems from the last issue of Paradox remain unsolved, and the grand prize unclaimed. Will you be able to solve them all before the next issue comes out? If not, feel free to contribute an article, or better yet, something new of your own!

— Kristijan Jovanoski

Q: Do you already know the latest stats joke?
A: Probably...

Words from the President

Hello and welcome to another year at MUMS. This semester, we have our usual array of activities including weekly seminars, trivia and games nights, and our much-loved Puzzle Hunt.

As this is the first issue of Paradox for the year, let me take this opportunity to encourage all of you to participate in MUMS activities. The Melbourne University Mathematics and Statistics society is a club that is run by students who share a common love for mathematics (the statistics bit may need a bit of work). Our aim is to expose students to mathematics that is fun, fascinating, and not covered in coursework. That is, we just do fun stuff!

If you want to know what MUMS is all about, the best way to find out is to head over to G24 in the Richard Berry Building (otherwise known as the MUMS room) to meet our regular members and ask them. The room is open to everyone, so don't feel shy to just pop in and say hello. The MUMS room is a place to rest and hang out with other students. Very little work is done in this room, and that's the way we like it. Of course, if you have a cool maths problem we enjoy those too!

MUMS also runs weekly seminars where we invite guest speakers to talk about interesting mathematics. This semester, they will be held at 1:00 pm every Friday in the Latham Theatre on the ground floor of the Redmond Barry Building. Listening to talks about maths during lunch may seem odd, but let me assure you that there is nothing better that you could possibly be doing. Look around the maths building for posters and abstracts!

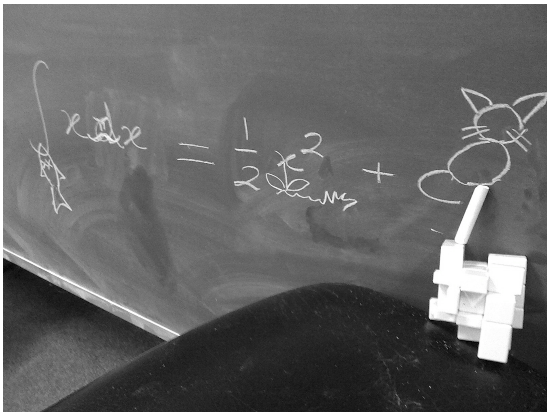
Lastly, the Puzzle Hunt is the biggest event run by MUMS and will begin shortly after the Easter break. It is a week-long puzzle-solving extravaganza involving teams from all over the world. If you've never participated before, then I strongly encourage you all to do so. The puzzles themselves are not mathematical, but they do take quite a bit of imagination to solve. For examples of past hunts, please visit our website.

But for now, enjoy the rest of Paradox. I hope to see all of you throughout the year. All the best!

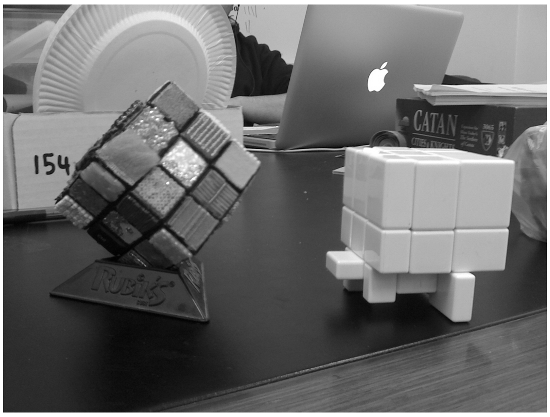
— TriThang Tran

The Adventures of Rubik's Turtle

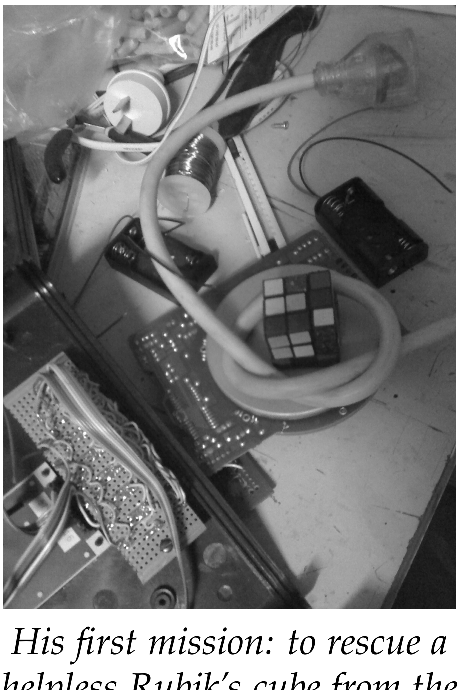
Episode 2: Cyril's Circus of Circuits



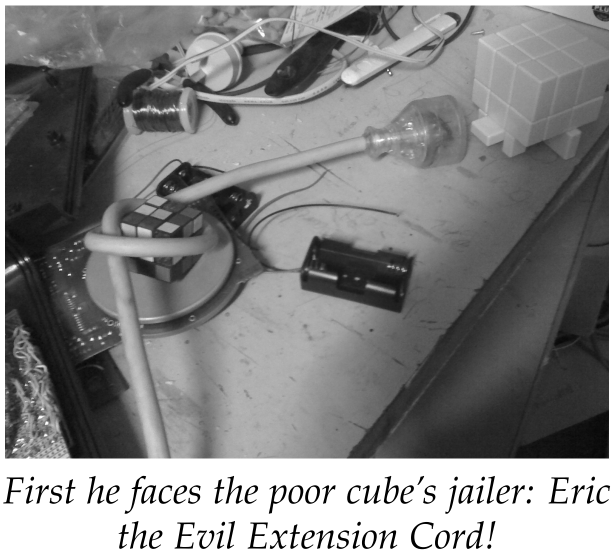
Previously, a poor orphan Rubik's cube is rejected by society and seems destined for a life of petty crime.



