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Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society

MUMS

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Cover:	Information theory, one of the most important scientific achievements of the last century. See page 19 for more.

In This Edition of Paradox

Regular Features

Special Features

Puzzles

The New Zealand physicist Ernest Rutherford (1871-1937) noticed that one student in his lab was incredibly hard-working. One evening, Rutherford decided to ask:
- Do you work in the mornings too?
- Yes, - proudly answered the student, eager for praise.
- But when do you think? - amazed Rutherford.

Words from the Editor

Welcome to the second issue of Paradox for 2012. It is that time of year when the first semester draws to an end and our assessments lurk in the shadows, waiting to creep up slowly and startle us when we finally take notice of them. What better way to be prepared for that frightening moment than with a copy of the latest Paradox in your hands!

While Paradox cannot do your revision for you, it can most assuredly provide a pleasant diversion from cramming and provide a more interesting outlet than mere procrastination would. Among other curiosities, in this issue you will find an introductory discussion to an idea that has since revolutionized the world, preceded by a brief glimpse into the fascinating life that generated the idea.

The next MUMS alumnus to be interviewed in this issue has taken a rather different journey through life than his predecessors, and he provides some valuable advice for budding mathematicians and researchers alike. Paradox would also like to congratulate a previous interviewee, Norm Do, for his recent thumping victorious debut on SBS's *Letters and Numbers* quiz show.

While our hero Rubik's Turtle embarks on his next courageous mission, Paradox now has a mission for all readers to undertake: to solve the elusive problem at the back of this issue! Who will be able to finally step up to the task and claim the glory of overcoming this challenge?

It is also that time of year when the MUMS Annual General Meeting is held with great anticipation, and I encourage anyone interested to put volunteer for a role on the MUMS Committee. As Paradox Editor, I have found the experience to be a great deal of fun, despite not being a mathematics student myself. Who knows what surprises the next year shall bring? As always, please feel free to contribute an article, a solution to the elusive problem, bad maths jokes, or something entirely original of your own!

— Kristijan Jovanoski

Mathematicians never die. They only lose some of their functions.

Words from the President

Hello and welcome to another edition of *Paradox*. MUMS has recently just finished running our annual Puzzle Hunt and I would like to congratulate team Zō o Hassha for saving the world from death by drop-bear attack. Team Xyzzy and The Sons of Tamarkin round off the rest of the top three this year. A huge thank you to all the organisers, puzzle writers, and puzzle testers for running an awesome hunt.

As semester one finishes and you buckle down for exams, just remember that there's another semester of MUMS activities to come! Keep an eye out for the University Maths Olympics, as well as our usual Games Nights and weekly seminars. Oh, and don't forget the end-of-semester Trivia Night!

This is also the last issue of *Paradox* for which I will be writing these words of wisdom. That's right, another year has gone by and the AGM is just around the corner. It takes place on the 18th May at 1:00 pm, so if you want to come and are reading this after the 18th, then oh well... you missed out on free pizza.

I have had a great time being at the helm of MUMS. It has been a huge privilege to work with some of the most intelligent, funny, awesome, and mathematically-inclined people I've known during my term as president. I would like to thank everyone one the committee for doing so much. And to the incoming committee — good luck!

So long and thanks for all the fish!

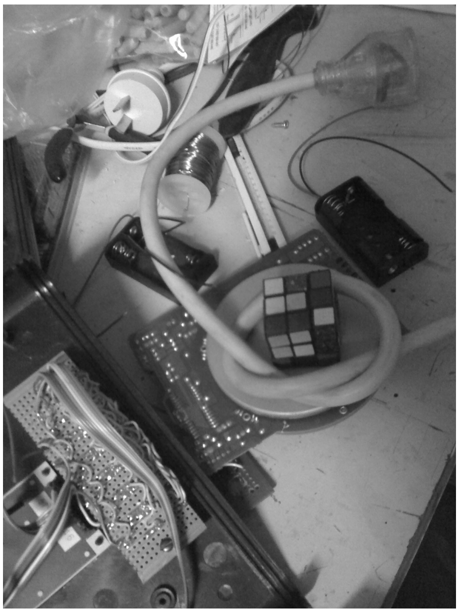
— TriThang Tran

Some engineers are trying to measure the height of a flag pole. They only have a measuring tape and are quite frustrated trying to keep the tape along the pole, as it falls down all the time. A mathematician comes along and asks what they are doing. They explain it to him.
"Well, that's easy..." He pulls the pole out of the ground, lays it down, and measures it easily.
After he has left, one of the engineers says: "That's so typical of these mathematicians! What we need is the height... and he gives us the length!"

The Adventures of Rubik's Turtle

— Dougal Davis and Jinghan Xia

Episode 3: The Staplers of DOOM! — Part I



Previously, our hero Rubik's Turtle is sent on a mission to rescue a helpless Rubik's cube from the evil Cyril and his Circus of Circuits.

But as they make their escape, he is captured by Cyril's gang and seems sure to meet his untimely demise!

A physics professor has been conducting experiments and has worked out a set of equations which seem to explain his data. Nevertheless, he is unsure if his equations are really correct and therefore asks a colleague from the mathematics department to check them.
A week later, the math professor calls him: "I'm sorry, but your equations are complete nonsense."
The physics professor is, of course, disappointed. Strangely, however, his incorrect equations turn out to be surprisingly accurate in predicting the results of further experiments. So, he asks the mathematician if he was sure about the equations being completely wrong.
"Well", the mathematician replies, "they are not actually complete nonsense. But the only case in which they are true is the trivial one where the field is Archimedean..."

