Paradox

In this edition of Paradox

On the cover (clockwise, from top-left): 1) Miriam Mirzakhani, the late Iranian differential geometer and Fields Medalist, 2) Every mathematician's favourite hot drink, 3) Every topologist's favourite snack, 4) A straw. The four things on this cover are homeomorphic (piercings and nostrils notwithstanding).

Words from the President - Madeleine Johnson

Hello and welcome to the second edition of Paradox for the year!

It's Week 12 and if you're reading this, it means you're still alive, congratulations! We hope you're surviving the approximately infinite assessments and unbounded stress around this time of semester and know that it's not too far off until that glorious winter break! In keeping things nice and mathematical, I thought I'd share a rundown on MUMS' semester by the numbers. In the first 11 weeks of this semester, we've held a whopping 22 events, and have another 5(!!) lined up to send off a great semester in Week 12. The Events team, lead by Elizabeth, held 10 of these – six picnics and four games nights. The Education subcommittee, led by Anand, held another 7 – four seminars and three study groups, which have been spearheaded by Sharon & Indira. The Diversity & Education subcommittee, lead by Hannah, held 5 – four women in maths coffee catch ups and the launch event for our new mentoring program. In Week 12, we're also holding three more study groups, a maths career panel, and a uni-first MATHS vs PHYSICS debate to celebrate the end of semester – check out our Facebook page for more information about these events.

MUMS Committee has also been running around attending important events on behalf of the club! This includes attending three School of Maths & Stats committee meetings as student reps, one UMSU International Head of Clubs meeting, one UMSU governance training session, several meetings with sponsors, one women in physics camp and one extremely official and important MUMS trip to IKEA. As President, I also meet with various staff members within the school to discuss things as varied as collaborating on events and programs, drafting responses to proposed University policy and generally trying to represent the best interests of students to the School. But how do we get all this done? MUMS has a committee of 22 fantastic and dedicated students, split up into six sub-committees – Executive, Finance, Communications & Publicity, Education, Events, Diversity & Inclusion. We've had almost 30 internal meetings this semester – two big general committee meetings with everyone, 5 Executive meetings, 4 Communications & Publicity, 4 Education, 5 Events, 5 Diversity & Inclusion and a whole bunch more meetings for various subsets of committee members working on specific projects such as our website (spearheaded by Hannah), planning for big Sem 2 events like SMO (Elizabeth), or launching our WeChat presence (thanks Sally!).

This semester, we were also very excited to elect five new Epsilons (first year reps) and a new postgrad rep in Week 4 – congratulations to Canis, Indira, Sally, Sharon and Yifan, as well as Lu Lin. If you're interested in running for a position on the MUMS Committee, stay tuned for news about our AGM (Annual General Meeting) mid-semester 2, when every position goes up for election! And there's lots more coming up next semester! Keep your ears out for bigger and better social events, the University Maths Olympics (UMO)... and the return of that international mathematical adventure, the Puzzle Hunt! Wishing you all the best of luck for your final assessments and exams! May your SWOTVAC be one of minimum stress and maximum productivity, may your exams be reasonable, and may you get assigned a seat under a heater in the ice palace that is REB at the end of Sem 1. Until next semester!

Yours mathematically,

Madeleine

Words from the Editor

It is an honour and privilege to deliver the second edition of the Paradox for 2019. It has been an eventful semester for the MUMS committee, as Madeleine has indicated!

It has been great fun putting together this edition of Paradox, and it wouldn't be possible without contributions from both the MUMS committee, and our guest contributors. This time, we have a distinctly geometry-themed Paradox. Blake Sims gives us an introductory look at fibre bundles. I continue my own discussion of differential geometry with my second instalment, titled "What is a Tensor?" – students of Applied Mathematical Modelling and Electrodynamics may want to peruse this as they gear up for their exams! Hannah Perry wrote a biographical article on the late geometer and Fields medallist, Miriam Mirzakhani. We continue this theme in our offbeat pages - decide which topological space YOU are!

In this edition, you will also find some excellent tips for performing fast and efficient partial-fraction decomposition, courtesy of Brae Vaughan-Hankinson. Our first-year readers will find this very helpful as SWOTVAC approaches. Daniel Johnston presents an exciting look into the crazy world of category theory. Celia Dowling and Hannah Perry have each written an article about the Women in Physics and Mathematics camp. Madeleine has interviewed a trader from Optiver for any students who are contemplating their next step in the exciting world of investment banking.

Regards,
Steven Xu

MUMS Committee

Executive: Madeleine Johnson *(President, Chair)*, Vanessa Thompkins
(Secretary), Luke Ireland *(Treasurer)*, Anand Bharadwaj *(Education Officer)*, Ivy Weng *(Publicity Officer)*
Finance: Luke Ireland *(Treasurer, Chair)*, Vanessa Thompkins *(Secretary)*, Lu Lin *(Postgrad Rep)*
Communications & Publicity: Vanessa Thompkins *(Secretary, Chair)*, Ivy Weng
(Publicity Officer), Steven Xu *(Paradox Editor)*, Daniel Johnston *(Postgrad Rep)*, Fiona Guan *(Undergrad Rep)*
Events: Elizabeth Ivory *(Undergrad Rep, Chair)*, Keshini Karunaharan *(Undergrad Rep)*, Matthew Walker *(Undergrad Rep)*, Billy Price *(Undergrad Rep)*, Yifan Guo *(Epsilon)*, Canis Nugroho *(Epsilon)*.
Education: Anand Bharadwaj *(Education Officer, Chair)*, Daniel Johnston
(Postgrad Rep), Dominic Yew *(Undergrad Rep)*, Sharon Qian *(Epsilon)*, Indira Senthil Ajeetha *(Epsilon)*
Diversity & Inclusion: Hannah Perry *(Undergrad Rep, Chair)*, Elinor Mills *(Undergrad Rep)*, Sally Cai *(Epsilon)*, Madeleine Johnson *(President)*

Partial Fraction Decompositions - Brae Vaughan-Hankinson

Introduction

A function f is a rational function iff it can be written in the form

$$f = \frac{p}{q},$$

with p and q polynomials and $q \neq 0$. For the sake of simplicity we will assume that $\deg p \leq \deg q$ (thus f is a *proper* rational function). They arise often in mathematical analysis, and typically one seeks to decompose a given rational function into simpler **partial fractions** with irreducible denominators. Doing this operation quickly and efficiently is an essential skill for exam-style situations. In order to perform a decomposition we first invoke the fundamental theorem of algebra, which allows us to factor our single variable polynomial q into irreducible factors φ_i, each with multiplicity r_i, and a constant κ:

$$q = \kappa \prod_{i=1}^{k} \varphi_i^{r_i}.$$

Without loss of generality we can take $\kappa = 1$ for simplicity; we can always seek q such that it is monic. Note that the sum of all degrees of φ_i with corresponding multiplicities is the degree of q itself, i.e.

$$\sum_{i=1}^{k} r_i \deg \varphi_i = \deg q.$$

Our decomposition will allow us to express f as

$$f = \sum_{i=1}^{k} \sum_{j=1}^{r_i} \frac{\pi_{ij}}{\varphi_i^{j}},$$

where π_{ij} are polynomials with $\deg \pi_{ij} < \deg \varphi_i$.

Typical Expression

If our functions p and q are polynomials over the field of real numbers $\mathbb{R}$ then our irreducibles φ_i are either linear or quadratic, and the respective π_{ij} are constant or linear. Decompositions of such form are the most commonly or

only encountered form by first- or second-years, e.g.

$$f_1(x) = \frac{p(x)}{(x+\alpha)(x+\beta)} = \frac{A}{x+\alpha} + \frac{B}{x+\beta}$$

$$f_2(x) = \frac{p(x)}{(x+\alpha)(x+\beta)^2} = \frac{A}{x+\alpha} + \frac{B_1}{x+\beta} + \frac{B_2}{(x+\beta)^2}$$

$$f_3(x) = \frac{p(x)}{(x+\alpha)(x^2+bx+c)} = \frac{A}{x+\alpha} + \frac{Bx+C}{x^2+bx+c}.$$

Note the quadratics are irreducible over $\mathbb{R}$ iff $b^2 - 4c < 0$.

The **method of undetermined coefficients** can be used to equate the coefficients of p with those on the right hand side after multiplying though by q. This gives a system of equations that can be solved using any of the standard methods. However, it is often time-consuming or tedious to do so by hand.

Heaviside the Hero

Instead of the arduous process mentioned above, we will invoke the **cover method** developed by Oliver Heaviside: first to compute A from our first example, we multiply through by its denominator $(x + \alpha)$

$$\frac{p(x)}{(x+\beta)} = A + \frac{B(x+\alpha)}{x+\beta}.$$

Now we take the limit as $(x + \alpha) \to 0$, yielding

$$\lim_{x \to -\alpha} \frac{p(x)}{(x+\beta)} = \lim_{x \to -\alpha} \left(A + \frac{B(x+\alpha)}{x+\beta} \right)$$

$$\implies \frac{p(-\alpha)}{-\alpha+\beta} = A + \frac{B(-\alpha+\alpha}{x+\beta}$$

$$\implies A = \frac{p(-\alpha)}{-\alpha+\beta}.$$

Easy! Rinse and repeat for B. This method is so-named because you "cover up" the factor, the coefficient for which you wish to find, and simply substitute the x-value for which it would limit to zero. In general for a decomposition of the form

$$\frac{p(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)} = \frac{A_1}{x-\alpha_1} + \cdots + \frac{A_n}{x-\alpha_n}$$

we find A_i by "covering" $x - \alpha_i$ and computing the result when $x \to \alpha_i$. Sometimes, however, we can have higher powers of irreducibles in the denominator, so our decomposition may take a form similar to:

$$\frac{p(x)}{(x+\alpha)^r \, s(x)} = \frac{A_r}{(x+\alpha)^r} + \cdots + \frac{A_1}{x+\alpha} + \frac{\sigma(x)}{s(x)},$$

with $\deg \sigma < \deg s$. If we multiply through by any $(x + \alpha)^k$ with $k < r$, we will always have a zero denominator when we take $(x + \alpha)^k \to 0$. Not good! But if we multiply through by $(x + \alpha)^r$ and take the zero-limit we will see that all coefficients except for A_r are annihilated, yielding

$$A_r = \lim_{x \to -\alpha} \frac{p(x)}{s(x)}.$$

But how do we make an irreducible quadratic factor limit to zero? And how do we compute the remaining coefficients A_i?

Get Complex...

Let p and q to be non-zero polynomials over the field $\mathbb{C}$, the **complex numbers**, with with $\deg p < \deg q$ and q monic. Recall we can express q as

$$q = \prod_{i=1}^{k} \varphi_i^{r_i}.$$

i.e. a product of factors φ_i with multiplicity r_i. Under this scheme any quadratic polynomial can be reduced to two linear factors with roots in $\mathbb{C}$. Thus every φ_i is **linear**! Then we can easily employ the cover method as outlined above to find all the **constant** coefficients π_{ij} to express our rational function f as

$$f = \sum_{i=1}^{k} \sum_{j=1}^{r_i} \frac{\pi_{ij}}{\varphi_i^j}.$$

In complex analysis, constants like this are referred to as **residues**.