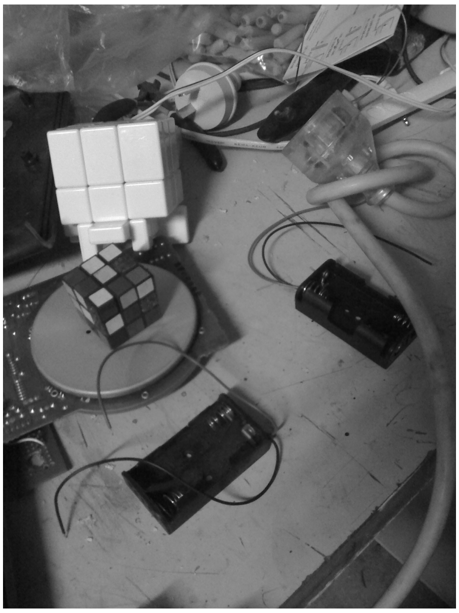
Until he meets the mysterious Mother-of-all-Rubiks, who teaches him to become the hero Rubik's Turtle*!*



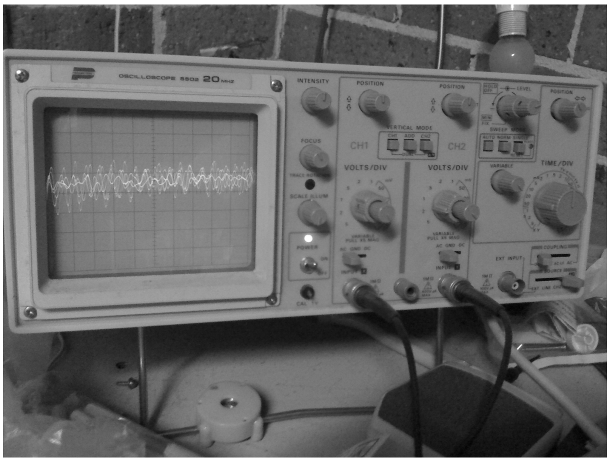
His first mission: to rescue a helpless Rubik's cube from the clutches of the evil Cyril and his Circus of Circuits!



First he faces the poor cube's jailer: Eric the Evil Extension Cord!

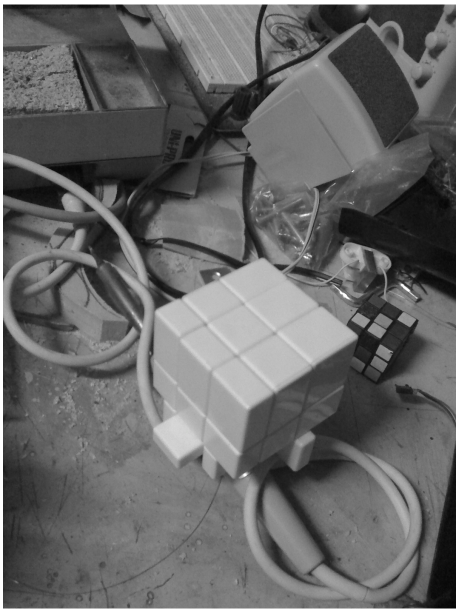


He defeats the fiendish power cable and frees the cube!

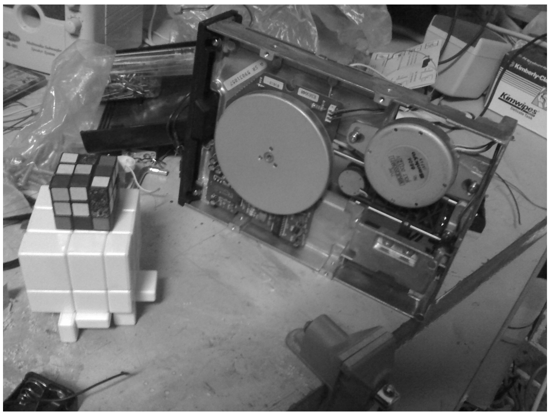


But before they can escape, the disturbance is noticed and the alarm is raised by Oliver the 'Orrible Oscilloscope!

Their escape is blocked by another of Cyril's minions: Sid the Sneaky Soldering Iron.

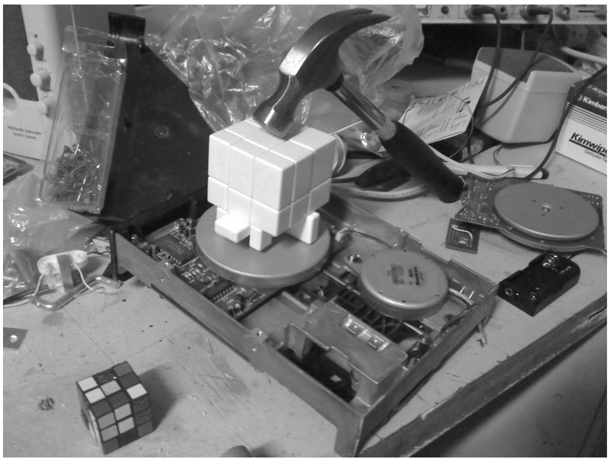


But Rubik's Turtle *defeats the villain and they continue their flight!*



Another scoundrel blocks their way: Derek the Dastardly Disc Drive…

…who is also overcome by the courageous Rubik's Turtle*!*



But before he has time to celebrate his victory, he is knocked unconscious by an attack from behind from Harry the Horrible Hammer!

When he awakes, he finds himself tied up, and face to face with the fiendish floppy disc and self-proclaimed ringmaster Cyril himself!

"You've caused us quite some trouble, boy! So now you shall die!" cries the wicked Cyril as the heated Sid moves to deal a fatal blow...

Is this the end for our new hero? Has he failed, in his very first mission? Find out, in Episode 3 of *The Adventures of* Rubik's Turtle!

— Dougal Davis and Jinghan Xia

A math professor, a native Texan, was asked by one of his students: "What is mathematics good for?" He replied: "This question makes me sick! If you show someone the Grand Canyon for the first time, and he asks you 'What's it good for?' What would you do? Well, you kick that guy off the cliff!"

Mathematical Miscellany

1. A sphere has two sides but there are one-sided surfaces.
2. In a group of 23 people, at least two have the same birthday with the probability greater than 1/2.
3. There are curves that fill a plane without holes.
4. A clock never showing the right time might be preferable to one showing the right time twice a day.
5. The next sentence is true but you must not believe it.
6. The previous sentence was false.
7. Among all shapes with the same perimeter, a circle has the largest area.
8. Among all shapes with the same area, a circle has the shortest perimeter.
9. One can divide a pie into eight pieces with just three cuts.
10. There is something the dead eat but if the living eat it, they die.
11. The billionth digit of π is 9.
12. $111,111,111 \times 111,111,111 = 12,345,678,987,654,321$.
13. You would have to count to one thousand to use the letter 'A' in the English language to spell a whole number.
14. Almost everything you can do with a ruler and a compass you can do with the compass alone.
15. In elliptical geometry, the sum of the angles in a triangle does not have to be equal to 180°.

Three statisticians go hunting. When they see a rabbit, the first one shoots, missing it on the left. The second one shoots and misses it on the right. The third one shouts: "We've hit it!"

Interview with a MUMS Alumnus

Background

Andrew Kwok(star) is a former member of MUMS. He studied at Melbourne University from 2003 until 2008, graduating with Honours in a Bachelor of Commerce (majoring in Actuarial Studies) and a Bachelor of Science (majoring in Statistics). After completing two internships in Hong Kong and a graduate program at AXA in Melbourne, he now works with AMP (which merged with AXA) as a Strategy Analyst in Sydney.