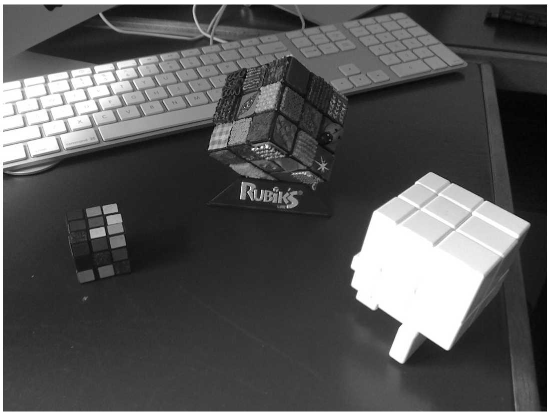
Cyril: "You've caused us quite some trouble, boy! So now you shall die!"

But just as our hero is about to suffer a fatal blow from Sid the Sneaky Soldering Iron, the smaller cube leaps out from hiding and saves the day!

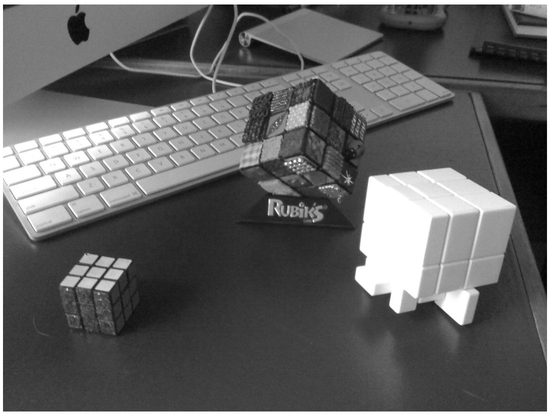
In the confusion, Rubik's Turtle overcomes his captor Eric the Evil Extension Cord...

Rubik's Turtle: "Quick, let's go!"
Small cube: "Run away!"

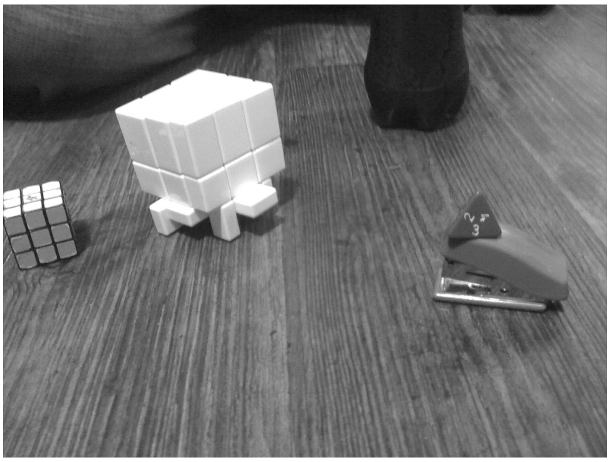
Q: How does a mathematician induce good behavior in their children?
A: "I've told you n times, I've told you $n+1$ times. . . "



Back home, Rubik's Turtle is congratulated by Mother-of-all-Rubik's...



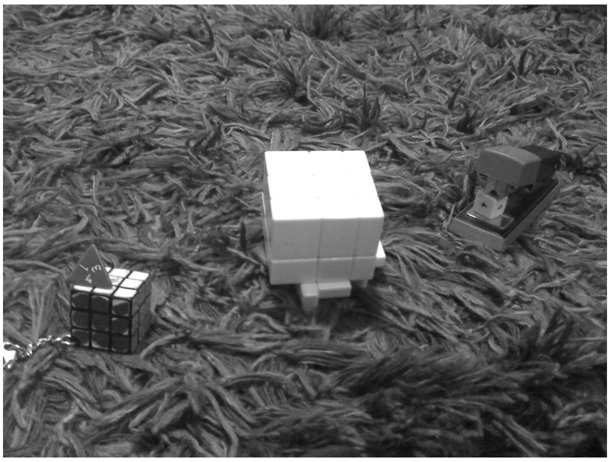
And in recognition of his bravery, she makes the small cube Rubik's Turtle's companion: Rubik's Sidekick!



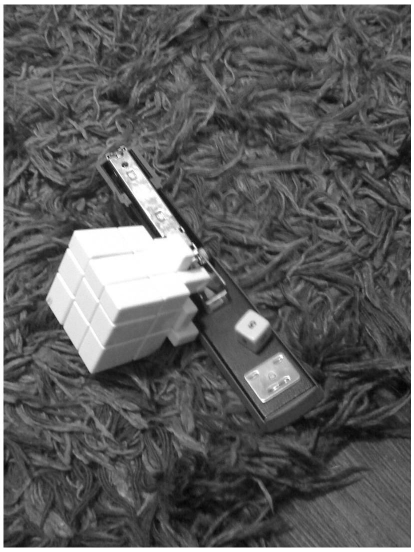
Their next mission: to retrieve the five mystic gems of Polyhedra from the infamous STAPLERS OF DOOM!

THE FIRST STAPLER OF DOOM, *however, is no match for our heroes, and they easily wrest the Tetrahedron from its grasp.*

Professor Niels Bohr, a famous Danish applied mathematician and physicist, had a horse shoe over his desk. One day a student asked if he really believed that a horse shoe brought luck. Professor Bohr replied, "I understand that it brings you luck, whether you believe in it or not."