... but Keep It Simple

For the case of **simple roots**, i.e. $r_i = 1$, we simply seek the residue of f at our i-th pole. This can easily be computed using the cover method outlined above. For a direct computation we can define $\varphi_i = x - x_i$. Then we have

$$\begin{aligned}
\pi_i &= \lim_{x \to x_i} (x - x_i) \frac{p(x)}{q(x)} \\
&= \lim_{x \to x_i} \frac{xp(x) - x_0 p(x)}{q(x)} \\
&= \lim_{x \to x_i} \frac{p(x) + xp'(x) - x_0 p'(x)}{q'(x)} \qquad \text{(by L'Hôpital's rule)} \\
&= \frac{p(x_i)}{q'(x_i)}.
\end{aligned}$$

Whether or not this computation is easier than the "cover method" must be determined on a case by case basis, but they are effectively equivalent and will give the same result.

Dealing with Multiplicity

For poles $\varphi_i = x - x_i$ with multiplicity $r_i > 1$ we can generalise the above computation. Suppose we have a decomposition of the form

$$f(x) = \frac{\pi_{ir_i}}{(x - x_i)^{r_i}} + \cdots + \frac{\pi_{ij}}{(x - x_i)^j} + \cdots + \frac{\pi_{i1}}{x - x_i} + \rho(x),$$

where ρ is a rational function having poles coprime with φ_i, and $1 \leq j \leq r_i$. We seek the j-th coefficient, which is the residue of $(x - x_i)^{j-1} f(x)$ at x_i. We first multiply through by $\varphi_i^{r_i}$:

$$(x - x_i)^{r_i} f(x) = \pi_{ir_i} + \cdots + \pi_{ij}(x - x_i)^{r_i - j} + \cdots + \pi_{i1}(x - x_i)^{r_i - 1} + \rho(x)(x - x_i)^{r_i}.$$

Since φ_i is linear and monic, differentiation follows the power rule. Differentiating $r_i - j$ times and dividing by the respective powers yields

$$\frac{1}{(r_i - j)!} \frac{\mathrm{d}^{r_i - j}}{\mathrm{d}x^{r_i - j}} (x - x_i)^{r_i} f(x) = \pi_{ij} + \cdots + \binom{r_i - 1}{r_i - j} \pi_{i1}(x - x_i)^{j-1}$$

$$+ \frac{1}{(r_i - j)!} \frac{\mathrm{d}^{r_i - j}}{\mathrm{d}x^{r_i - j}} (x - x_i))^{r_i} \rho(x).$$

Taking the limit as $x \to x_i$ will annihilate everything on the right-hand side apart from π_{ij}. This formula can become especially unwieldy at high multiplicities, but it is completely general and will work every time

$$\pi_{ij} = \frac{1}{(r_i - j)!} \lim_{x \to x_i} \frac{\mathrm{d}^{r_i - j}}{\mathrm{d}x^{r_i - j}} (x - x_i)^{r_i} f(x).$$

Applications

So we've covered the underlying theory and how-to of decomposition, but what is a concrete example of its usefulness, other than just doing integrals in MAST10006 or MAST20030? One example which itself has wide uses is the **Chinese Remainder Theorem** (CRT). This theorem from number theory states that if we know the remainders of some number after Euclidean division with a set of coprime divisors, we can determine what the remainder is when divided by their product. But this doesn't just apply to integers! The ring of polynomials allows for a notion of **degree** and therefore Euclidean division as we have thoroughly discussed, and so we can apply CRT to polynomials too.

Suppose we have some function p which we know satisfies a set of congruences, namely

$$p \equiv \alpha_i \mod \varphi_i,$$

for $i \in [1, k]$. Then define $q := \prod_{i=1}^{k} \varphi_i$ so that we can decompose $1/q$ into k polynomials with $\deg \pi_i < \deg \varphi_i$,

$$\frac{1}{q} = \sum_{i=1}^{k} \frac{\pi_i}{\varphi_i}.$$

Then we can multiply through by pq to obtain

$$p = \sum_{i=1}^{k} \frac{pq\pi_i}{\varphi_i} \equiv \sum_{i=1}^{k} \frac{\alpha_i \pi_i}{\varphi_i} \mod q$$

If we wish to find the unique polynomial such that $\deg p < \deg q$ then we need only consider the remainder term β_i from the division of $\alpha_i \pi_i$ by φ_i. Then

$$p = \sum_{i=1}^{k} \frac{q\beta_i}{\varphi_i}.$$

A special case of CRT is where each $\varphi_i(x) = x - x_i$ is linear and monic, in which case the remainder for each congruence relation is $\alpha_i = p(x_i)$ (recall cover method shown above; exercise left to the reader). Then applying CRT is equivalent to polynomial interpolation, i.e. we are given a set of points and we can then find the unique polynomial which connects all those points. This is called Lagrange interpolation, and versions of this are widely used for public-key cryptographic protocols for secret sharing in which we desire to have a threshold number of keys in order to recover the secret. Other important applications of CRT include: the Jordan-Chevalley decomposition of a linear operator into its semisimple (diagonalisable) and nilpotent components, which is of great importance in the study of the algebraic geometry of groups; and the Good-Thomas prime-factor algorithm for simplifying the computation of a discrete Fourier transform of size $n_1 n_2$ into two computations of size n_1 and n_2, provided n_1, n_2 are coprime.

Conclusion

The above methods are at their core based on the same principle, and in their simplest representation rely on the fundamental theorem of algebra. Good judgement will determine which form of the method is the most pertinent to the problem at hand, and good judgement is best developed through lots of practice! Hopefully some of you have found this overview to be a useful toolbox for an exam scenario, as well as offering a small glimpse into practical complex analysis for those of you who are yet to discover it, and into wider and more abstract applications.

- Brae Vaughan Hankinson

An Introduction to Category Theory - Daniel Johnston

Category theory is a modern field of mathematics first introduced in 1945. It provides us with a powerful abstract language that allows us to generalise many notions in mathematics. It has also found use in other fields of science including computing, physics and even biology.

What is category theory?

As one would expect category theory is the study of categories!

But what is a category?

Well, a category is defined in terms of two pieces of data:

- Objects - A, B, C, D, . . .

- Arrows - f, g, h, . . .

To begin with, you might like to think of objects as sets and arrows as functions. In fact, this will give us the category called **Set** (we usually write the names of categories in bold text).

Now, to complete our definition of category we also need to know the conditions on these objects and arrows. Although it may seem like a bit to remember at first, it doesn't take long to get used to.

- Each arrow goes from one object to another. For instance we write $f : A \to B$ to mean that A is the domain of f and B is the codomain of f.

- We can always compose two arrows $f : A \to B$ and $g : B \to C$ to get a new arrow $g \circ f : A \to C$.

- For any three arrows f, g and h we always have that $(f \circ g) \circ h = f \circ (g \circ h)$.

- Every object X has an identity arrow $1_X : X \to X$.

- The identity arrows "do nothing" in the sense that $f \circ 1_A = f = 1_B \circ f$ for any arrow $f : A \to B$.

Now if we do indeed consider the case where the objects are sets and the arrows as functions then these are all just standard properties. The composition $g \circ f$ would be the function $g(f(x))$ and the identity arrow for a set X would be the function that sends everything to itself, i.e. $1_X(x) = x$ for all $x \in X$.

At this point, it may seem that the definition of category was specifically designed to describe sets and functions. But alas, we can more or less make a category out of whatever we want!

So, let's now construct our own category called **MUMS**.

The objects of **MUMS** will be the committee members of MUMS.

And then we will draw arrows between any two committee members such that the arrow goes from the shorter to

the taller committee member.

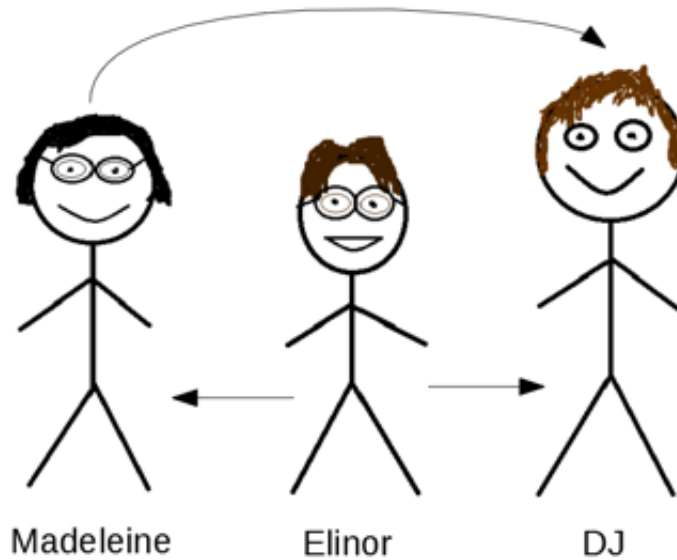To see what we mean by this, consider the following three committee members - Madeleine, Elinor and DJ.

Figure 1: Three of the MUMS committee members

Elinor is the shortest of the three so we draw an arrow from Elinor to the other two committee members. And then, DJ is taller than Madeleine so we draw an arrow from Madeleine to DJ.

Everyone is also given their own identity arrow (not drawn in figure 1) so that we can fully satisfy the definition of a category. This is all shown more compactly in the following diagram (M = Madeleine, E = Elinor, D = DJ).

$$1_M \circlearrowright M \longrightarrow D \circlearrowleft 1_D$$
$$\uparrow \quad \nearrow$$
$$1_E \circlearrowright E$$
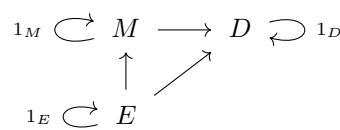
Finally we need to describe how to compose arrows.

There is an simple way to do this. Namely, composing the arrow $M \to D$ with $E \to M$ should give you the arrow $E \to D$. That is, we define $E \to M \to D = E \to D$.

This means that on our diagram above we can take any route from $E \to D$ and always end up with the same arrow. A diagram that satisfies this path-independent property is known as a "commutative diagram". If we apply this structure to every arrow of **MUMS** you should now be able to convince yourself that **MUMS** is a valid category.

Why is category theory useful?

Now you might be wondering what the point of category theory is. For mathematicians, it provides a way to generalise many mathematical objects and explore the links between them. However, category theory is also employed heavily outside of mathematics. In particular, the use of category theory is widespread in computer science and many universities offer courses on category theory for computer scientists.

For an example, let's consider some computer program that performs operations on integers and strings (a string is just a bunch of text such as "apple" or "89F@*"). Let's say that our program can perform three different functions:

(1) "length" which inputs a string and returns how many characters are in the string. For instance length(cat)=3. Note that this function converts a string to an integer.

(2) "times 10" which multiplies any integer by 10.

(3) "numbername" which converts an integer to its English word e.g. numbername(3)=three.