MUMS

Why did you join MUMS?

I'd kind of been involved in it since high school. In Year 9, we participated in the SMO (School Maths Olympics) and came first that year, beating Scotch [College]. We had a healthy competitive rivalry with Scotch all the way until Year 12. When uni started, I became more involved in activities and eventually became first year rep, then Treasurer the next year (2004) and then President the following year (2005). Also, I already had friends there who were closely involved in MUMS, through my IMO (International Mathematics Olympiad) training days.

Did you really bribe the first years to win MUMS presidency? No, I remember you were wildly popular in MUMS. Do you want to tell me what that was all about?

I don't know why [laughs]. High grades? Most people were friends from high school whom I knew well. I have no idea.

Yes, but your nickname was KWOKSTAR! Does it have anything to do with the fact that your average grade is 99?

[Laughs] No, it's not that high! Let's say around the mid 90s . . . it's harder to score high marks in commerce, especially non-maths ones.

What did you get up to in the MUMS room?

Sometimes we played games, sometimes we did work, sometimes we just hung out and chatted. My favourite MUMS event was probably the UMO (University Maths Olympics). The best thing about MUMS is the friends you make, and all the activities that happen throughout the year. The MUMS room is just a fun place to go where you can hang out!

Career

Tell me about your internships in Hong Kong.

The first was with a local consulting firm, the second was actuarial consulting with a multinational firm. The first internship gave me my first real experience of working in an actual firm. Since working is so different from studying, you quickly realise there's all these other skills that you need. I learnt a variety of skills including communication, writing reports, time management, etc.

Once you have some work experience, you find it much easier to get work experience at bigger, multinational firms. My first consulting role helped me gain the actuarial consulting role as part of the second internship. That internship was awesome! I had the opportunity to apply the actuarial knowledge I'd learnt at uni, learnt a lot about financial markets across Asia, developed more technical skills such as Excel and Database and also made a real contribution on live projects.

Why did you leave Hong Kong for the graduate program at AXA?

Melbourne was the headquarters for AXA's Asia-Pacific operations, so I believed there would be a lot of different opportunities available working in the head office. They have a really good graduate program: it's a three-year program with five rotations. Each rotation is a full-time role, so straight away you're doing real work and learning a lot on the job. They have a strong support program to continue with your actuarial examinations. In addition, the culture at AXA felt right for me. There was also the opportunity to rotate to Hong Kong!

Which rotations did you choose, and what did you do in each?

In total I did three rotations: two actuarial rotations (one in finance, one in pricing), then one rotation in Group Strategy.

The first rotation was in the Finance Department where my role involved calculating profits for AXA's income protection business in Australia and New Zealand. This is a type of life insurance policy that provides you income if you're unable to work for a period of time. During reporting periods, I would run projection models to estimate the future claims that we'd likely need to make compared to the revenue we were likely to receive in the future. Based on these long-term projections, I could then estimate the profit earned during each period. I also assisted in writing regular reports to provide more detailed analysis and commentary on the experience. During non-reporting periods, it'd be a matter of helping various teams perform ad-hoc analyses as well as improve projection models and overall processes to increase efficiency.

The second rotation was in the Product Department where I assisted in pricing and risk management for a capital guaranteed product called North. The idea of the guarantee is that if markets go up, your investment will also go up, but if the markets go down over a period (usually 1 year or 2 years), then the investment will stay flat. In return for receiving the guarantee, the investor needs to pay a fee. Part of my role was assisting the team in calculating what fees we needed to charge to ensure that we could cover the guarantee and other costs associated with managing the product. This area requires quite complicated postgraduate mathematics in areas such as stochastic calculus in order to calculate the fee, since you need to allow for a large variety of possible future scenarios.

At the moment your permanent role is with strategy. Why did you prefer that rotation?

There were a variety of reasons. One of the benefits of the graduate actuarial program is that they encourage you to try one rotation in a non-actuarial team. So I decided to give Strategy a try. While the actuarial rotations were very analytical and focused on particular areas, Strategy gave me a high level overview of the company, where I needed to understand how each part of the company operated. It provided a good opportunity to meet people from across the business. I still had the opportunity the continue developing analytical skills (being an actuary by profession, they trust that I have strong analytical skills), but I was also able to improve other skills including stakeholder management and strong communication skills through writing reports and preparing presentations.

During my rotation in Strategy, AXA sold off their Australian and New Zealand businesses to AMP. When the merger happened, AMP offered me a permanent role as a Strategy Analyst in Sydney, where I've been working since July last year.

Does one have to train as an actuary to do what you do, or can they just study maths?

For the finance and pricing rotations, it is important to have strong actuarial knowledge and skills. During these rotations, I continued studying for actuarial examinations in order to qualify as a Fellow of the Institute of Actuaries of Australia (FIAA) and as a Chartered Enterprise Risk Actuary (CERA).

In Strategy, you'll find people working from a variety of backgrounds by no means limited to actuarial since this work requires a variety of skills, such as strong communication (both written and verbal), the ability to think strategically as well as strong stakeholder management and negotiation skills.

And does studying maths help with that? Before you said that studying was very different to working.

Not really, actually! Studying mathematics can help you develop strong analytical skills; programming is also useful. Analytical skills including problem solving are useful, but other skills are equally important, such as strong interpersonal skills, time management, communication etc. You can't be a brilliant analytical person but find it difficult to communicate your findings with others. People need to be able to work with you on a regular basis.

So learning maths at uni encourages analytical and problem-solving skills that are employers seek, but you can't just be a bookworm?

Yes, strong analytical skills and great academic results may get you to the interview stage (plus if you have some work experience it always helps) but often it's the other skills that you display during the interview stage that will help secure you the job. Have a personality! I think it's a combination of competent technical knowledge (in whatever field you're in) and whether you'd think they'd work well in your company (which is partly a gut feeling). Working in a company with the right culture is important.

I'm sure final year students will appreciate your tips. So why did you study maths at uni?

I was interested in maths and I was good at it in high school. But I always wanted to work in a company and commerce was an area that interested me. Actuarial was a good compromise between wanting to work in a company and doing something analytical. For people who want to study commerce but enjoy mathematics, I think finance or actuarial studies is a good option.

Random and Interesting

Did you ever go on exchange?

During uni I started learning Japanese, and became more interested in Japanese culture. Hence, I went on exchange to Tokyo University in Japan to study Japanese language, Japanese linguistics, and international relations. I strongly recommend going on exchange! For me, it's definitely been one of the most fun times of my life so far. There were heaps of awesome experiences, I made many great friends from across the world and got the chance to immerse myself in a completely different culture while developing my foreign language skills. I think you also develop a broader, more international perspective, which helps you observe and appreciate things from multiple angles.

Why do you come down to Melbourne to get your hair cut? Are the haircuts in Sydney really that bad?