Second Stapler of DOOM*: (through the gem in its mouth) "MMPHRGH!" Rubik's Sidekick: "Oh no, he's scary!"*



Rubik's Turtle: "Aha! Take that you fiend! The Cube is ours!"

Our heroes are feeling pretty confident until they see the dizzying heights at which The Third Stapler of DOOM *makes its lair.*

Rubik's Turtle scales the cliffs of Mathsassignment and prepares to make a fearful plunge...

Will our hero succeed in his desperate attempt to retrieve the mystic Octahedron from The Third Stapler of DOOM? Or will he plummet tragically to his death, leaving Rubik's Sidekick to fight on alone? Find out next issue, in Episode 4 of *The Adventures of Rubik's Turtle*!

Mathematical Miscellany

Mersenne numbers, named after Marin Mersenne (the 17th century French monk who began studying these numbers), are positive integers that are one less than a power of two: $M_p = 2^p - 1$.

Mersenne claimed that $2^{67} - 1$, the 67th Mersenne number was prime, but this was proven not to be the case in 1903 October meeting of the American Mathematical Society (AMS). An American mathematician, Frank Nelson Cole (1861-1927), announced a talk titled *On the Factorisation of Large Numbers*.

He walked up to the blackboard without saying a word, calculated by hand the value of 2^{67}, and carefully subtracted 1. Then he multiplied two numbers: 193707721 and 761838257287. Both results written on the blackboard were equal.

Cole silently walked back to his seat, and this is said to be the first and only talk held during an AMS meeting where the audience applauded. There were no questions. According to Cole, it took about 3 years, each Sunday, to find this factorisation.

For the curious:

$2^{67} - 1 = (193707721)(761838257287) = 147573952589676412927$

The largest known prime number, $2^{43112609} - 1$, is a Mersenne prime.

Once upon a time, Albert Einstein's wife was pressed into a public relations tour of some American research centre. Dutifully, she plodded through lab after lab filled with gleaming new scientific technology. The chief scientist condescendingly explained one of the large contraptions to her:

- Mrs. Einstein, we use this equipment to probe the deepest secrets of the universe.

- Is THAT all? - she snorted. - My husband did that on the back of old envelopes!

Interview with a MUMS Alumnus

Background

Damjan Vukcevic is a former President of MUMS. He studied at Melbourne University from 2001 to 2004, graduating with a Bachelor of Science (Honours) degree, majoring in pure mathematics and statistics. Damjan then went on to complete his PhD in statistical genetics at Oxford University, and is now employed as a statistician at Experian Hitwise.[1]

MUMS

How were you involved with MUMS?

I feel like I was involved before I even began uni! During high school I participated in the Maths Olympiad training camps and the National Mathematics Summer School. Through both of these I became friends with many people who were already in MUMS. I used to come to the MUMS room once a week to meet with Chaitanya Rao[2], my mentor for the IMO [International Mathematics Olympiad] team (I was a Reserve Member in 2000). It felt strange coming to uni in my school uniform, with everyone else dressed much more casually.

From then on I felt like I did the standard MUMS thing: rose through the ranks, did a year of presidency (2004), and then gracefully bowed out. It's a nice cycle. I got to do everything from chalking and lecture bashing, to shaping the entire events calendar and trying new innovations. We ran the inaugural Puzzle Hunt in 2004. It was an ambitious project that required the dedication and combined creative powers of the whole committee. We were thrilled that it was so successful in its first year and I'm excited to see it still going strong almost a decade on.

What was your favourite thing about MUMS?

It was the one place I found where people were willing to discuss pretty much any topic. Mathematics was a common interest, of course, but there was a diversity of interests and people were happy to share their thoughts and ideas.

[1]Editor's note: As this interview went to print, Damjan commenced a new job with the Murdoch Children's Research Institute, starting up a new Statistical Genetics group.

[2]Chaitanya Rao was co-President of MUMS in 1998.

What was your favourite MUMS event?

Definitely the Maths Olympics (both of them)![3] They are fun and intellectually challenging, and I think it is amazing that they involve so many different people, from high school students to professors, all working together on the same activity. I wonder if any other event at uni can lay claim to that?

Education

Why did you decide to drop your Engineering degree?

It was mainly because I wanted to do more mathematics subjects than the double degree would allow, and I didn't want to overload heavily to do it. I also had a closer look at the Engineering subjects and talked to some friends who had already been through them, and decided I would probably find the maths subjects more interesting.

When did you first become interested in maths?

I attended my first Maths Olympiad training camp in 1998 and I think it changed my life. Before that I was just coasting along at school. The Olympiad programme showed me just how exciting maths really is! It also put me in contact with others like myself for the very first time. The same was true of my experience with the National Mathematics Summer School. With both of these programs, I continued my involvement as a tutor and lecturer, and found it just as rewarding as being a student.

How did you decide on your majors?

My third year majors were in pure mathematics and applied statistics, which seemed to be an uncommon combination from what I saw. Initially my focus was on pure mathematics, since that's what I was interested in and enjoyed doing. Then in second year I discovered probability and statistics, and the love affair grew from there. This field has strong mathematical foundations, yet still allows me to work on applied problems in many different fields. I was also impressed at its sometimes almost magical ability to solve certain problems.

For any budding statisticians out there, I highly recommend doing pure maths in addition. The foundations of statistical methods are built on some

[3]Every year, MUMS runs two Maths Olympics: one for students, and one open to everyone at the University.

quite sophisticated mathematical ideas and it definitely helps to have a firm grasp of them.

What other advice would you give to current statistics students?