We can now define a category where the data types (integer and string) are the objects and the functions are the arrows. The structure of the category can thus be represented diagrammatically as follows:

$$
\text{times 10} \;\circlearrowleft\; \text{Integer} \quad\underset{\text{length}}{\overset{\text{numbername}}{\rightleftarrows}}\quad \text{String}
$$
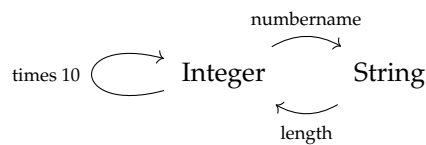
To properly make this a category we can also add composites of these functions and define some functions that do nothing (these will be the identity arrows).

From this example it should now be evident that we can describe the structure of programs using categories. This is a very basic example but with more advanced category theory one can begin to completely break down and describe many modern and complicated computer programming languages.

We now turn to an example where category theory can be applied in mathematics (applying mathematics to mathematics hehehe). If anything, category theory is extremely useful in providing general and precise definitions of things in mathematics. For our example, we consider defining the notion of n^m i.e. n raised to the power of m for integers $n, m \geq 0$.

But you may already be thinking "oh but that's easy, don't we just define n^m to be $\underbrace{n \times n \times ... \times n}_{m \text{ times}}$?"

Now, this is more or less correct, but how do we deal with defining 0^0?
On the one hand, 0 raised to any power should always be 0 yeah? However, we've also been taught since high school that anything to the power of 0 is 1. So what's the correct definition?

Typically to settle such matters we could look at the graph of a relevant function and see what happens as we approach the point of interest. In figure 2, we see the plots of the functions $f(x) = 0^x$ and $g(x) = x^x$.

But these plots still don't clear up the issue! As x approaches 0 $f(x)$ tends towards 0 and $g(x)$ tends towards 1. Which one is correct?!

To solve our issue once and for all we see how n^m is defined in category theory. We have that

*n^m is the number of arrows (or functions) in **Set** from a set with m elements to a set with n elements*
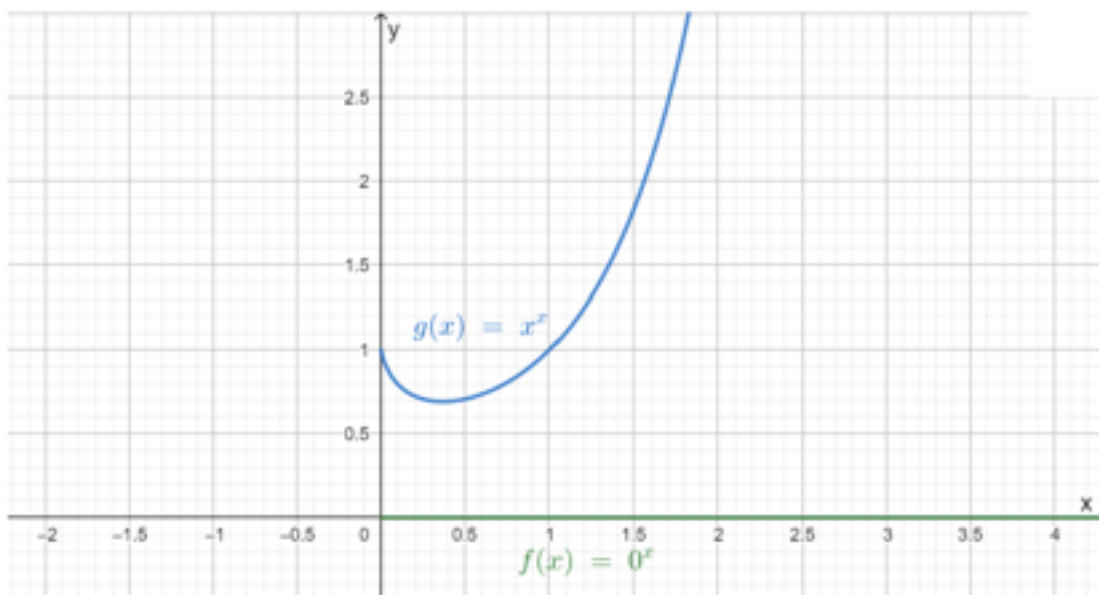
Figure 2: Plot of $f(x) = 0^x$ and $g(x) = x^x$

To test our new categorical definition, let's try to compute 2^2 ($n = m = 2$). Below we have listed all possible functions (labelled f_1, f_2, f_3 and f_4) from $\{1, 2\}$ to $\{1, 2\}$. Note that both of these sets have two elements.

x	1	2
$f_1(x)$	1	1

x	1	2
$f_2(x)$	1	2

x	1	2
$f_3(x)$	2	1

x	1	2
$f_4(x)$	2	2

As we would expect, there are exactly 4 such functions allowing us to deduce the absolutely amazing result $2^2 = 4$!

Okay, so maybe this result isn't so amazing, but what if we now apply our categorical definition to 0^0. In other words, how many functions are there from a set of size 0 to a set of size 0?

Well to begin with, we need to think of a set of size 0. Luckily, there is such a set - the empty set $\emptyset$!
But how many functions are there from $\emptyset$ to $\emptyset$? You may be tempted to say 0 but just like any other set in **Set**, there is an identity arrow $1_\emptyset$ from $\emptyset$ to $\emptyset$.
And so, we have exactly 1 arrow from $\emptyset$ to $\emptyset$ meaning that $0^0 = 1$!!!!

Now I don't know if this has truly convinced you that $0^0 = 1$ but at least one can now appreciate that defining things in a very general "categorical" way can be very clear and unambiguous. For this reason, the language of category theory is where a lot of mathematicians turn to for definitions.

How to sound smart
Arguably more than any other field of mathematics, the language of category theory can make you sound like you know a lot more than you actually do. Although this can annoy many people, speaking in this language is a surefire way to convince your parents and friends that you know some high-tier abstract mathematics. So to finish off, we're going to go through a few different phrases that'll make you sound real hip (in a mathematical sense).

As previously discussed, many things can be defined using the language of category theory. Although a lot of these definitions can be quite precise and compact, they sound like utter jiberish to people who've never studied category theory. So, we now give a couple of definitions of mathematical objects that you may or may not heard of.

The set $\mathbb{Z}$ is just the initial object in the category of commutative unital rings.
A group is just a category with one object in which all of the arrows are isomorphisms.
The tensor product is just the left adjoint of the internal hom functor

Note that we have used the word "just" in all of these definitions to sound extra pompous. Even though some of these words may sound crazy, most of them mean rather simple things in the language of category theory. For instance, an initial object is an object that has exactly one arrow going to every object in a category. Similarly a terminal object is an object with one arrow coming from every object in a category. If we think back to our category **MUMS** then DJ (the tallest committee member) would be the terminal object and the shortest committee member would be the initial object.

Hopefully this introduction has been able to persuade you of the magnificence of category theory. Although many mathematicians palm it off as "abstract nonsense" it is still a beautiful topic and undeniably has many applications within and outside of mathematics.
- Daniel Johnston

What is a Tensor? - Steven Xu

This is the second installment in a four-part introduction to Riemannian Geometry. I aim to clarify some concepts that scare many undergraduate students. These ideas are covered and assessed in Applied Mathematical Modelling (MAST30030), Geometry (MAST30024), Methods of Mathematical Physics (MAST30031) and various physics subjects - chiefly Electrodynamics (PHYC30016). They also turn up implicitly in Vector Calculus (MAST20009) and Linear Algebra (MAST10007). The first instalment, "What is a Vector?" was published in Paradox Edition 1 2019. To summarize the key points:

- A manifold M is a topological space that locally looks Euclidean. Informally, take pieces of "flat" $\mathbb{R}^n$ space, stretch/squeeze/twist them as desired, then glue them together, "nicely" so the glued sections look locally Euclidean.

- The above property guarantees the existence of a **chart**, a map $\phi : M \to \mathbb{R}^n$, that attributes each point on the manifold to a point in a flat n-dimensional "page". For example, a world map (Figure 3) depicts the points of the earth's surface (approximately a sphere) onto a 2-dimensional page (i.e. some subset of $\mathbb{R}^2$). For curved manifolds (like the earth) distortion is unavoidable. In the Mercator projection (right), Greenland looks the same size as Africa. The Peters projection (left) preserves area, at the expense of distorting aspect ratios of countries near the poles.

- Charts give rise to **covariant basis vectors**. Each **geometric vector** at point p on the manifold is some linear combination of these.

- A [geometric] vector at a point $p \in M$ can be conceptualized as the limiting process as two points approach each other at p. The set of all possible vectors at p form a vector space $V = T_pM$, called the tangent space - there is one such vector space at *every* point on the manifold.

- A covector is a linear map $T_pM \to \mathbb{R}$, conceptualized as a hypothetical "caliper" that measures the separation between a pair of points. A basis covector ε^i is aligned and calibrated such that it measures "1" for e_i, and "0" for all other basis vectors (i.e. $\varepsilon^i(e_j) = \delta^i_j$). The set of possible covectors at a point forms the **dual** to the tangent space, called the **cotangent space**.

- One can define an inner product by its action on each pair of basis vectors $e_i \circ e_j = g_{ij}$, and given such an inner product (specified by positive-definite symmetric matrix $g = [g_{ij}]$, called the **metric**) represent ε^i using a vector e^i, whose components are the i-th row of matrix g^{-1}. The two are "equivalent" in the sense that $\varepsilon^i(e_j) = e^i \circ e_j = \delta^i_j$. We notate this "musical isomorphism" by $(e^j)^\flat = \varepsilon^j$ and $(\varepsilon^j)^\sharp = e^j$ (the **flat isomorphism** turns a vector into a co-vector - the **sharp isomorphism** does the opposite).

- A **Riemannian** manifold is equipped with an inner product on each tangent space, such that g varies smoothly.