[Laughs] That's more out of convenience! I come back to Melbourne on a regular basis, sometimes for work, sometimes to visit family. My hairdresser in Melbourne is a family friend.

I hear you are quite popular with the ladies at AXA ... your family friend must be really good!

No comment! I think your questions are starting to get sidetracked ...

— Lu Li

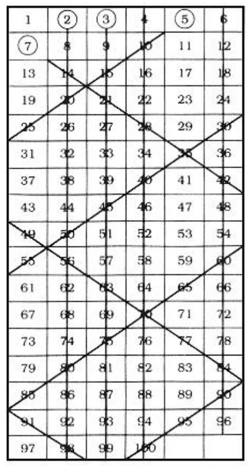
If you are interested in being interviewed for Paradox, please send an email to paradox.editor@gmail.com. Include your name, occupation, and relation to or interest in MUMS.

An actuary is someone who brings a fake bomb on a plane to decrease the chances that there will be another bomb on the plane.

Biography: Eratosthenes (276-195 BC)

The ancient Greek mathematician Eratosthenes was considered a leading all-rounder in his time, for he was also a widely-acclaimed astronomer, athlete, music theorist, and poet. A Renaissance man centuries before the Renaissance, he was called Pentathlos, champion of multiple skills, by some of his contemporaries, and his breadth of knowledge secured him the prestigious post of Librarian in the Great Library of Alexandria, Egypt, the greatest repository of classical knowledge until it was tragically destroyed in 642 AD.

Mathematics best remembers Eratosthenes of Cyrene for the simple algorithm he developed for finding prime numbers, now known as the Sieve of Eratosthenes. It is able to find all prime numbers up to any given limit by iteratively marking as composite (not prime) the multiples of each prime, starting with the multiples of 2.



The sieve.

The multiples of a given prime are generated by starting from that prime and marking the sequence of numbers with the same difference, equal to that prime, between consecutive numbers. The Sieve of Eratosthenes, despite seeming rather simple and primitive, is actually one of the most efficient ways to find all of the smaller primes below numbers approaching tens of millions.[1]

One of his lasting achievements was coining the word 'geography' and inventing the discipline of geography that remains with us today. He also developed a system of latitude and longitude, and was the first person to not only calculate the circumference of the Earth (using the length of stadiums of his time as units of measurement), but also calculate the tilt of the Earth's axis with remarkable accuracy for his time.

These achievements were evidently not enough for him as he also had enough time to sketch the course of the Nile from the sea to Khartoum, and he correctly predicted that its source lay in great upland lakes that he would never

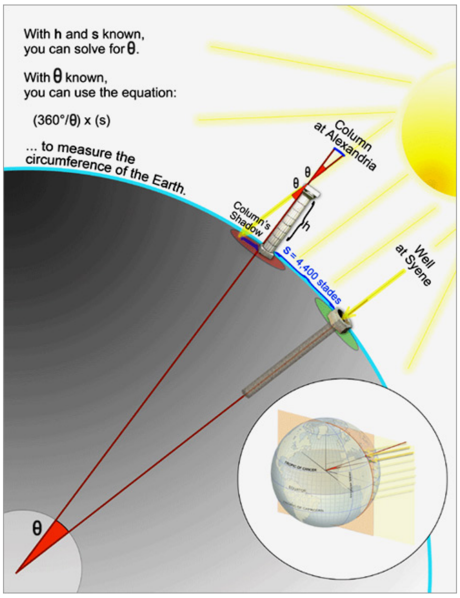
[1] Euler's proof of the zeta product formula, discussed later in this issue of Paradox, contains a version of this sieve in which each composite number is eliminated exactly once.

visit. He also deduced the length of the year as 365.25 days and suggested that calendars should have a leap day every fourth year. He criticised Aristotle's belief that humanity was divided into Greeks and barbarians, and he believed that there existed good and bad in all peoples. Eratosthenes became blind in old age and possibly committed suicide by starvation at the ripe old age of 82.

Interestingly, despite being so well-accomplished in so many different areas, Eratosthenes was widely considered to fall short of the highest rank, and was mocked by some of his detractors as a jack-of-all-trades and master of none:

> *[Eratosthenes] was, indeed, recognised by his contemporaries as a man of great distinction in all branches of knowledge, though in each subject he just fell short of the highest place. On the latter ground he was called Beta, and another nickname applied to him, Pentathlos, has the same implication, representing as it does an all-round athlete who was not the first runner or wrestler but took the second prize in these contests as well as others.*[2]

— Kristijan Jovanoski



How Eratosthenes calculated the circumference of the Earth.
(Source: NOAA National Ocean Service Education)

[2] Heath, T.L. (1921) *A History of Greek Mathematics*.

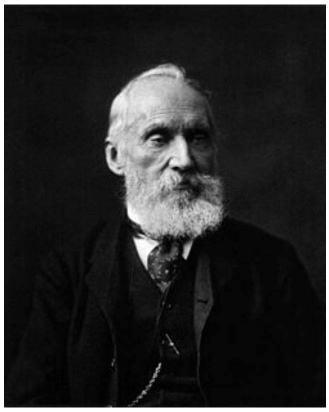
An Inquiry into a Public Preconception of Actuarial Science

Whenever I tell strangers that I am studying actuarial studies, I am instantaneously received with immense flattery as they compliment my mathematical abilities and my general supposed superior intellect. I greet this with mixed reactions; such praise inevitably catalyses the aggrandisement of my self-esteem yet I am also filled with curiosity regarding why exactly am I labelled "smart" by the layman? Yes, an actuary employs "complicated" mathematics to solve problems, but so do many other occupations, such as physicists, econometricians, and engineers. A qualified actuary must pass a series of tough professional exams, but so must land surveyors, accountants, and software engineers. Surely there exist other explanations? Therefore, I felt it incumbent upon myself to investigate this matter further by delving deep into history to uncover some possible clues.

The Tripos

My search brought me to the (old) Mathematical Tripos, a series of difficult examinations sat by undergraduates studying Mathematics at the University of Cambridge. From 1753 to 1909, the aggregate score of each student was tallied and the results of the successful candidates were published in order of merit. Those who gained first class honours earned the title of "Wrangler", with the highest scoring student named "Senior Wrangler", the second highest "Second Wrangler" and the lowest scoring Wrangler was aptly named the "wooden spoon". Although the actual marks were never made publicly, Forfar[1] noted that in one year, out of a possible 17,000 marks, the Senior Wrangler obtained 7634 marks, the second Wrangler obtained 4123 marks, and the wooden spoon obtained around 1500 marks. The lowest scoring candidate who received the degree obtained just 237 marks. These results may well provide some insight into the difficulty of the examinations and why earning the title of Senior Wrangler was considered to be quite an outstanding intellectual accomplishment.