- Tackle both the theoretical and applied aspects of statistics while you are a student. Eventually you might specialise in only one of these, but having a good knowledge of the other is a great benefit.
- Start analysing *real* data early on. That's the best way to get to grips with practical issues. Assignment problems often don't expose them well enough.
- Learn to program. Computers have more than revolutionised statistics; they have *enabled* it. They will be an indispensable tool for the rest of your career. However, even specially designed statistical software only gets you so far before you need to start tweaking and customising, and so programming skills are essential if you want to take things further.
- Collaborate with others. We spend so much time at school and uni working on our own, but to solve real world problems we need to work together. This is particularly true for statistics, since you will usually be trying to solve other people's problems, and will be even more important in the future as we start seeing very large datasets emerge in many areas. Even if there is little opportunity to do this formally at uni, you can learn the same sorts of skills by, for example, joining the MUMS Committee!

What did you do for Honours?

My honours research project was in bioinformatics with Terry Speed at WEHI (Walter Eliza Hall Institute). By that point I was focusing mainly on statistics subjects. I developed a statistical model for a certain class of proteins from one of the malaria parasites, which was of interest at the time to the biologists at WEHI. It was my first taste of real science.

Tell us about your PhD at Oxford University!

While I was doing my undergraduate degree, many people told me that I should go overseas on an exchange semester. It was good advice, but life in Melbourne, and Melbourne University in particular, was so fun and involving that I was reluctant to give it up. However, as I neared the end of my degree, and as a number of my friends slowly departed Australia to do PhDs overseas, I decided my time had come.

Oxford is a fantastic place to study, in so many ways. It is full of bright students and excellent academics; famous visitors frequently come to visit and give talks, and of course the countless picturesque buildings and historic traditions are always a treat. I feel sorry for all the tourists who visit. All they get to see is the college buildings (and often only the outside), but the magic is all in the people and their interactions. That's the *real* Oxford experience. Studying overseas was a marvellous learning experience. Definitely do it if you have a chance!

You also managed to get a *Nature* publication out of your PhD — you must have been excited! What was it about?

Our project was a large collaboration between multiple research groups across the UK. The whole project involved about 200 researchers. Our group of twelve at Oxford formed the majority of the statistical analysis group. It was my first experience at working with so many people and with such a large dataset.

The project itself was a pioneering study of the genetic factors underlying a number of common human diseases, including diabetes, heart disease, arthritis and a few others. Taking advantage of the latest in genetic technology, we were able to compare the genetic makeup of about 17,000 individuals and look for mutations that made you more likely to get one of those diseases. It was the first study of its type at that scale and it was a success. We even made the 6 p.m. news on the day we published, so yes, it was very exciting!

Career

How did you decide on what you wanted to do in your career?

Well, that's been as much a product of serendipity as it is of deliberate choice. I make an effort to meet people, get advice and recommendations, and then grab opportunities that come my way.

Tell me about your current job as a statistician at Experian Hitwise.

After five years in Oxford, I wanted to return home. I missed my friends and family, but also our lovely sunny weather!

When I started looking around for job opportunities, this one came up and looked interesting. It combined many of my interests: statistical modelling, data analysis, working with large datasets and working in teams. The company collects data and reports on internet browsing behaviour. If you are familiar with TV show ratings (by Nielsen), then you can think of what we do as similar to that, but for the Internet.

Remarkably for a company whose product is statistics, I'm the only statistician in the company! My colleagues come primarily from a software engineering background and I'm learning a lot from them. Hopefully they are learning from me too.

What do you do in a typical day at work?

On a good day, I might be developing a new estimation method, which would involve reading papers, nutting out some ideas, and trying them out with some data. Eventually I would program a prototype, and present the results to my team and the product managers.

On a less ideal day I would be helping to investigate issues with our data and software, trying to determine if they are caused by bugs, sampling biases, missing data, or something else, and then working out how to fix it.

In any case, there is always a lot of interaction with other people, whether it be through meetings, presentations, or just informal discussions. Being the only statistician there, I end up working with many people across the company. I've learnt that applied statistics is a team sport and that working in teams is much more fun than working on your own when you've got the right team.

You've had quite a diverse range of projects throughout your education and employment!

During Honours I was working on bioinformatics, at Oxford it was statistical genetics, and now it is 'web analytics'. It is all statistics under the hood, and underneath that I can feel my mathematical heart beating!

What are your plans for the future?

To keep learning, work on interesting problems, maintain links to both industry and academia, and promote statistics as a profession.

Random

You're involved in a lot of sporting activities!

I like to stay active and so have done a mixed bag of sports. I was keen on Taekwondo early on in my undergraduate studies and did it for a few years. I also played volleyball both here and in Oxford, and now go running regularly.

However, what I enjoyed most, and still do, is dancing. Initially I was too shy for anything like that, but once I got dragged along to a dance class (a common route for many of us), I was hooked. Pretty soon I was dancing most days of the week, and eventually I spent more time at dance classes than at lectures. The Dancesport Club became my second home (overtaking the MUMS Room). Later on, I even started dragging MUMS friends along. I'm pretty sure that at one point at least half the MUMS Committee were also dancers!

Would dancing happen to be your favourite sport because you met your wife there?

Haha! Yes, it's one of those stories. I was a dancer from Melbourne and she was a dancer from Monash, and one day there was a big, magical joint dance ball...

— Lu Li

If you are interested in being interviewed for Paradox, please send an email to paradox.editor@gmail.com. Include your name, occupation, and relation to or interest in MUMS.