In this article, we will:
- Define the tensor product
- Introduce the Einstein notation for tensors
- Show that the tensor product and inner product transform covariantly (and explain what this means)
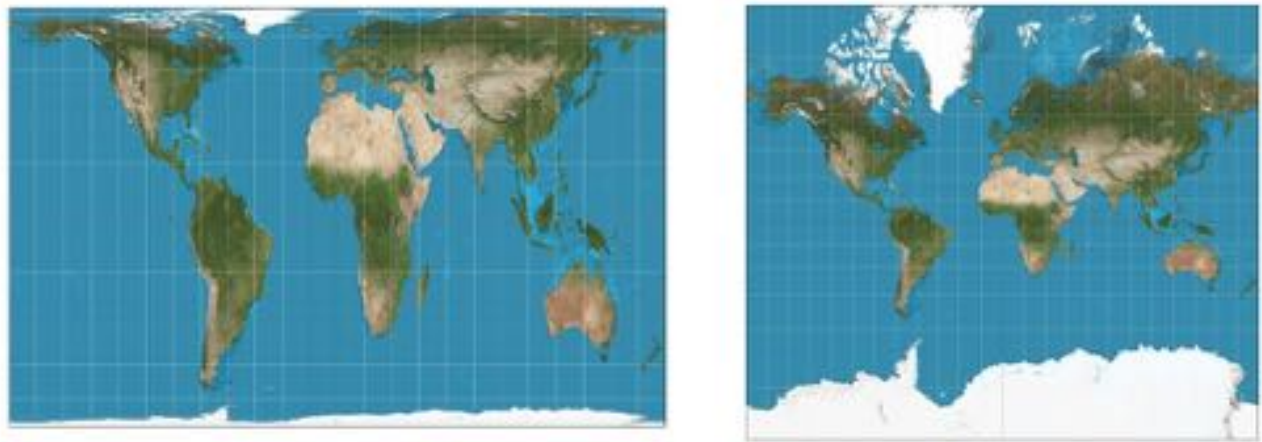- Define k-forms and the exterior derivative

Figure 3: The Peters projection (left) and Mercator projection (right)

1   The Tensor Product

Given a vector space V with basis elements $\{e_i\}_{i=1}^n$, construct an n^m dimensional vector space, whose basis elements are the ordered m-tuples of e_i. For example, taking $m = 2$, and $V = \mathbb{R}^3$ (the familiar three dimensional Euclidean space), setting $\{e_1, e_2, e_3\} = \{\hat{i}, \hat{j}, \hat{k}\}$, we have a nine-dimensional space:

$$span\{e_1e_1, e_2e_1, e_3e_1, e_1e_2, e_2e_2, e_3e_2, e_1e_3, e_2e_3, e_3e_3\} = span\{e_ie_j\}_{i=1,j=1}^{3,3}$$

We denote the resulting vector space $V^{\otimes 2} = V \otimes V$, where the $\otimes$ symbol denotes a **tensor product** of vector spaces. If we choose $m = 3$, we get a 27-dimensional vector space, $V \otimes V \otimes V = V^{\otimes 3}$.

We define the **tensor product** (**outer product**) to send a pair of vectors $u, v \in V$ to $V \otimes V$ in the obvious way - specifically, the e_ie_j component is u^iv^j, where u^i and v^j are the e_i and e_j components of u and v, respectively. The tensor product can extend to dissimilar vector spaces, for example the product between vector $v \in V$ and covector $u \in V^*$ (V^* denotes the dual space) gives us an element of $V \otimes V^*$ which has basis $\{e_i\varepsilon^j\}_{i=1,j=1}^{n,n}$ A product comprising n copies of V and m copies of V^* is called the (n, m)-tensor space, its elements are (n, m)-tensors; n and m, are, respectively the contravariant and covariant degree ("order", "rank", "valency")

Visualize each element $e_i\varepsilon^j$ as mathematical entity that linearly maps some vector (say, basis vector e_k) to another vector, namely $\varepsilon^j(e_k)e_i = \delta_k^j e_i$. Extend this map by linearity - a general second-order $(1, 1)$-tensor $T =$

$\sum_{j=1,i=1}^{n,n} T_j^i \varepsilon^j e_i$ is a linear combination of these $\varepsilon^j e_i$ basis elements. It maps a vector $w = \sum_{k=1}^{n} w^k e_k$ to:

$$w \circ T = \sum_{j=1,i=1}^{n,n} \sum_{k=1}^{n} w^k \varepsilon^j(e_k) T_j^i e_i = \sum_{k=1,j=1,i=1}^{n,n,n} w^k \delta_k^j T_j^i e_i = \sum_{j=1,i=1}^{n,n} w^j T_j^i e_i$$

We have **contracted** a *vector*, with components w^k, with $(1,1)$-tensor, with components T_j^i, to produce a *vector* with components $\sum_{j=1}^{n} w^j T_j^i$ - observe that the summation leaves one free index, namely i. We can instead contract a vector $v = \sum_{k=1}^{n} v^k e_k$ with the *first* index of T:

$$\sum_{k=1,j=1,i=1}^{n,n,n} v^k T_j^i (e_k \circ e_i) \varepsilon^j = \sum_{k=1,j=1,i=1}^{n,n,n} v^k g_{ki} T_j^i \varepsilon^j \in V^*$$

The outcome is a co-vector, with components $\sum_{k=1,i=1}^{n,n} v^k g_{ki} T_j^i$; summation over the i and k indices leaves only j as a free index. In both cases, *contraction removes two indices from the expression.* Contraction between vector and co-vector indices gives $\varepsilon^i(e_j) = \delta_j^i$, regardless of the chart (immediately follows from the definition of ε^i). By contrast, contraction between vector indices invokes the inner product ($e_k \circ e_i = g_{ki}$, where g_{ki} are metric components in the chosen chart). Contracting two co-vectors inserts $g^{ij} := (g^{-1})^{ij}$, the inverse metric. Given an inner product g on the vector space, we can raise or lower the indices; given contravariant and covariant components v^i and u_i respectively, we define $v_i = g_{ij} v^j$, and $u^i = g^{ij} u_j$.

Every linear map $V \to V$ is some linear combination of the 9 basis elements. Writing the components in a 3×3 grid we recognize tensor contraction as matrix multiplication:

$$T \circ w = \begin{pmatrix} T_1^1 & T_2^2 & T_3^3 \\ T_2^1 & T_1^2 & T_2^3 \\ T_1^1 & T_2^2 & T_3^3 \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}$$

$$v \circ T = \begin{pmatrix} v^1 & v^2 & v^3 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} T_1^1 & T_2^2 & T_3^3 \\ T_2^1 & T_1^2 & T_2^3 \\ T_1^1 & T_2^2 & T_3^3 \end{pmatrix}$$

Higher-order tensors are higher-dimensional generalizations of matrices; the 27 components T_{jk}^i of a $(1,2)$-tensor could be written in a $3 \times 3 \times 3$ cube. The world abounds in tensor-valued quantities. Scalars are zeroth-order tensors - they have no tensor indices, and $3^0 = 1$ component. Vectors (in our 3-dimensional space) are first-order tensors, they have one index, and therefore $3^1 = 3$ components. Second order tensors have $3^2 = 9$ components, and a classical example is the stress tensor. When a solid under stress is cut, the surfaces separate under the [now unbalanced] forces. The force separating the two surfaces is a vector, and it depends on the direction of the cut - specifically, one can show that it depends linearly on the unit normal vector of the cut surface. The stress vector

$t(n)$ is therefore the a contraction between a unit-normal vector $n = \sum n^k e_k$, and a second-order tensor, called the stress tensor $T = \sum_{i=1,j=1}^{i,j} T_i^j \varepsilon^i e_j$ (i.e. $t(n) = \sum_{i=1}^{3} n^i T_i^j e_j$). A third-order tensor can be conceptualized as a linear map that converts a vector it into a second-order tensor - or alternatively a linear map that converts a second-order tensor to a vector. In Paradox Edition 4 this year, we will meet a third order tensor called Torsion, and a fourth-order tensor called Riemannian curvature.

Having specified a coordinate chart, and therefore the basis vectors $\{e_i\}_{i=1}^n$, some authors drop the basis elements, and talk about the components "being" the tensor in question. The an (n,m)-tensor has components with n contravariant and m covariant indices. For the remainder of this article, and the others in this series, we will adopt the **Einstein summation convention**. Each term in a **tensor expression** comprises a product of basis vectors (with covariant indices) or covectors (with contravariant indices), and numbers (with any number of covariant or contravariant indices. An index label (say, j) may either appear exactly once (as a **free index**), or exactly twice (as a **repeated index**), in which case the summation $\sum_{j=1}^n$ is implied to act over the term. Summation/equality of tensor expressions is only meaningful when the free indices match; an equation asserts equality *for all combinations of integer values of free indices* in the range $\{1...n\}$.

2   "Tensors transform Tensorially!"

Physical quantities, such as force or velocity are **chart independent** - it would be absurd, for example, if the evolution of a system depended on the framework with which we use to describe it. In the previous article, we defined a vector at a point $p \in M$ without any reference to a coordinate chart - it is the limiting process as one point approaches (or leaves) another point at p. The covariant basis vector e^i associated to chart coordinate x^i in a given chart $x = \{x^i\}_{i=1}^n$ by moving the point thus: hold all coordinates except x^i fixed, and vary $x^i \mapsto x^i + \epsilon$. In a different chart, say $y = \{y^j\}_{j=1}^n$ the same point moves from coordinates $\{y^j\}_{j=1}^n$ to:

$$y^j \mapsto y^j + \epsilon \frac{\partial y^j}{\partial x^i}$$

In other words, the vector e_i in the x-chart equals $\frac{\partial y^j}{\partial x^i} \tilde{e}_j$ in the y-chart. This justifies the terminology *covariant* basis vector - the list of basis vectors "co-vary" with the Jacobian [matrix] $J_i^j := \frac{\partial y^j}{\partial x^i}$ of the coordinate transformation $x \mapsto y$; specifically, $e_i = \tilde{e}_j J_i^j$. Substitute this relationship to find the transformation law for vector *components*, v^i, from the covariant basis (say $\{e_i\}_{i=1}^n$) associated to one chart (say, x), to another (say basis vectors $\{\tilde{e}_i\}_{i=1}^n$ in the y chart):

$$e_i v^i = \underbrace{\left[\tilde{e}_j \frac{\partial y^j}{\partial x^i}\right]}_{e_i} v^i = \tilde{e}_j \underbrace{\left[\frac{\partial y^j}{\partial x^i} v^i\right]}_{\tilde{v}^j}$$
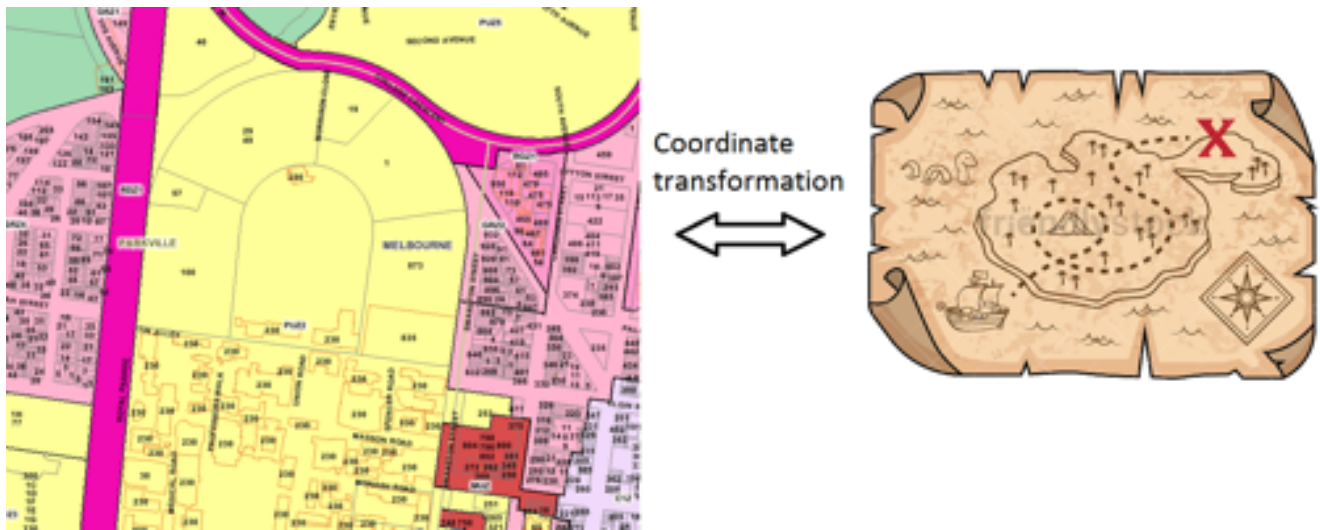
Figure 4: The official Planning Map of Parkville (left), and my secret map of Parkville (right)