[1] See Forfar, D.O. (1996). What Became of the Senior Wranglers?, *Mathematical Spectrum*, **29**(1) for more details. The article draws on this paper throughout.



Naturally, much attention centred around those who achieved the top rankings. However, since the exams where largely a test of speed in the application of familiar techniques together with memory work, many Senior Wranglers were in fact diligent, hardworking students rather than exceptionally original. One example pertains to the mathematical physicist Lord Kelvin, who thought that he would unquestionably be the undisputed Senior Wrangler. He asked his servant to run down to the Senate House and find out who was Second Wrangler on the day the results were published. To his great despair, his servant returned and announced "You, Sir."

Lord Kelvin had been beaten by Stephen Parkinson, who did not possess the same level of mathematical originality as Lord Kelvin (who later won the Smith's Prize for original research), but was nevertheless exceptionally intelligent and trained himself to solve problems at great speed. Other prominent inventive and original students who did not gain the top ranking include Hardy[2] (fourth), Russell[3] (seventh), Malthus[4] (ninth) and Keynes[5] (twelfth).

To Actuarial Science

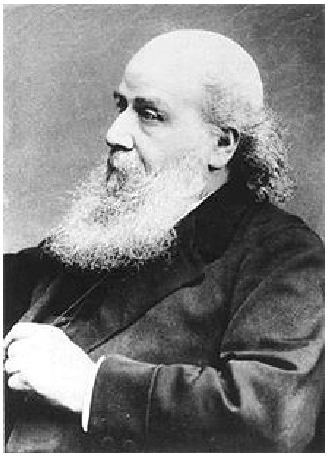
Although many of the top Wranglers were eventually appointed as professors of mathematics, this was not always the case. Indeed, many saw the Tripos as paving a pathway for other professions such as those in medicine, law, the church, and actuarial practice. Since actuarial science relies heavily on mathematics, in particular probability theory and statistics, many esteemed Wranglers entered the actuarial profession. This, to some extent, provides a credible answer to the compliments actuaries receive about their mathematical abilities; it is justified on the grounds that many actuaries are in fact top mathematics students who decide to dedicate their lives instead to actuarial practice.

[2]Pure mathematician and author of *A Mathematician's Apology*. See *Great Lives* in Paradox Issue 1, 2011 for more about Hardy's life.

[3]Logician and one of the founders of analytic philosophy.

[4]Early economist known for his controversial publication *An Essay on the Principle of Population*, which hypothesised the destruction caused by excessive population growth.

[5]Macroeconomist advocating the use of fiscal and monetary measures to navigate the economy.



For instance, although James Joseph Sylvester, a Second Wrangler, was better known for his contributions in pure mathematics,[6] he nevertheless made significant contributions to the actuarial profession. He played a central role in the establishment of the world's first actuarial professional body in 1844, the Institute of Actuaries, with the responsibility for setting the professional standards for an actuary.[7] In the Royal Charter, the role of an actuary was formally defined as a practitioner dealing in matters in connexion with "financial questions particularly in reference to those numerous and important questions involving the scientific application of the doctrine of probabilities and the principles of interest".

Sylvester, among 94 others, were instantly fellows of the Institute and worked hard immediately after the Institute was conceived. Among his most important roles was preparing a syllabus. It was decided that there would be three exams; the first, a mathematical exam in "Arithmetic and Algebra, the elementary doctrines of Probability, Simple and Compound Interest, and in the theories of Life Assurance and Annuities", the second, on the elementary principles of constructing the life table and the third, on bookkeeping and questions related to actuarial practice.

Similar to mathematics, symbols and notations are also vital in actuarial practice as a means of conveying the greatest amount of information in as few strokes as possible. Hence, they have to be efficient, memorable, and malleable. To the actuary, these symbols are superficial, but to the bystander, they most likely appear inscrutable and exceedingly alienating. It may well be due to the discouraging nature of such symbols to the untrained eye, which causes the layman to perceive the actuary as intelligent for he understands such symbols with much ease. Not only do actuaries employ the usual $\int$, $\sum$ and e^x with their usual mathematical interpretations, but also they have invented their own elaborate system to denote frequently used formulae; e.g.

$$a_{\overline{n}|} = \frac{1 - v^n}{i};$$

[6] Among some of his achievements are coining the words 'graph' and 'discriminant'.

[7] See *James Joseph Sylvester: Jewish Mathematician in a Victorian World* by Karen Hunger Parshall for more information.

$$(\bar{I}\bar{a})_{x:\overline{n}|} = \int_0^n te^{-\delta t}{}_t p_x \, dt;$$

$${}^{p}_{t|}A^{(m)1}_{x:\overline{n}|} = \sum_{r=mt}^{m(t+1)-1} v^{p(r+1)/m} \cdot {}_{r/m}p_x \cdot {}_{1/m}q_{x+r/m}.$$

To demonstrate the powerful simplification process of such notation, the second formula represents the mathematical expectation of the present value random variable calculated at a force of interest of δ p.a., where a payment at a rate of \$$t$ p.a. is paid continuously for a maximum of n years, provided that an individual now aged x is still alive in t years' time.

Many other Wranglers also proceeded into actuarial science: Pell (1849 Senior Wrangler), Sprague (1853 Senior Wrangler) and Friend (1780 Second Wrangler) and within Europe, many prominent mathematicians such as Cramer, Lundberg, and Lah also made great contributions to the field. Although the Old Tripos system was abolished following much scrutiny (in particular from Hardy) and replaced with a new system, where only the class of degree was publicly available, an examiner "unofficially" now reveals the identity of the Senior Wrangler by tipping his hat when reading out the student's name.

In recent times, many Wranglers continue to pursue actuarial science; McCutcheon (Wrangler 1962), now an Emeritus Professor of Actuarial Studies, has made significant contributions to the field, although he is known to many actuarial students as co-author of *An Introduction to the Mathematics of Finance*, considered by some to be a timeless textbook. Therefore, it is ostensibly clear that much of the merit and distinguishing prestige attached to the modern-day actuary is actually largely attributed to the successes of their predecessors.

— Timothy Lee

Q: What do you call an actuary who is talking to someone?
A: Popular.

The Riemann Hypothesis

1644.

Pietro Mengoli poses the following problem: determine the exact value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \simeq 1.645.$$

Do you know what the value is? Can you guess who proved it?

1735.

Leonhard Euler announces that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Wow. Here follows Euler's original proof. Some steps require detailed justification, but Euler knew that his answer was correct because he checked some partial sums. He did not present a completely rigorous proof until 1741; however his original proof is typically clever. A modern approach would use Fourier series.