Two men are having a good time in a bar. Outside, there's a terrible thunderstorm. Finally, one of the men thinks that it's time to leave. Since he has drunk a lot, he decides to walk home. His friend asks:
- But aren't you afraid of being struck by lightning?
- Not at all. Statistics shows that, in this part of the country, one person per year gets struck by lightning, and that one person died in the hospital three weeks ago.

Biography: Claude Shannon (1916-2001)

Claude Elwood Shannon was an American mathematician most famous for having founded information theory with one landmark paper published in 1948. But he also wrote what has been widely claimed as the most important master's thesis of the century, which is credited with founding both digital circuit and digital computer design theory. He was also an avid cryptographer during World War II and afterwards, making numerous contributions to code breaking throughout his life.

His most revolutionary idea, elaborated upon further in the next Paradox article, was that the information content of a message consists simply of the number of 1s and 0s it takes to transmit it. It was gradually adopted by communications engineers and led to the technology underpinning today's Information Age. Not only are now all computer lines measured in bits per second, but also Shannon's theory made it possible for bits to be used in computer storage needed for images and other data.

As is often the case with famous mathematicians, Shannon also had a whimsical side. At Bell Laboratories, he was known for riding through the halls on a unicycle whilst juggling three balls. He invented a juggling machine as well as rocket-powered Frisbees, motorized Pogo sticks, and a device that could solve the Rubik's Cube puzzle. One of his most famous inventions was a magnetic mouse called *Theseus*,[1] created in 1950. With the same dimensions as an average living mouse, *Theseus* was controlled by a relay circuit that enabled it to move around a maze of 25 squares.

The maze's configuration was flexible and could easily be modified by Shannon. *Theseus* was designed to search through the corridors of the maze until it found its target and then, having travelled through the maze, the mouse would then be placed anywhere it had been before and its prior experience would allow it to then go directly to the target.

If it was placed somewhere it had not been before, *Theseus* was programmed to search until it reached a known location and then to proceed ot the target directly as usual, adding the new knowledge to its memory and hence learn-

[1]Named after the mythical founder-king of Athens in ancient Greece.

ing. Shannon's mouse appears to have been the first artificial learning device of its kind and a pioneer in artificial intelligence experience.

Shannon and the maze he used to test *Theseus*.

Shannon and his wife Betty would also go with M.I.T. mathematician Ed Thorp to Las Vegas on weekends, where they made a fortune from blackjack using game theory methods based on information theory principles codeveloped with fellow Bell Laboratories associate John Kelly Jr. They later applied the same ideas, eventually encapsulated as the *Kelly criterion*, to the stock market with even better results.

Since then, their ideas have been embraced by mainstream investment theory and prominent successful billionaire investors such as Warren Buffett have been known to use their methods. Unfortunately, Shannon never lived to experience the technological marvels that ultimately spawned from his ideas since his mind was ravaged by Alzheimer's disease (the most common form of dementia) in the final years of his life.

— Kristijan Jovanoski

Benjamin Franklin could still the waves of a stream just by waving his walking stick. It worked because his stick was hollow and contained oil. It is told that Franklin learned the principle behind this trick when on a boat trip he noticed that the water behind the ship became calmer after the cook cast greasy water overboard.

Information Theory, a Basic Introduction

How much information is contained in a message? Say you've flipped a coin a million times and want to communicate the resultant sequence to another person. You can do this using one million bits, where a *bit* (a contraction of 'binary digit') is a zero or one. However, if your coin is biased, then it is possible to choose some sort of encoding such that the sequence takes less than one million bits to send on average by using your knowledge of the distribution to select a more efficient code.

As an example, suppose your coin is biased so that you have a 1% chance of getting heads and a 99% chance of getting tails. Then the expected number of heads in a million tosses is 10,000 and numbers between 1 and 1,000,000 can be encoded by 20 bits each ($2^{20} = 1048576 > 1000000$), so if we decide to transmit just the position of each head rather than the result of each toss, we will be able to convey the sequence in $20 \times 10000 = 200000$ bits on average.

Could we find an even more efficient encoding? What is the limit on the efficiency of an encoding? It turns out that there is a way to calculate such a limit, and this limit will be the number of bits used by an optimal encoding, which we call the information content of a message.[1]

Information Content (Entropy)

If we consider our message as a sequence of results from a discrete random variable, that is, the variable has a finite number of outcomes with a certain probability associated to each and the sum of the probabilities being one (e.g., the aforementioned coin), then the average number of bits required to encode one result is:

$$\sum_x P(x) \times -\log_2 P(x)$$

where x varies over the possible outcomes of the random variable and $P(x)$ is the probability of that outcome occurring.

Using this formula for the example above, we get:

$$1/2 \times -\log_2(1/2) + 1/2 \times -\log_2(1/2) = 1 \text{ bit}$$

[1]A derivation for this is beyond my ability (and the scope of this article) so I'll just give the result.

for a fair coin, and:

$$1/100 \times -\log_2(1/100) + 99/100 \times -\log_2(99/100) \approx 0.080793 \text{ bits}$$

for the biased coin. So the optimal number of bits on average to encode the million long sequences above is one million for a fair coin and 80,793 for the biased coin.

This formula is motivated by using $\log_b(n)$ as the uncertainty of a random variable with n outcomes with uniform probability distribution (e.g. a fair die with n faces). The b is usually set to 2 to count the bits needed to specify the outcome.[2] For example, if $n = 8$, then you would need 3 bits.