This expression confirms that the coordinates transform in the opposite (*"contravariant"*) direction, $\tilde{v}^j = J_i^j v^i$: to find the components v^i in the x-basis from the components $\tilde{v}^i$ in the y-basis, we need the *inverse* of the Jacobian matrix. By definition, $\varepsilon^j(e_i) = \delta_i^j = \tilde{\varepsilon}^j(\tilde{e}_i)$, so the transformation law for the covector basis is reversed, since:

$$\tilde{\varepsilon}^s(e_i) = \tilde{\varepsilon}^s(\tilde{e}_t J_i^t) = \tilde{\varepsilon}^s(\tilde{e}_t)J_i^t = J_i^s = J_k^s \varepsilon^k(e_i)$$

Since the equation holds for all $i \in 1...n$, it follows that $\tilde{\varepsilon}^s = J_k^s \varepsilon^k$. In analogy to the previous discussion, that the *components* of a covector transform covariantly, that is: $w_j = \tilde{w}_k J_j^k$. Recalling that tensors comprise tensor products of vectors, we affirm by linearity that tensors transform in a similar way, for example, a tensor in $V \otimes V^* \otimes V^*$:

$$T_{jk}^i e_i \varepsilon^j \varepsilon^k = \left[T_{jk}^i \frac{\partial y^s}{\partial x^i} \frac{\partial x^j}{\partial y^t} \frac{\partial x^k}{\partial y^u} \right] \tilde{e}_s \tilde{\varepsilon}^t \tilde{\varepsilon}^u$$

Covariant indices go "downstairs", contravariant indices go "upstairs"; **this convention will remain consistent throughout this article, all other Paradox articles I write, and any other paper I write during my lifetime**. Henceforth, we will change the basis vector and covector notation to remind us of the charts from whence they came:

$$e_i \mapsto \frac{\partial}{\partial x^i} \qquad \tilde{e}_j \mapsto \frac{\partial}{\partial y^j}$$

$$\varepsilon^i \mapsto dx^i \qquad \tilde{\varepsilon}^j \mapsto dy^j$$

Charts are useful because they enable us to specify regions, curves or vectors on a manifold using concrete num-

bers. One can specify a vector with a list of components in a stipulated coordinate chart. But does it transform like a vector? It does if we simply *define it to do exactly this*. Consider this analogy – your local city council keeps a Planning Map, which depicts the precise boundaries of each property in your suburb. It uses chart coordinates that conveniently fit your suburb onto the legal document - but the precise scale and rotation is otherwise arbitrarily fixed. If you draw a line that subdivides an existing property, you amend this legal document. The document confers a physical meaning to the line. In order to accurately draw this line on a different map, one must first calculate its new length and angle in accordance with to the aforementioned transformation law. This crucial point is often missed: *simply declaring a set of components in a specified chart to be a vector, suffices to make it a vector*.

Figure 4 shows the official Planning Map of Parkville (left), and my own map of Parkville (right). Although the two maps *appear* different, **they depict the same physical location**. The tangent vectors along the dotted path correspond to physical tangent vectors on the university campus. With the appropriate coordinate transformation, they can be transcribed to the Planning Map (I have omitted the details of this coordinate transformation to keep the location of the red "X" confidential).

3 Operations on Tensors "transform tensorially"!

Physical quantities are related by operations that convert vectors into vectors (for example $\nabla \times (A) = B$), scalars into vectors (for example $\nabla(\phi) = -E$), or vectors into scalars (for example $\nabla \cdot (D) = \rho$). These quantities are *defined* using some underlying coordinate chart, and any physicist of good conscience must check that the outcome is chart-independent. To do this, we:

1) Compute the result using some general chart, say $\{x_i\}_{i=1}^n$

2) Tensorially transform the input into components in some other chart, say $\{y^j\}_{j=1}^n$

3) Compute the result in the y chart, and switch back to x coordinates. Compare with 1)

Is the inner product chart independent? Consider the inner product of two vectors:

$$v \circ w = v^i \frac{\partial}{\partial x^i} \circ w^j \frac{\partial}{\partial x^j} = v^i w^j g_{ij}$$

Changing coordinate chart $x \mapsto y$, we have:

$$v \circ w = \left[v^i \frac{\partial y^s}{\partial x^i} \frac{\partial}{\partial y^s} \right] \circ \left[w^j \frac{\partial y^t}{\partial x^s} \frac{\partial}{\partial y^t} \right] = v^i w^j \left[\tilde{g}_{st} \frac{\partial y^s}{\partial x^i} \frac{\partial y^t}{\partial x^j} \right]$$

Here $\tilde{g}_{st}$ denotes the matrix containing the inner products between y-chart basis vectors. When we assert that the inner product is chart-independent, we equivalently say that the metric g transforms tensorially (i.e. $g_{ij} = \tilde{g}_{st} \frac{\partial y^s}{\partial x^i} \frac{\partial y^t}{\partial x^j}$). A common way to specify an inner product is to give the components of the metric tensor in one chart, but in doing so you forfeit the freedom to specify g in any other frame.

Does the derivative transform tensorially? Consider a scalar (say, temperature) $T(x)$, given as a function of coordinates $\{x^i\}_{i=1}^n$ in some chart. We can differentiate it to get a components that conveniently have a downstairs index (i.e. in the denominator of the derivative quotient), so let's contract it with covector basis elements: $\frac{\partial T}{\partial x^k} dx^k$. Suppose we do this in a different chart $\{y^i\}_{i=1}^n$, giving us $\frac{\partial T}{\partial y^i}$. By the multivariable chain rule:

$$\frac{\partial T}{\partial y^i} dy^i = \frac{\partial T}{\partial x^k} \frac{\partial x^k}{\partial y^i} dy^i = \frac{\partial T}{\partial x^k} dx^k$$

We confirm that taking the **gradient** of a scalar adheres to chart independence - **it produces a tensor with one covariant index** (i.e. a covector).

But now suppose we differentiate the components of a tensor $T = T_{ijk} dx^i dx^j dx^k$, with respect to coordinate x^s (the analysis extends to $(0, k)$-tensors; $k = 3$ is used here for illustration). We call this the **comma derivative**. In the x coordinate chart, we denote it thus:

$$T_{ijk,s} := \frac{\partial T_{ijk}}{\partial x^s} dx^s dx^i dx^j dx^k$$

Express this result in the $\{y\}$ coordinate chart:

$$= \frac{\partial T_{ijk}}{\partial y^t} \frac{\partial y^t}{\partial x^s} dx^s dx^i dx^j dx^k = \frac{\partial T_{ijk}}{\partial y^t} dy^t dx^i dx^j dx^k = \frac{\partial T_{ijk}}{\partial y^t} \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} dy^t dy^{i'} dy^{j'} dy^{k'}$$

Compare this against the outcome if we began in the y coordinate chart. We start with:

$$T = T_{ijk} dx^i dx^j dx^k = T_{ijk} \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial k^i}{\partial y^{k'}} dy^{i'} dy^{j'} dy^{k'}$$

When you differentiate using the product rule:

$$\left[\frac{\partial T_{ijk}}{\partial y^s} \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} + T_{ijk} \left(\frac{\partial^2 x^i}{\partial y^s \partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} + \frac{\partial x^i}{\partial y^{i'}} \frac{\partial^2 x^j}{\partial y^s \partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}} + \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial^2 x^k}{\partial y^s \partial y^{k'}} \right) \right] dy^s dy^{i'} dy^{j'} dy^{k'}$$

This entity has n^4 independent components, each of which comprises of four terms. The first term matches the previous result - it transforms tensorially. Our previous expression differs from this quantity by the terms in the round brackets, which arise as an artefact of our decision to differentiate in y coordinates rather than x - this immediately confirms that *the comma derivative is chart dependent*. This immediately disqualifies it from inclusion in any physical law.

4   The Exterior Derivative and k-forms

We **antisymmetrize** a $(0, k)$-tensor by taking $k!$ copies (one for each permutation of the index labels), and average them - but first, switch the signs of odd permutations. For example the antisymmetrization of a $(0, 3)$-tensor, say T_{abc}, gives a $(0, 3)$-tensor with $[a, b, c]$ component equal to $\frac{1}{6}(T_{abc} + T_{bca} + T_{cab} - T_{acb} - T_{bac} - T_{cba})$. Suppose we antisymmetrize the aforementioned comma derivative with respect to free indices (s, i', j', k'). Closely examine the first term in the round brackets:

$$T_{ijk} \frac{\partial^2 x^i}{\partial y^s \partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}}$$

Antisymemtrization produces $4! = 24$ terms, of which the above expression is one. Of the remaining 23, here is one with an **odd** permutation (I have swapped $s \leftrightarrow i'$), hence the minus sign:

$$-T_{ijk} \frac{\partial^2 x^i}{\partial y^{i'} \partial y^s} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}}$$

Crucially, **the two are numerically equal** (equality of mixed partial derivatives) **and therefore cancel**. In fact, all 24 terms in the antisymmetrization can be paired up in this way (they are numerically equal, but differ by a single swap between the first two indices). In a similar way, antisymmetrization annihilates the other terms in the round brackets, leaving the only the first term, which has the desired coordinate transformation properties:

$$\text{Antisymm}\left[\frac{\partial T_{ijk}}{\partial x^s}\right] \rightarrow \text{Antisymm}\left[\frac{\partial T_{ijk}}{\partial y^s} \frac{\partial x^i}{\partial y^{i'}} \frac{\partial x^j}{\partial y^{j'}} \frac{\partial x^k}{\partial y^{k'}}\right]$$

The above construction is called the **exterior derivative** of a $(0, 3)$-tensor T. The general case, $((0, k)$-tensor) follows in parallel - you take the comma derivative in any arbitrary chart, and then antisymmetrize the resulting $(k + 1)$ index entity. **In contrast to the comma derivative, the exterior derivative is demonstrably chart-independent**, making it a valid tool for formulating physical theories. An antisymmetric $(0, k)$-Tensor is called a **k-covector**. A k-covector comprises a linear combination of antisymmetrized $dx^{i_1} ... dx^{i_k}$ basis vectors (called **k-blades**), where $\{i_m\}_{m=1}^k = S$ is a size-k subset of $\{i_m\}_{m=1}^n$ (there are C_k^n such subsets, and therefore C_k^n k-blades from which to construct a k-covector). A tensor field of smoothly varying k-covectors is called **k-form**. Exterior differentiation converts a k-form to a $(k + 1)$-form.