Proof. By Taylor expansion of $\sin(x)$,

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots.$$

Note in particular that the coefficient of x^2 in the Taylor expansion of $\frac{\sin(x)}{x}$ at $x = 0$ is

$$[x^2]\frac{\sin(x)}{x} = -\frac{1}{6}. \quad (1)$$

One can check that the roots of $\frac{\sin(x)}{x}$ (in $\mathbb{C}$) are $n\pi$, for $n \in \mathbb{Z} \setminus \{0\}$ (use the expression $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$). Factorizing into linear factors, analagously to the fundamental theorem of calculus,

$$\frac{\sin(x)}{x} = \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\left(1+\frac{x}{3\pi}\right)\cdots \quad (2)$$

$$= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots.$$

Now

$$[x^2]\frac{\sin(x)}{x} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

so, recalling equation (**??**),

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

Solving this problem made Euler famous, for it had stumped the leading mathematicians of his day. It is now known as the *Basel problem*, after Euler's hometown in Switzerland.

Using Fourier series, you can also prove that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

Using modern complex analysis, it is not too difficult[1] to determine the value of $\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$ in terms of *Bernoulli numbers*, for $k \in \mathbb{Z}_{>0}$:

$$\zeta(2k) = (-1)^{k+1}\frac{B_{2k}(2\pi)^{2k}}{2(2k)!}, \qquad k \in \mathbb{Z}_{>0}. \tag{3}$$

The series

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

certainly converges for $s = 2, 3, \dots$, however the values are not as easy to compute when s is odd. It was not until 1979 that Apéry proved that $\zeta(3)$ is irrational.

The harmonic series

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

diverges. However, $\zeta(s)$ converges for all $s \in \mathbb{R}_{>1}$, by the integral test. We can now extend the domain of the function ζ:

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots, \qquad s \in \mathbb{C}, \mathrm{Re}(s) > 1,$$

[1] You might learn this in a Masters course. See Flajolet and Sedgewick, *Analytic Combinatorics*, pp. 268-9.

since the series converges absolutely. Note that the function ζ is complex analytic.

The domain of ζ can be further extended to $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\} \setminus \{1\}$ as follows.[2] Let $s \in \mathbb{C} \setminus \{1\}$ be such that $\sigma := \mathrm{Re}(s) > 0$. By a telescoping series,

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \left[n^{-s} - \int_n^{n+1} x^{-s} dx \right] = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s}) dx. \quad (4)$$

For $n \in \mathbb{Z}_{>0}$ and $x \in [n, n+1]$,

$$\begin{aligned} |n^{-s} - x^{-s}| = \left| s \int_n^x y^{-1-s} dy \right| &\leq |s| \int_n^{n+1} |y^{-1-s}| dy \\ &= |s| \int_n^{n+1} |y^{-1-\sigma}| dy \leq |s| n^{-1-\sigma}. \end{aligned} \quad (5)$$

As

$$\sum_{n=1}^{\infty} |s| n^{-1-\sigma} = |s| \sum_{n=1}^{\infty} n^{-1-\sigma}$$

converges absolutely, the comparison test implies that the series in equation (**??**) converges absolutely. Thus, we can extend ζ using the definition

$$\zeta(s) := \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s}) dx, \qquad s \in \mathbb{C} \setminus \{1\}, \mathrm{Re}(s) > 0. \quad (6)$$

Moreover, ζ is complex analytic in this region, being a uniform limit of complex analytic functions (namely the partial sums) on any compact subset of $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\} \setminus \{1\}$.

1737.

Euler publishes the following amazing result, now known as the *Euler product formula*: for $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

His method is remarkably simple, and is an example of a *sieve*.

[2] This is from Noam Elkies' notes, found at `http://www.math.harvard.edu/~elkies/M259.02/zeta1.pdf`.

Proof. Let $s \in \mathbb{C}$ be such that $\mathrm{Re}(s) > 1$.

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots. \tag{7}$$

$$\implies \frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots. \tag{8}$$

(**??**) − (**??**) sifts out the multiples of 2:

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \dots. \tag{9}$$

$$\implies \frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \dots. \tag{10}$$

Considering (**??**) − (**??**), we have sifted out the multiples of 2 and 3:

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \dots.$$

Continuing in this way, we sift out the multiples of p for each prime p, and we are left with

$$\zeta(s) \prod_{p \text{ prime}} (1 - p^{-s}) = 1 \qquad \therefore \zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

□

The Euler product formula shows immediately that there are infinitely many prime numbers, since the harmonic series $\zeta(1)$ diverges. So far we have done nothing high-powered. A genius scribbled a few things down and worked out some cool stuff. Little did people know how much the game was going to change.

1859.

Bernhard Riemann publishes an eight-page paper entitled *On the Number of Primes Less than a Given Magnitude*. He establishes the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s), \tag{11}$$

where Γ is the gamma function.[3] Equation (**??**) extends ζ to $\mathbb{C} \setminus \{1\}$, and ζ is still complex analytic![4] The function ζ is called the *Riemann zeta function*.

[3]The gamma function extends the (shifted) factorial function to a function that is complex analytic over $\mathbb{C}$.

[4]Geometrically, $s \mapsto 1 - s$ is a reflection across the line $\{s \in \mathbb{C} : \mathrm{Re}(s) = \frac{1}{2}\}$.

It is easy to show that ζ has a simple pole at $s = 1$, with residue equal to 1.

Proof. It suffices to prove that

$$\lim_{s \to 1^+} (s-1)\zeta(s) = 1,$$

which derives from the fact that, for $s \in \mathbb{R}_{>1}$ and $k \in \mathbb{Z}_{>0}$,

$$\frac{1}{(k+1)^s} < \int_k^{k+1} \frac{ds}{x^s} < \frac{1}{k^s}.$$

□

From equation (**??**), we see that

$$\zeta(-2n) = 0, \qquad n \in \mathbb{Z}_{>0}.$$

These are called the *trivial zeroes* of the Riemann zeta function. Substituting $s = 0$ into equation (**??**) does not yield a zero, since the simple pole of ζ cancels with the simple zero of the sine function. Similarly, if $s \in 2\mathbb{Z}_{>0}$ then Γ has a simple pole at $1 - s$.

Riemann also establishes that all nontrivial zeroes of the Riemann zeta function lie in the region

$$\{s \in \mathbb{C} : 0 \leq \mathrm{Re}(s) \leq 1\}.$$

This is easy enough to prove.[5]

Proof.

- Assume that $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 1$. We prove that $\zeta(s) \neq 0$ by proving that $\log \zeta(s)$ converges.