The logarithm is used because it has the property that the overall uncertainty of two independent sources of uncertainty is the sum of their individual uncertainties. That is, if we have two uniform random variables, one with n outcomes and the other with m, then considered together, they have mn outcomes of equal probability and so their uncertainty is:

$$\log_b(mn) = \log_b(m) + \log_b(n)$$

Now $\log_b(n) = -\log_b(\frac{1}{n})$, and $1/n$ is the probability of the outcome occurring. So this can be generalised to any discrete random variable by using $-\log_b 1/P(x)$, where $P(x)$ is the probability of outcome x occurring.[3] Finally, to get the average uncertainty, we weight each outcome by its probability of occurrence, obtaining $\sum_x (P(x) \times -\log_b P(x))$ as the formula for the uncertainty of a discrete random variable. We set $b = 2$ for an answer in bits.

As an aside, this information content/uncertainty is also termed *entropy*, as its formula is remarkably similar to the formulae for entropy in thermodynamics, the most general of which is the Gibbs entropy:

$$S = -k_B \sum p_i \ln p_i$$

where k_B is the Boltzmann constant, and p_i is the probability of a microstate. In fact, thermodynamic entropy can be interpreted as the amount of further information required to fully described the microscopic state of the system,

[2] Any other number can be used, but e and 10 are common, with information units of *nats* and *dits*, respectively.

[3] This also has the property that a certain event has uncertainty (and therefore information content) $\log_b 1 = 0$, which makes sense, as we know the event happens no matter what and hence it conveys no information about anything.

beyond the macroscopic variables of classical thermodynamics such as pressure, volume, temperature, etc. For example, adding heat to a system increases its thermodynamic entropy, which can be interpreted as increasing the number of possible microstates for the system and therefore the amount of information required to describe it completely.

More Examples

Say we have a six-sided die with faces labelled A-F and the probability of rolling A being 1/2, of B 1/4, of C 1/8, of D 1/16, of E 1/32 and of F 1/32. The average number of bits of information conveyed by one roll of the die is: $-\frac{1}{2}\log_2(\frac{1}{2}) - \frac{1}{4}\log_2(\frac{1}{4}) - \frac{1}{8}\log_2(\frac{1}{8}) - \frac{1}{16}\log_2(\frac{1}{16}) - \frac{1}{32}\log_2(\frac{1}{32}) - \frac{1}{32}\log_2(\frac{1}{32}) = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{5}{32} = \frac{31}{16}$.

In this case, it is quite easy to find an optimal code by representing A as 0, B as 10, C as 110, D as 1110, E as 11110 and F as 11111.

Now suppose we have a fair six-sided die, then the average amount of information in each roll is $\log_2 6 \approx 2.58$ bits. If we encoded the result of each roll, we'd use 3 bits per roll. On the other hand, we can group the rolls by 3, each of which has $6^3 = 216$ possibilities, less than $2^8 = 256$. This gives us an average of $8/3 \approx 2.67$ bits per roll. In this case, the optimal code is much harder to find.

Some Notes

- The information content is at its maximum when each outcome has an equal change of occurring.
- In a rough sense, information theory[4] states that no lossless compression scheme can compress messages to have more than, on average, one bit of entropy per bit of message.
- Another implication of the theorem is that lossless compression which shortens some messages increases the length of at least one message.

—Mel Chen

"The number you have dialed is imaginary. Please rotate your phone by 90 degrees and try again..."

[4] More specifically, Shannon's source coding theorem.

Solutions to Previous Problems

We had a number of correct solutions to the problems since last issue. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

Dom Kersch solved problem 3 and may collect \$3.
Steven Xu solved problems 1, 2, 4, 5, and may collect \$12.

1. Evaluate $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2012^2}\right)$.

   Solution: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{2012^2}\right)$
   $=\left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\cdots\left(\frac{2011}{2012}\cdot\frac{2013}{2012}\right)$
   $=\frac{1}{2}\left(\frac{3}{2}\cdot\frac{2}{3}\right)\left(\frac{4}{3}\cdot\frac{3}{4}\right)\cdots\left(\frac{2012}{2011}\cdot\frac{2011}{2012}\right)\frac{2013}{2012}$
   $=\frac{1}{2}\cdot\frac{2013}{2012}=\frac{2013}{4024}$

2. A tromino is an L-shaped tile made of three connected unit squares. how many ways are there of tiling a $3\times n$ chessboard with trominoes where n is a positive integer? (Every square must be covered and overlaps are forbidden).

   Solution:

a						
a	a					
0						

a	a					
*	a					

a	a					
a						
*						

   The three diagrams show that the only ways of covering the top left square with a tromino (shown with a's). In the left diagram, it is impossible to cover the 0, and in the other diagrams there is only one way to cover the $*$ as shown in the following diagrams with b's:

a	a					
b	a					
b	b					

a	a					
a	b					
b	b					

   In both cases, a grid with 2 fewer rows needs to be tiled.

So the number of ways of tiling a $3 \times n$ is double the number of ways of tiling a $3 \times (n-2)$ grid.

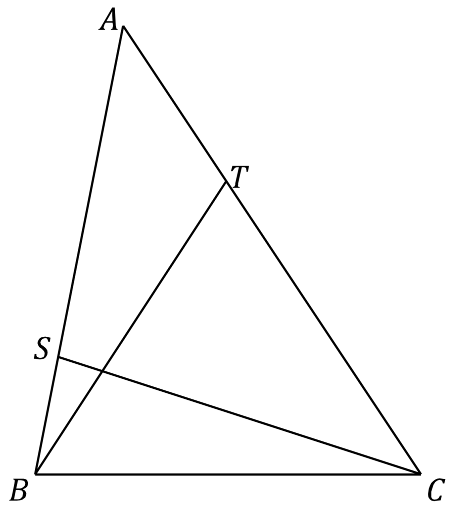
Let the number of ways of tiling a $3 \times k$ grid be $f(k)$. Then $f(k) = 2f(k-2)$.