In this article, we have defined tensor products, tensor contractions, and revisited the notion of chart-independence, contravariant and covariant transformation laws. We have defined the exterior derivative, and shown it to be chart-independent. We will later re-define the vector cross product in a chart-independent way, by constructing the Hodge star operator (Hodge dual). The next installment of this introduction to differential geometry will be titled "What is a Derivative?". We will use the exterior deriative to define the divergence, curl, gradient and Laplacian operators on Riemannian manifolds. We will define two more differential operators, namely the covariant derivative and the Lie derivative.
- Steven Xu

# How to explore higher dimensions - Blake Sims

"...The tools mathematicians use to see the invisible"

Have you ever wondered what the fourth dimension looks like? We live our lives in three dimensions; three independent directions we can move in: forward/back, left/right, and up/down. However, it's possible there is an extra dimension which is hidden from us. This might be due to our human perspective being too large or too small to comprehend this extra dimension (or extra dimensions!). We could be like ants confined to the surface of the Earth; unable to comprehend the notion of 'height'. However, we are highly intelligent ants that can use mathematics to learn about the universe of shapes that live in higher dimensions. Mathematics provides the tools required to discover the properties of higher dimensional objects. One such tool is called a bundle.

1   What is a bundle?

A bundle is a collection of flat spaces, 'bundled' together onto another curvy space. To begin, start with an interesting curvy space and a collection of flat spaces like straight lines or flat planes. Next, 'bundle' all of the flat spaces together onto the curvy space, so each point on the curvy space has an attached flat space. The resulting object is called a bundle.
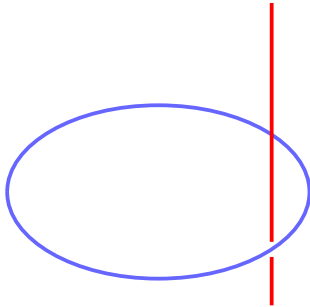
For example, let's make a bundle of lines on the circle. We start off with the following two ingredients:

Space: Circle   Flat Spaces: Straight Lines
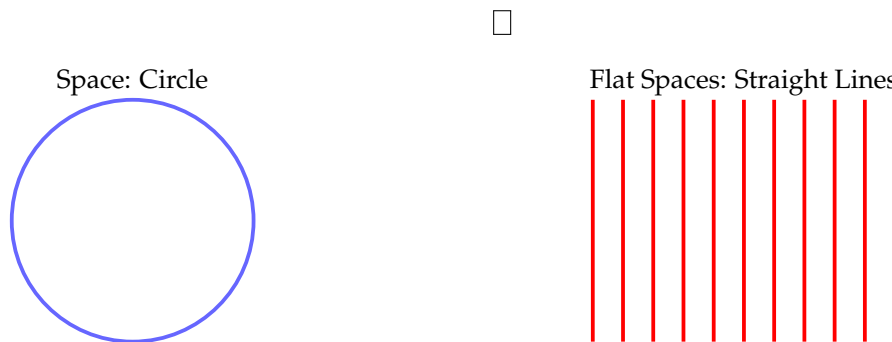
Using these two ingredients, we can bundle the lines onto the circle in two different ways. We can stick them on vertically up and down at each point (see the figure to the left below), or they can be stuck on tangent to the circle at each point (see the figure to the right below).

Sticking the lines on vertically   Sticking the lines on tangentially
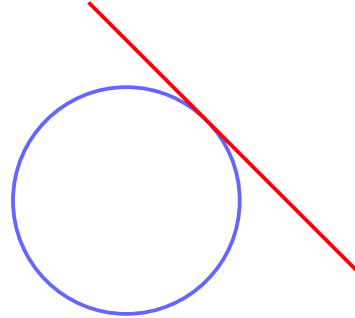
By 'bundling' all the lines onto the circle at each point in these two ways, we will end up with two bundles on the circle, depicted in Figure 5.
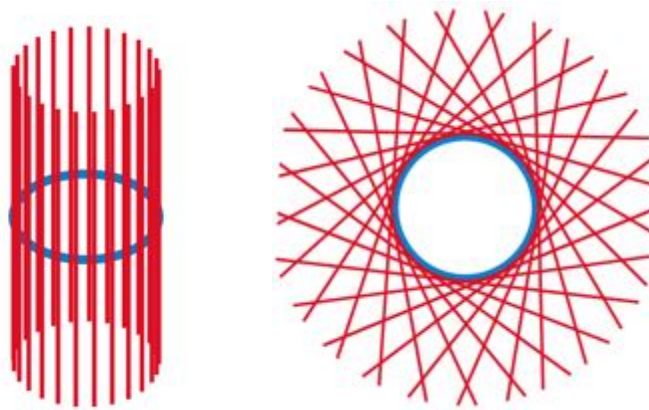

Figure 5: Two bundles of lines on the circle. By sticking the lines onto the circle vertically at each point we obtain the bundle on the left, resembling a cylinder. By sticking the lines on tangentially we obtain the bundle depicted on the right, called the tangent bundle.

Although these two bundles look different, mathematicians consider them to be the same. Two bundles are considered the same if one can be deformed into the other in a particular way. For example, we can rotate each vertical line in the left bundle by 90 degrees and the line will end up tangent to the circle, just like the tangent bundle. This is analogous to the fraction $2/4$ being the same as $1/2$. Although they look different, they are really the same mathematical object.

With this notion of 'sameness' in mind, we have only one bundle on the circle! So are there any more bundles on the circle? We can create one more by twisting the cylinder bundle as seen in Figure 6.
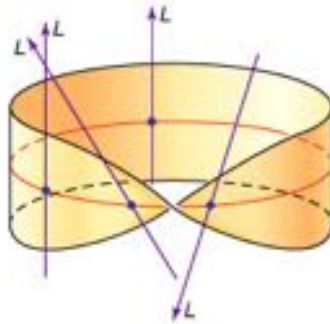
Figure 6: The Möbius bundle over the circle (red). Notice that the idea of "up" changes as you move around the circle. Starting from the back and moving anti-clockwise we begin to see that what was up now becomes down! This property is called non-orientability.

Because of this twist it is a special type of bundle called a non-orientable bundle. Orientability means you can make sense of which direction is up and which is down. So, for highly intelligent ants living on the non-orientable bundle in Figure 6, the ideas of "up" and "down" do not make any sense. This was the idea behind Escher's art piece "Möbius Strip II" seen in Figure 7.
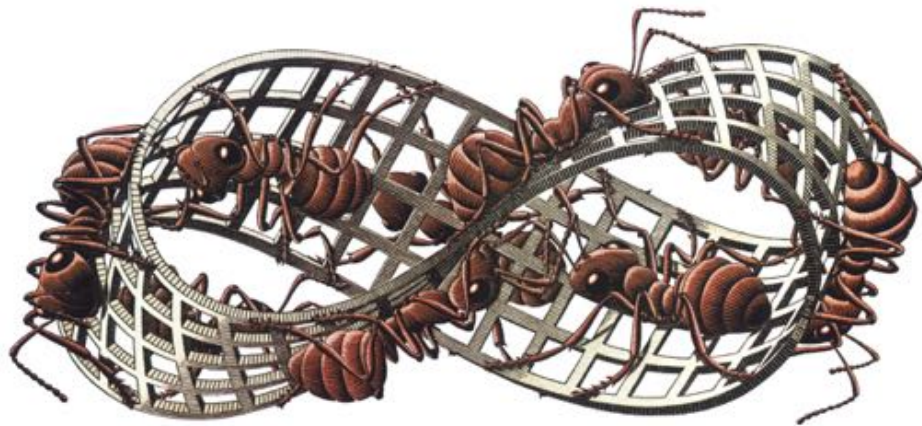
Figure 7: Escher's art piece "Möbius Strip II" showing what it would be like to live on a non-orientable space.

So there is only one orientable bundle on the circle - how many orientable bundles are possible on the sphere? To begin, here are the ingredients we need to build bundles on the sphere:

□

Space: Sphere
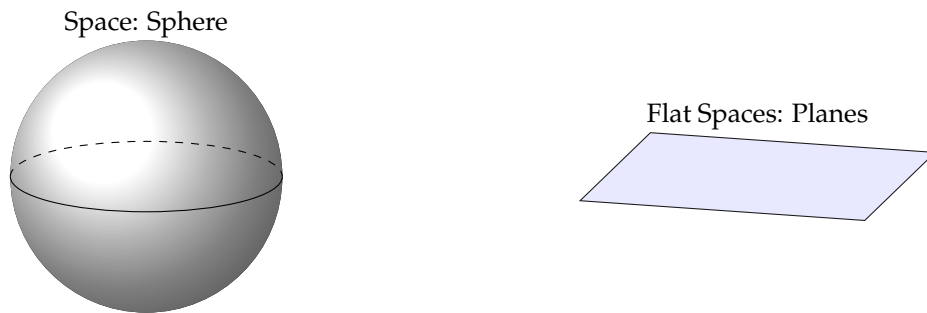
Flat Spaces: Planes

Similarly to the circle, we just need to decide how to bundle the flat spaces onto the sphere. For example, analogously to the circle, we can attach planes onto the sphere so they are tangent to the sphere (see how a plane is attached at the north pole in Figure 8).

Figure 8: We can bundle the flat planes onto the sphere by attaching planes so that they are tangent to the sphere. In the figure above, we have attached a plane to the north pole so that it is tangent to the sphere there.

Doing this at all points on the sphere gives you the tangent bundle on the sphere. For example, we see three planes bundled onto the sphere in Figure 9. The tangent bundle to the sphere actually lives in the fourth dimension.

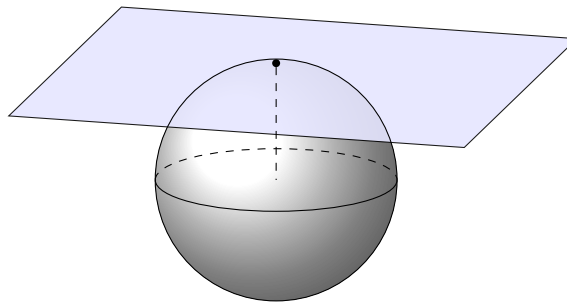Thus, we find ourselves in the world of higher dimensional objects. To navigate this territory we need tools from mathematics. For example, if we wish to know if there are other bundles we can build on the sphere we will need a tool called the clutching function.

2   Clutching functions

These tools were named by mathematicians in analogy with a clutch in a car, which is a mechanical device used to connect two rotating shafts. Similarly, a clutching function is used in mathematics to connect/glue two pieces of a bundle together.
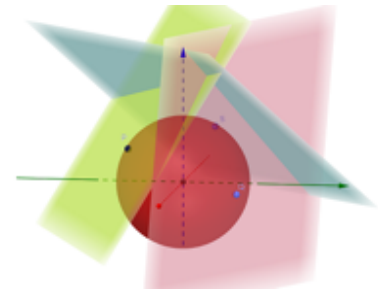

Figure 9: Three tangent planes on the sphere, seen intersecting one another. They intersect one another because there is not enough room in three dimensions - we need an extra fourth dimension to see this bundle properly!