  Using the Euler product formula, and using the expansion

  $$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n},$$

[5]This is taken from Carl Erickson's notes, found at
http://www.math.harvard.edu/~erickson/pdfs/primes_and_riemann.pdf.

we compute:

$$\log \zeta(s) = -\sum_{p \text{ prime}} \log(1 - p^{-s}) = \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}$$
$$= \sum_{p \text{ prime}} \frac{1}{p^s} + \sum_{p \text{ prime}} \sum_{m=2}^{\infty} \frac{1}{mp^{ms}},$$

which converges absolutely.

Now that we know that there are no zeroes with real part greater than 1, the functional equation (**??**) tells us that there are no nontrivial zeroes with real part less than 0. Thus, all nontrivial zeroes lie in the region

$$\{s \in \mathbb{C} : 0 \leq \mathrm{Re}(s) \leq 1\}.$$

□

In this paper, Riemann writes that the following is "probably true":

Conjecture (Riemann hypothesis)**.** *Let* $s \in \mathbb{C} \setminus \{1\}$ *be a nontrivial zero of* ζ*. Then* $\mathrm{Re}(s) = \frac{1}{2}$.

Motivated by the Riemann hypothesis, the *critical line* is

$$\left\{s \in \mathbb{C} : \mathrm{Re}(s) = \frac{1}{2}\right\}.$$

As you might guess from the title of the paper, Riemann was trying to understand something quite specific, namely the distribution of primes. Already in 1859, Riemann was able to provide a formula pinpointing the exact locations of primes, contingent on knowing the zeroes of the Riemann zeta function!

Riemann gives the formula

$$\Pi_0(x) = \left(\mathrm{li}(x) - \ln 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)} \ln t\right) - \sum_{\rho} \mathrm{li}(x^{\rho}), \tag{12}$$

where the summation is over nontrivial zeroes ρ of the Riemann zeta function, summing in increasing order of $|\text{Im}(\rho)|$, since the series is only conditionally convergent.[6] The function Π_0 can recover the *prime-counting function*

$$\pi(x) := \#\{\text{primes } \leq x\},$$

and the function li is the *logarithmic integral function*, extended to $\mathbb{C}$. The formula has a part which we can understand, and an 'error' term which is controlled by the nontrivial zeroes of the Riemann zeta function.

1896.

Jacques Hadamard and Charles Jean de la Vallée-Poussin prove independently that the Riemann zeta function has no zeroes with real part equal to 1. Since we'es established that there are no zeroes with real part greater than one, the functional equation (**??**) implies that all nontrivial zeroes of the Riemann zeta function lie in the *critical strip*,

$$\{s \in \mathbb{C} : 0 < \text{Re}(s) < 1\}.$$

This was a crucial step in the first proofs of the prime number theorem, provided by Hadamard and de la Vallée-Poussin.

1900.

David Hilbert presents ten problems at the International Congress of Mathematicians, in Paris, later publishing the remaining thirteen on his list. Hilbert predicts that these problems will be highly influential in 20th century mathematics.

His incredible foresight makes the release of these problems one of the greatest moments in the history of mathematics. His eighth problem, the Riemann hypothesis, is one of the few that to this day have not been at least partially resolved.

2000.

The Clay Mathematics Institute copies Hilbert's idea, making a list of seven of the most important problems in mathematics, and offering a prize of one million dollars each. The Riemann hypothesis is the only problem to appear both on the Clay Math list and on Hilbert's list from 100 years earlier, and is widely regarded as the most important problem in mathematics.

[6] Pairing ρ with $\bar{\rho}$ shows that the summation is real.

> "If I were to awaken after having slept for a thousand years, my first question would be: has the Riemann hypothesis been proven?"
>
> — David Hilbert

Empirical evidence

It is known that the first 1.5 trillion zeroes lie on the critical line![7] How, you might ask?[8]

Note that $s \in \mathbb{C} \setminus \{1\}$ is a zero of the Riemann zeta function if any only if $\bar{s}$ is, since $\overline{\zeta(s)} = \zeta(\bar{s})$. Thus, we need only study zeroes in the 'top half' of the critical strip. Suppose we want to check up to height T, i.e. study zeroes in the region

$$X := \{s \in \mathbb{C} : 0 < \mathrm{Re}(s) < 1, 0 \leq \mathrm{Im(s)} \leq T\}.$$

First define the function Z, which has simple poles at 0 and 1 but is complex analytic on $\mathbb{C} \setminus \{0, 1\}$, by

$$Z(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{13}$$

The function Z is much 'nicer' than the Riemann zeta function, for the following reasons:

- The zeroes of Z are precisely the nontrivial zeroes of ζ, since Γ has a simple pole at each point in $\mathbb{Z}_{<0}$.
- The functional equation (**??**) becomes[9]

  $$Z(s) = Z(1 - s),$$

  which implies that Z is real-valued on the critical line. This allows us to relate the zeroes on the critical line to sign changes.

Now we can just count!

[7] `http://www.claymath.org/millennium/Riemann_Hypothesis`
[8] Thanks to Keith Conrad for his post on mathoverflow.
[9] Use the 'reflection' and 'duplication' identities for the gamma function.

1. Use the *argument principle* from complex analysis to compute x, the number of zeroes in X (with multiplicity). Specifically, let $\gamma = \partial X$, oriented anti-clockwise, and define an entire (complex analytic on $\mathbb{C}$) function f by
$$f(s) = s(1-s)Z(s), \qquad s \in \mathbb{C}$$
Then f and Z have the same zeroes (recall that Z has poles at 0 and 1), so[10]
$$x = \frac{1}{2\pi i} \int_\gamma \frac{f'(s)}{f(s)} ds. \tag{14}$$
This is easy enough to derive, given Cauchy's residue theorem. Let $z_N \in X$ be a zero, and let k be its multiplicity. Then there exists a complex analytic function g such that $g(z_N) \neq 0$ and, in a neighbourhood of z_N,
$$f(z) = (z - z_N)^k g(z),$$
so
$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_N} + \frac{g'(z)}{g(z)},$$
which has residue equal to k. The sum of the residues is therefore x, so equation (**??**) follows from Cauchy's residue theorem.

2. Use a computer (but be careful) to compute y, the number of sign-changes of the function
$$[0,T] \to \mathbb{R}$$
$$t \mapsto Z\Big(\frac{1}{2} + it\Big).$$

Let n be the number of zeroes in X that lie on the critical line (with multiplicity). Then
$$x \geq n \geq y. \tag{15}$$
Thus, if we find that $x = y$, then every zero in X lies on the critical line. Hence, using computers, we can use this method to check up to any arbitrary height, given some time. It will not work if we hit a multiple zero on the critical line (in which case $n > y$), but this has not happened yet: they have all been simple zeroes so far.

[10] There need to be no zeroes or poles of f on γ. Technically, we need to choose the height T such that ζ has no zeroes with imaginary part equal to T. The integral is easy to compute numerically, since it has to be an integer multiple of 2π. As an aside, note that the argument principle generalizes to meromorphic functions.