If n is even, then $f(n) = 2f(n-2) = 4f(n-4) = \ldots = 2^{\frac{n}{2}} f(0) = 2^{\frac{n}{2}}$.

If n is odd, then $f(n) = 2f(n-2) = 4f(n-4) = \ldots = 2^{\frac{n-1}{2}} f(1) = 0$.

So the answer is $2^{\frac{n}{2}}$ for n even and 0 for n odd.

3. Let ABC be a triangle with $\angle ABC = 80^\circ$ and $\angle BAC = 40^\circ$. Let S and T be points on segments AB and AC respectively with $\angle BCS = 20^\circ$ and $BT = SA$. Find $\angle STA$.



Solution: $\angle CSB = 180^\circ - \angle BCS - \angle SBC = 180^\circ - 20^\circ - 80^\circ = 80^\circ = \angle CBS$. So ΔBSC is isosceles and $BC = CS$.

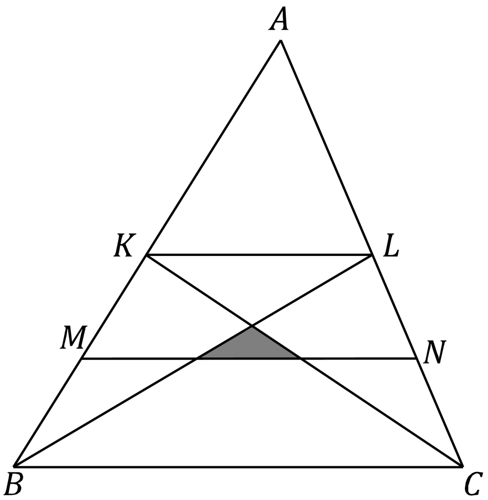
Also $\angle SCA = \angle CSB - \angle CAS = 80^\circ - 40^\circ = 40^\circ = \angle SCA$.

So $CS = SA \Rightarrow BC = CS = SA = BT$ so ΔCBT is isosceles so $\angle BTC = \angle BCT = 60^\circ$ so ΔCBT is equilateral.

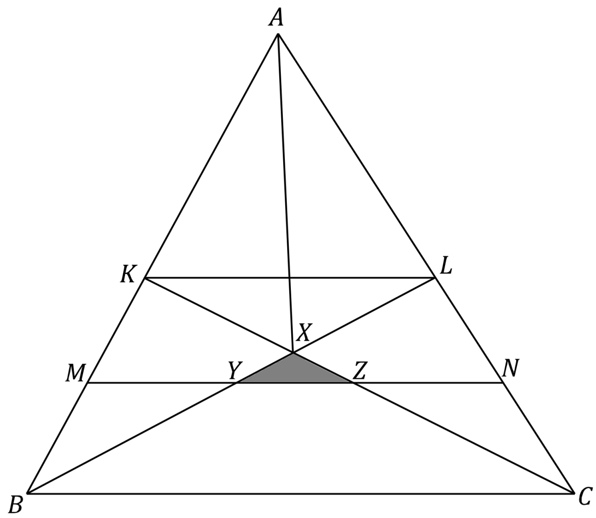
So $CT = CB = CS$ so $\angle CTS = \angle CST = \frac{180^\circ - \angle TCS}{2} = \frac{180^\circ - 40^\circ}{2} = 70^\circ$.

So $\angle STA = 180^\circ - \angle CTS = 180^\circ - 70^\circ = 110^\circ$.

4. Let ABC be a triangle with area 1 and let K, L, M, N be the midpoints of AB, AC, KB, LC respectively. Find the area of the triangle formed by lines KC, LB and MN.



Solution: Label X, Y and Z on the diagram as shown:



Since K and L are the midpoints of AB and AC respectively, KL is parallel to BC. Since M is the midpoint of KB and N is the midpoint of LC, MN is the median line of KL and BC (it is parallel to both lines and is the same distance from each of them). So Z and Y are the midpoints of CK and BL respectively.

Since ZN is parallel to KL, triangle ZNC is similar to triangle KLC. So lengths $\frac{ZN}{KL} = \frac{CN}{CL} = \frac{1}{2}$. So $ZN = \frac{KL}{2}$.

Similarly $MY = \frac{KL}{2}$.

Since MN is parallel to KL, triangle AMN is similar to triangle AKL. So lengths $\frac{MN}{KL} = \frac{AN}{AL} = \frac{3}{2}$. So $MN = \frac{3KL}{2}$.

Since BC is parallel to KL, triangle ABC is similar to triangle AKL. So lengths $\frac{BC}{KL} = \frac{AC}{AL} = 2$. So $BC = 2KL$.

So $ZY = MN - MY - ZN = \frac{KL}{2} = \frac{BC}{4}$.

Since BC is parallel to YZ, triangle XYZ is similar to triangle XBC. So $|XYZ| = \frac{|XBC|}{16}$. (Note that $|QRS|$ denotes the area of triangle QRS.)

Since L is the midpoint of AC, $|ALB| = |CLB|$ and $|ALX| = |CLX|$ so $|AXB| = |ALB| - |ALX| = |CLB| - |CLX| = |BXC|$. Similarly $|AXC| = |BXC|$.