It turns out that we can create any bundle on the sphere using a clutching function. We do this by gluing a bundle on the northern hemisphere to a bundle on the southern hemisphere along their intersection - the equatorial circle. For very deep reasons, this amounts to wrapping the equatorial circle around another circle. For example, see Figure 10 for the tangent bundle.
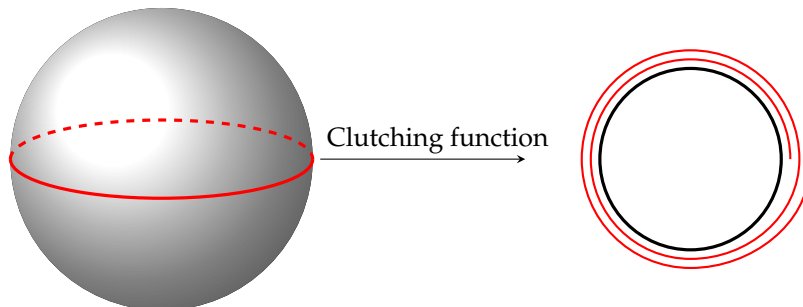
Figure 10: The clutching function used to build the tangent bundle on the sphere. The clutching function is used to glue/connect the northern hemisphere bundle to the southern hemisphere bundle in order to create a bundle over the whole sphere.

So our question "how many bundles are possible on the sphere" has a clear answer - an infinite number. Every bundle corresponds to how many times the equatorial circle wraps around another circle. So far we have observed only one orientable bundle on the circle, and an infinite number of orientable bundles on the sphere. Using the same tools and methodologies mathematicians can count how many bundles are possible on the hyper-sphere - the fourth dimensional sphere (as depicted in Figure 11), or indeed any sphere in any dimension. The results are shown in Table 1.
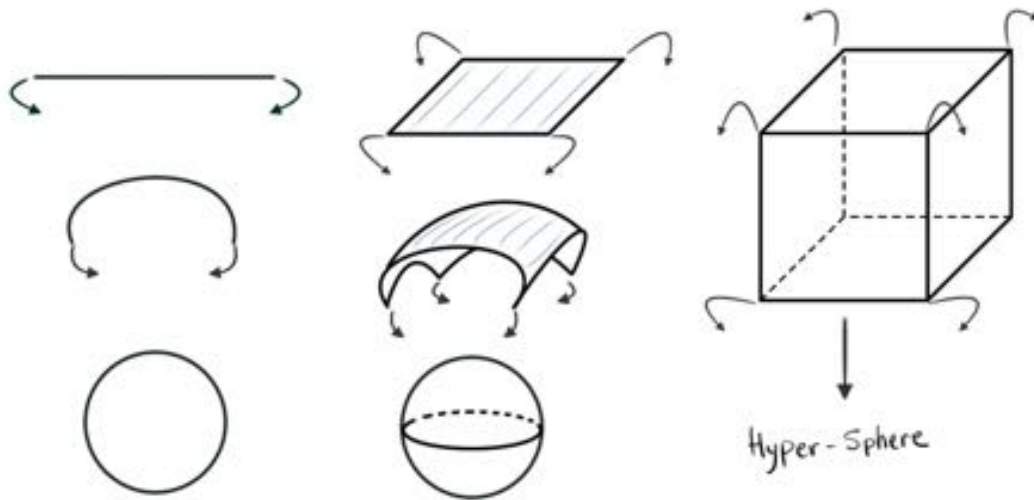
Figure 11: Spheres come in different dimensions. The sphere in the second dimension, i.e. the circle, is a *line* that curves back on itself. The sphere in three dimensions is an *area* that curves back on itself. So the sphere that lives in the fourth dimension, i.e. the hyper-sphere is a *volume* that curves back on itself.

Table 1: How many bundles are possible?

Dimension of the Sphere	2	3	4	5	6
Number of bundles	1	∞	1	$2 \times \infty$	2

3 What have we learned?

We can see a large difference between the circle and the sphere - the circle has only one bundle whilst the sphere has an infinite number. The hyper-sphere has just one bundle, so is more similar to the circle than the sphere in that regard. Continuing further into the invisible realm of higher dimensional objects, the sphere that lives in the fifth dimension has even more bundles than the ordinary sphere. And then the black sheep: the sphere in the sixth dimension has *two* bundles!

I am currently using these tools to search for spaces that have bundles with a particular property I am calling "being turnable". This property can be described as the ability to continuously turn the bundle by 180 degrees. It is unknown if there are such spaces, other than spheres, that have bundles which are turnable. Whilst there are no immediate implications for this research, it will add a small amount to our human understanding of something humans cannot possibly see, even with the most powerful telescope, microscope or scientific apparatus.

- Blake Sims

Interview: Working at Optiver - Madeleine Johnson

MUMS president Madeleine Johnson interviews Tim Crowe, a Trader at Optiver. This is a sponsored article. If you have any questions or comments about this, please feel free to get in touch with us at mu-ms@unimelb.edu.au

What did you study at uni?

I graduated from UoM in 2016 with a BSc in Mathematics and Statistics. I specialised in Applied Mathematics but also had a keen interest in Statistics and Stochastic Processes.

What's your role at Optiver?

I joined Optiver as a Trading Intern in Nov 2016, before moving immediately into a Graduate role. I started in a team trading Hong Kong index derivatives, before spending a year trading in the Japanese market and I now focus on short term Korean index options. Optiver is a market making firm; this means we are happy to both buy and sell the financial products that we trade, regardless of how busy the conditions are. By always providing competitive prices, we ensure that the market is a fairer place for all participants such as Superannuation funds, banks and individual investors. For an example of an uncompetitive market, imagine a foreign exchange booth at the airport – that difference between the buy and sell price is often very large. As an options trader, my role is to provide markets in many different products for which volatility is the most important input to our pricing. This means that a large part of my role is using mathematics to determine how volatile the market is likely to be under different conditions in the future and complementing this with an understanding of economics and the market I'm trading.

What skills do you use at your work and what are some of the various ways that maths is used at Optiver?

In terms of technical skills, I find myself using statistics and data analysis quite frequently. A lot of our trading has been built around statistical tools, so being able to draw inference from the past but also understand the limitations of data is key. Probability was a favourite area of mine at university and it's something I use every day, often as simply as discussing with my team the likelihoods of certain scenarios and expected values of different trades. There's a lot of interesting mathematics built into our systems, from differential equations underpinning our option pricing models to linear algebra and statistics in our automation strategies.

What's the most useful thing you learnt at uni, that you use as a trader?

My degree helped me to think critically about problems, understanding how to break them down into more manageable pieces. I often think back on Applied Mathematical Modelling when I'm tackling a new project. Although by no means a prerequisite to joining Optiver, I found my basic coding background to be really valuable. It helps being comfortable using a programming language to dive into data and conduct some statistical analysis.

What is something you learnt as a trader, that wasn't taught at uni? Did you have much finance knowledge before becoming a Trader?

While I was at UoM I chose my breadth studies in Economics and Finance, although I would be in a minority of traders who had any Commerce background before joining Optiver. Every trader here goes through a 3-month

training program which gets them up to speed with the necessary mathematics, financial knowledge and programming skills needed to have an impact. The biggest lessons I've learned since starting on the trading floor have been more behavioural; for example, developing the ability to make quick decisions and communicating well within a team in busy conditions.

How can current students make themselves more employable?

Everyone here has a considerable aptitude and interest in quantitative problem solving. Beyond that, Optiver focuses on people with a strong sense of curiosity, a determination to succeed and an ability to work well in a collaborative environment. There's no one thing that I'd recommend, it's just about pursuing your interests and showing how you learn and grow from your experiences.

What's your favourite aspect of your work at Optiver?

Working in a very faced-paced environment and being surrounded by a team of highly intelligent and driven people. The busy days where there's a lot happening in the market are the most challenging to be accurate with our decisions, but that is something that most of us thrive on. It's a great place to be able to work on interesting problems where everyone around you wants to see you succeed and will challenge you to do so. Early in my career I expressed interest in an options pricing problem and was quickly put on a project working with our head of research. The outcome was a new pricing model that is used everywhere in Optiver APAC plus I had the opportunity to go to Taiwan and introduce it to our team there. Compared to my other work experience and internships, the readiness of others to help you flourish is a stark contrast.

And what's your least favourite aspect of your work at Optiver?

Your day is often at the whim of the market. On one hand this can be exciting because you never know what's around the corner, but it also means that you'll find yourself trading some uninteresting days every now and then. These are the days when you hope you've got a cool project to get stuck into.

How diverse is your workplace and how is Optiver working on improving this?

There's no doubt that our gender diversity is not where we want it to be, though it is improving each year. One of the main challenges our industry faces is the relatively low number of females studying relevant STEM degrees. Optiver is keen to encourage more young Australians to pursue their passion for Maths and related fields. To this end, we've invested significant amounts to support organisations that encourage a broader audience to study STEM subjects, such as the National Computer Science School and the Australian Mathematics Trust through the Informatics, Mathematics and European Girls' Maths Olympiad. To ensure all candidates have a consistent experience we also include 'unconscious bias' training for everyone involved in the recruitment process. We also provide multiple opportunities for candidates to chat with current and recent graduates to hear an honest perspective on their experience. Recruitment is only half the story, so recently we've placed more focus on ensuring equal opportunities to employees. One outcome has been increased paid parental leave of 18 weeks for the primary caregiver. (Note: Some help from HR required on this one)

What did you want to do when you were a kid? Does it relate at all to what you're doing now?

When I was a little kid I wanted to play soccer for a living! Soon enough that plan went out the window and through high school and university my career interests tended to alternate from engineering, law or consulting. Trading never crossed my mind until a friend suggested it and I applied to intern at Optiver. One month into the internship I decided I loved it and at the end of the program I joined straight away as a grad.

- Madeleine Johnson

Guide to preparing your CV - Celia Dowling, Steven Xu

The annual Women in Physics and Mathematics camp was conducted in Queenscliff this year. Associate Professor Emma Ryan-Weber and Professor Virginia Kilborn from the Swinburne University astrophysics department ran a fantastic workshop on Curriculum Vitae for STEM jobs. Celia Dowling, a third year science student (and aspiring physicist!), shares her notes/take-home messages from the workshop. All students – regardless of gender identity – would benefit from these hot tips!