Applications and generalizations

The Riemann hypothesis would provide further information about the distribution of primes. For $x \geq 2$, the *offset logarithmic integral function* is

$$\mathrm{Li}(x) := \int_2^x \frac{1}{\ln t} dt = \mathrm{li}(x) - \mathrm{li}(2).$$

The *prime number theorem* says that $\pi(x)$ is approximated by $\frac{x}{\ln(x)}$, or equivalently by Li(x). Specifically,

$$\lim_{x \to \infty} \frac{\pi(x)}{\mathrm{Li}(x)} = 1.$$

The question is, "how large is the error?" In 1976, Lowell Schoenfeld proved that the Riemann hypothesis is equivalent to

$$|\pi(x) - \mathrm{Li}(x)| < \frac{1}{8\pi}\sqrt{x}\ln(x), \qquad x \geq 2657. \tag{16}$$

The Riemann hypothesis also provides upper bounds on the growth rate of certain arithmetic functions, such as the Mertens function (related to the Möbius function) and the sum-of-divisors function,

$$\sigma(n) = \sum_{d|n} d.$$

So what's better than the Riemann hypothesis? The generalized Riemann hypothesis! The Riemann zeta function can be generalized to *Dirichlet L-functions*, and the generalized Riemann hypothesis is essentially the *L*-function version of the Riemann hypothesis. Some consequences of the generalized Riemann hypothesis:

1. The Miller-Rabin primality test is guaranteed to run in polynomial time.
2. The Shanks-Tonelli algorithm (for finding square roots of a quadratic residue modulo an odd prime) is guaranteed to run in polynomial time.
3. Goldbach's weak conjecture: any odd number greater than 5 can be written as the sum of three (not necessarily distinct) primes.

4. Let p be prime, and let g_p be the smallest primitive root modulo p (with nonzero elements modulo p considered as integers in the range $1, 2, \ldots, p-1$). Then
$$g_p = O(\ln(p)^6). \tag{17}$$

5. (Artin's conjecture on primitive roots) Let $a \in \mathbb{Z} \setminus \{-1\}$ be a nonsquare. Then there exist infinitely many primes p such that a is a primitive root modulo p.

The verdict?

Some of you may remember that the last two Clay-Mahler lecturers briefly addressed the Riemann hypothesis in public lectures at the University of Melbourne, though it was not the subject of their lectures. In 2009, Terence Tao opined that it would not be proven by 2050, while in 2011 Peter Sarnak assured us that the hypothesis was certainly true.

Sure, the first 1.5 trillion zeroes may lie on the critical line, but maybe something goes wrong when the imaginary part gets really big! From a theoretical perspective, we can look at whether or not analogues of the Riemann hypothesis are true. Most famously, Pierre Deligne (1974) proved an analogue of the Riemann hypothesis for projective varieties over finite fields (one of the Weil conjectures). Some zeta functions satisfy a version of the Riemann hypothesis, while others apparently do not. The Riemann zeta function shares certain properties with the ones that do, which is some evidence in favour of the Riemann hypothesis.

Recent survey articles by Sarnak (2008), Conrey (2003) and Bombieri (2000) provide moderate evidence in favour of the hypothesis, while Ivić (2008) provides some reasons to remain skeptical. Solutions may be submitted to the MUMS room.

— Sam Chow

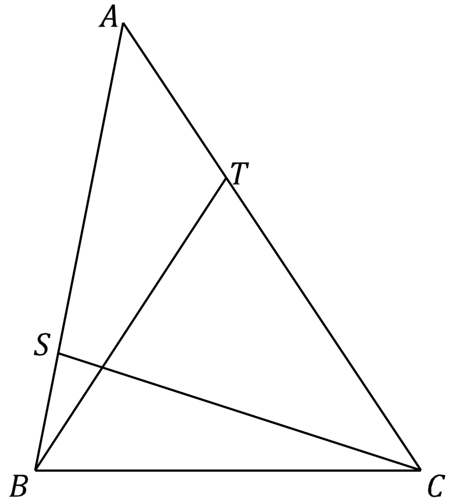
Any respectable mathematician about to speak at a conference announces their talk as *Proof of the Riemann Hypothesis*. Then when the conference actually takes place, they speak about something completely different. It's a standard precaution, just in case they die on the way to the conference...

Paradox **Problems**

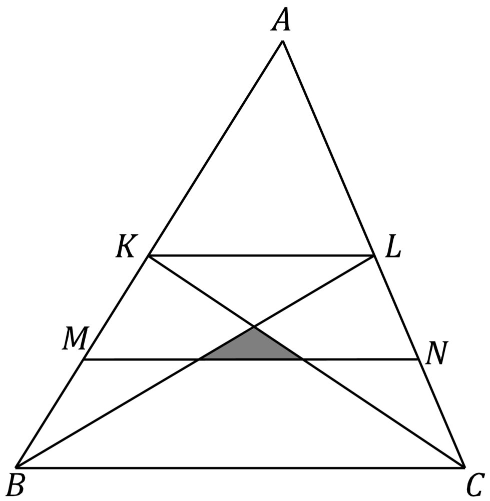
Below are some puzzles and problems for which cash prizes are awarded. Anyone who submits a clear and elegant solution may claim the indicated amount. Anyone who is able to do so for all seven questions will be awarded a $50 voucher from the Melbourne University Bookshop!

Either email your solutions to the Editor (paradox.editor@gmail.com) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

1. ($2) Evaluate $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2012^2}\right)$.
2. ($3) A tromino is an L-shaped tile made of three connected unit squares. how many ways are there of tiling a $3 \times n$ chessboard with trominoes where n is a positive integer? (Every square must be covered and overlaps are forbidden).
3. ($3) Let ABC be a triangle with $\angle ABC = 80^\circ$ and $\angle BAC = 40^\circ$. Let S and T be points on segments AB and AC respectively with $\angle BCS = 20^\circ$ and $BT = SA$. Find $\angle STA$.



4. (\$3) Let ABC be a triangle with area 1 and let K, L, M, N be the mid-points of AB, AC, KB, LC respectively. Find the area of the triangle formed by lines KC, LB and MN.



5. (\$4) There are 33 knights on a chess board. Prove that one of the knights is attacking at least two other knights.

6. (\$4) What is the smallest positive integer n such that $n \nmid 2^{2^{2^{2^2}}} - 2^{2^{2^2}}$?

7. (\$5) $2n$ people sit around a table with k chocolates distributed among them. A person may give a chocolate to their neighbour, but only after first eating one themselves. Nominating a *head* of the table, what is the minimum k such that, irrespective of the initial distribution of the lollies, there is a way for the *head* to get a chocolate? What is the minimum k such that *everyone* can get a chocolate?

— Andrew Elvey-Price

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