So $1 = |ABC| = |AXC| + |BXC| + |AXB| = 3|BXC| = 48|XYZ|$ so the area is $\frac{1}{48}$.

5. There are 33 knights on a chess board. Prove that one of the knights is attacking at least two other knights.

Solution:

1a	2a	3a	4a	5a	6a	7a	8a
3b	4b	1b	2b	7b	8b	5b	6b
2c	1c	4c	3c	6c	5c	8c	7c
4d	3d	2d	1d	8d	7d	6d	5d
9a	10a	11a	12a	13a	14a	15a	16a
11b	12b	9b	10b	15b	16b	13b	14b
10c	9c	12c	11c	14c	13c	16c	15c
12d	11d	10d	9d	16d	15d	14d	13d

In the diagram there are 4 squares a, b, c, d for each number between 1 and 16 such that each pair (a, b), (b, d), (c, d) and (a, c) is attacking each other.

Since there are 33 knights and only 16 numbers, by the pigeon-hole principle, there are atleast 3 knights on the same number. But then one of these three knights is attacking the other two (e.g., if they are on squares a, b, and c then the one on square a is attacking those on squares b and c).

6. What is the smallest positive integer n such that $n \nmid 2^{2^{2^{2^2}}} - 2^{2^{2^2}}$?

Solution:

Define a_i recursively so that $a_0 = 1$ and $a_{i+1} = 2^{a_i}$. Then the number in the question is $a_5 - a_4$.

Lemma: if s is odd and $\phi(s)|a_k - a_{k-1}$ then $s|a_{k+1} - a_k$.

Proof of Lemma: if s is odd then $2^{\phi(s)} \equiv 1 \pmod{s}$ so $2^{a_k - a_{k-1}} \equiv 1 \Rightarrow a_{k+1} = 2^{a_k} \equiv 2^{a_{k-1}} = a_k \pmod{s} \Rightarrow s|a_{k+1} + a_k$ as required.

$a_2 = 4$ and $a_3 = 16$ so $a_3 - a_2 = 12$ so $1, 2, 3, 4, 6, 12$ are factors of $a_3 - a_2$.

$\phi(1), \phi(3), \phi(5), \phi(7), \phi(9), \phi(11), \phi(13), \phi(15), \phi(17), \phi(19), \phi(21)$ are equal to $1, 2, 4, 6, 6, 10, 12, 8, 16, 18, 12$ respectively.

So by the Lemma: $1, 3, 5, 7, 9, 13, 21$ are factors of $a_4 - a_3$. Also 2^4 is clearly
a factor of $a_4 - a_3$. So $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 18, 20, 21$ are factors of $a_4 - a_3$.

So again by the Lemma: $1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21$ are all factors of $a_5 - a_4$. Again $16|a_5 - a_4$ so every positive integer which is less than 23 is a factor of $a_5 - a_4$.

Suppose that $23|a_5 - a_4$. Let k be the order of 2 mod 23 (the lowest power to which 2 must be raised to give something which is 1 mod. Then $k|22$ so k is 1, 2, 11 or 22. If k is 1 or 2 then $23|2^2 - 1 = 3$, which is not true. So $11|k$. $23|a_5 - a_4 = 2^{a_4} - 2^{a_3} \Rightarrow k|a_4 - a_3 \Rightarrow 11|a_4 - a_3$

Let k be the order of 2 mod 11. Then $k|10$ so k is 1, 2, 5 or 10. If k is 1 or 2 then $5|2^2 - 1 = 3$, which is not true. So $5|k$. So $11|a_4 - a_3 = 2^{a_3} - 2^{a_2} \Rightarrow k|a_3 - a_2 \Rightarrow 5|a_3 - a_2 = 12$. This is a contradiction.

Hence, $23 \nmid a_5 - a_4$, and so the answer is 23.

— Andrew Elvey-Price

When PhD candidates the mathematician Arne Beurling was supervising came to him with their finished theses he would read the last few pages of the thesis, then pull out a paper from his desk, look at it for a few moments and then say "Well, that seems to be the right answer, You can submit it."

Paradox **Problems**

Below is the only problem that has not yet been completely solved for the past five issues. No monetary incentive has been enough to inspire a solution, so now there is glory to be had too! Anyone who submits a clear and elegant solution may bask in this glory. A cash prize of $15 will be awarded to the first correct submission, followed by $10 for the second, and $5 for the third.

Either email your solutions to the Editor (paradox.editor@gmail.com) or drop a hard copy into the solutions dropbox outside the MUMS Room (G24) in the Richard Berry Building; please include your name and contact details.

Problem:

$2n$ people sit around a table with k chocolates distributed among them. A person may give a chocolate to their neighbour, but only after first eating one themselves. Nominating a *head* of the table, what is the minimum k such that, irrespective of the initial distribution of the chocolates, there is a way for the *head* to get a chocolate? What is the minimum k such that *everyone* can get a chocolate?

Hint: The first part of this question has already been discussed and solved in a previous issue of Paradox, but it is up to you to find out which one! With that solution in mind, there may be hope yet for those who wish to solve the rest of this puzzle...

— Andrew Elvey-Price

In the period that Einstein was active as a professor, one of his students approached him and said: "The questions of this year's exam are the same as last years!"
"True," Einstein said, "but this year all of the answers are different."

Paradox would like to thank Mel Chen, Andrew Elvey-Price, Dougal Davis, Lu Li, Damjan Vukcevic, and Jinghan Xia for their contributions to this issue.