General points

- As an undergraduate, don't be afraid if your CV is short at this early stage of your career. Be motivated to seek experiences that will flesh it out over time.
- Keep your LinkedIn profile up to date, including a professional profile photo. LinkedIn is a crucial resource!
- Have an "industry" and an "academic" version of your CV. The content will be the same, but the formatting will differ to ensure that the high-priority content is seen first in each case. In the "Industry" version, prioritise your employment history, and both your "Hard" and "Soft" skills. For the academic version, place your academic achievements, publications, grants and educational history near the top.
- For each application, fine-tune your CV to the internship, job or research opportunity in question – in particular, you want to edit your personal statement / introductory statement to address the key selection criteria (if appropriate). Research the organization, and also check that your background aligns yourself with the organization. Emphasise the sections where you are strong
- Think broadly about your content. Don't underestimate the relevance that something may have to your application. Remember that the actual position might differ significantly from the impression conveyed in the job description. Also, a recruiter may not necessarily be hiring a candidate with a single job in mind, but might also consider their potential eligibility for promotion, and suitability for other roles within the organization at a later date
- It is not mandatory to include the years of appointments and achievements – but do it if it paints a positive picture
- Remember to keep updating your CV as you progress through your studies and career. You will accumulate achievements over time – prioritise these.
- Use reverse chronological order where relevant. If you have noteworthy items further down list (i.e. further in the past), make sure you highlight it in your personal statement

Content

- Personal home address may be redundant, but up-to-date contact details are crucial – e-mail and phone number should suffice
- Personal statement is a brief introduction including key skills, and your career goals.
- Education history (unless a transcript is specifically requested, include the phrase, "Academic transcript available upon request"), including university work and other qualifications
- Licenses/certification (this may include VISA status if you are applying internationally, working with children's check)
- List of publications, conference talks and poster presentations

- Awards and other "markers of excellence" - GPA, WAM, and special prizes / Dean's Honours
- Fellowships and grants
- References (unless references are specifically requested, include the phrase, "References are available upon request")
- Both "Soft" skills and "Hard" skills need to be mentioned (see below).
- Previous experience and employment. Include your duties and leadership experience, or other skills developed. Succinctly highlight the relevance to the job in question, where possible. Long-term positions will be appraised positively, and even low-end jobs can demonstrate sustained dedication and effort
- Volunteer work and other "service" type roles
- Professional Affiliations – detail the ones relevant to the job (for example, The Australian Institute of Physics). If you belong to a number of "left field" groups, just include "member of student societies". Student clubs have lesser weight, although participation in committees (e.g. The MUMS Paradox Editor) can be valuable.

The "Hard" skills are domain-specific. Include any concrete evidence of your proficiency where possible. Think broadly – these skills may not necessarily show up on your academic transcript or previous job titles. Examples include:
- Coding (include languages and specify the amount of coding experience),
- Operating 3D printer
- Using specialized software (R, SPSS). Be specific! How much data did you analyse, how proficient were you?

In contradistinction, "Soft" skills are transferable, and common to a wide spectrum of roles. Promote these honestly, and give evidence where possible. If you misrepresent yourself, it will be apparent during the interview. Soft skills can be cultivated, so if you can identify weaknesses at this stage of your career, make sure you work on them. Some examples of soft skills are:
- Effective communication; receiving and giving effective feedback
- Leadership – this includes interpersonal skills when dealing with people both above and below you in command. Authenticity and confidence in team situations
- Initiative during teamwork
- Empathy (emotional intelligence, inclusion, respect)
- Organisation and time management,
- Creativity and adaptability to unfamiliar circumstances
- Having "vision"; strategic nous
- Work ethic (but also work-life balance!)
- Growth mindset; continued introspection into own performance

Things to avoid
- Photo, gender, hobbies – these play no role in the selection process
- ATAR (especially in post-graduate years) – university marks supersede this information
- Spelling and grammatical errors

- Avoid any grandiose and unrealistic claims about yourself, unless you have firm evidence.

By all accounts, the Curriculum Vitae workshop was a huge success. Celia hopes to arrange a similar workshop (open to all STEM students) through her role in the Physics Students Society.

Remember that UMSU runs interview and CV workshops on campus. The Careers and Opportunity Studio is open Tuesdays and Thursdays from 10am to 5pm at the ground floor of the Stop 1 building; they can provide helpful feedback and advice! Keep up to date with the university careers news.

- Steven Xu, Celia Dowling

Women in Physics and Mathematics Camp - Hannah Perry

This past weekend, Mad, Yifan and I had a great time at the Women in Physics and Maths Weekend. We stayed at the Whitehall Guest House in Queenscliff along with an awesome group of enthusiastic people from all career stages – from undergrads to professors and industry professionals!

On arrival after finding our room and settling in, we collected a notebook and some stationary, then got stuck in to decorating our notebooks and nametags. These notebooks came in handy for when we wanted to take notes during the various informative sessions held over the weekend.

On the first morning there were sessions on soft skills and CVs. The first session involved identifying soft skills and their importance in hiring, and discussing how we judged our current skill level and how we demonstrated or were improving our soft skills. From the valuable conversations we had during this workshop I realised there were ways I was already working towards or perhaps even proficient in some of these soft skills. In the CV session we had the chance to get feedback on our current CV/resume and detailed information on what should be included in an academic CV and how they differed from industry-focused CVs. From this workshop I was able to get much more detailed information on what's expected in an academic style CV than I have been able to find anywhere else.

Next we had activities, where MUMS held an origami table making a modular stellated icosahedron and other things! It was great when were joined on our origami making table by senior lecturers in physics and the former chief scientist of the BoM, giving us the opportunity to have interesting conversations and ask budding questions! Other activities at the camp included Zentangle drawing, knitting and crafts, walks, cycling, and some cool-looking app-controlled robots called Spheros!

Later we heard from a careers panel with experience in working for (and leading!) organisations like Bureau of Meteorology, Telstra, and Data Science at Unimelb. It was really good to hear about the variety of pathways and career journeys from the four successful and accomplished panelists. Something I found really interesting about their career pathways was how it wasn't necessarily a linear pathway from what they started off imagining themselves doing and where they ended up – some weren't following a pre-planned progression, some had many interests while studying and changed their focus during their degrees, and some had many different roles and responsibilities throughout their career.

In the evening, we played boardgames from a brilliant selection, including ones like Bananagrams, Catan, Ticket to Ride, and Railroad Ink. No previous experience was necessary – it was my first time playing these games! There was also other activities available like trivia and telescope/astrophotography.

The next day we had a sessions on career engineering and dealing with microaggressions. The career engineering involved thinking about past and future dream careers, and then planning a major aim and the smaller smart goals that are the steps to accomplishing our aim, on the way to our overarching dream career. It was challenging but worthwhile to think and plan this deeply, and I found it was important to pick a specific potential dream career to focus on to be able to set definite aims and smart goals. In the microaggressions session we got to discuss

some of our experiences (pretty cathartic!) and brainstorm potential responses to deal with some of the seemingly-small-but-buildable instances of sexism and other types of discrimination. I loved the suggestion of responding to subtly sexist remarks with sarcasm as a way to make the person realise what they remarked maybe wasn't so "21st century", but also see the sad importance of keeping a record of the more serious instances as a precaution in case of continued or escalating discrimination or harassment. To me, it was an ideal environment to have these tough but important discussions in and listening to the experiences of this great group of women who've managed to be successful despite the challenges was encouraging.

One of the best things was that we met so many cool and passionate physics students that have opened up small and big collaboration opportunities as well as new friendships and mentors!

Overall, a fantastic weekend that any female or non-binary student of maths or physics should strongly consider applying for next year!

-Hannah Perry

The Life of Miriam Mirzakhani - Hannah Perry

The Fields Medal is seen as one of the most prestigious awards in mathematics. Since 1936 it has been awarded every four years to up to four mathematicians under the age of 40 who have produced work deemed to be outstanding. In 2014 Maryam Mirzakhani was the first – and so far, only – woman to win the award.

Mirzakhani received her Fields Medal in recognition of her significant contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces. Specifically, her PhD focused on hyperbolic surfaces (a surface which has a constant negative Gaussian curvature – an example of this that Maryam often considered was the surface of a double-holed doughnut) and counting closed geodesics (a generalisation of "straight line", in the sense of a segment of the line being the shortest path along the surface between two points) that loop back and don't intersect with themselves. She also contributed research to topics including Teichmüller theory, ergodic theory, and symplectic geometry.

She didn't start off wanting to become a mathematician though. As an avid reader as a child she had contemplated channelling her creativity into a career in writing. She was also initially slightly discouraged from maths with lack of confidence from poor grades and teacher who wasn't very encouraging during the beginning of her middle school mathematics education at a selective girls' school for gifted and talented students in Iran. Though her experience improved immensely the next year with a more encouraging teacher and sparked her interest in the challenges of mathematics.

While she was at high school she successfully requested that the students at her school be offered similar opportunities for maths problem solving classes as were taught at the boys' school, the preparation and practice leading to the exceptional achievement of her and a friend competing as members of the team representing Iran in the 1994 International Mathematical Olympiad. They were the first female participants on the Iranian team.

She went on to complete her Bachelor of Science in Mathematics at Sharif University in Tehran, followed by her PhD at Harvard University involving the highly influential work for which she earned her Fields Medal. She progressed through her roles as an assistant professor at Princeton, research fellow at Clay Mathematics Institute and professor at Stanford.

Mirzakhani described her approach to problem solving as "slow" as she liked to take her time to think deeply about a problem, sometimes over many years. Collaborators and colleagues recall her optimism and "deep intellectual curiosity" in the face of challenging unsolved mathematical problems. She "doodled" diagrams to help her understand and focus on difficult problems, to the delight of her young daughter who called them "paintings". Her husband, Jan Vondrak, who works in applied maths at Stanford, jokingly said "She has these huge pieces of paper on the floor and spends hours and hours drawing what look to me like the same picture over and over." d Krys' family was keeping his line open for awhile Alongside her achievements she remained humble and focused. While she felt greatly honoured by the awards she had no desire to be the sole face of women in mathematics: "There are really many great female mathematicians doing great things."

She died from cancer in 2017, aged just 40. To respect her memory and recognise her brilliance, influence and achievement, the International Council for Science declared her birthday, 12th May, to be the International Women in Mathematics day.

References:

https://news.stanford.edu/2017/07/15/maryam-mirzakhani-stanford-mathematician-and-fields-medal-winner-dies/

https://www.quantamagazine.org/maryam-mirzakhani-is-first-woman-fields-medalist-20140812/

https://www.mathunion.org/imu-awards/fields-medal

https://www.abc.net.au/news/science/2018-01-07/
maryam-mirzakhani-a-brilliant-mathematician-remembered/9240684

https://www.youtube.com/watch?v=swLWqlKMl5M

- Hannah Perry

Tag Yourself - Topology Edition!

Moebius band
- Easily disorientated
- Ambidextrous at all tasks
- Gets into a bit of a spin at the slightest provocation
- Enjoys postmodern art — is a regular patron at the Escher Museum
- Speaks with a non-descript European accent. Hates hearing her name mispronounced
- Says some crazy things sometimes — edgy AF

Torus
- Commutes to work every day
- Loves Krispy Kremes
- Drinks >4 cups of coffee daily
- Sometimes throws a curveball (but everything seems to straighten out in the end)

Projective Plane
- Kind of twisted
- Seems to be around everywhere, but not really known to anybody
- Tries to conform

Klein Bottle
- Is Moebius band's older brother. Considerably less edgy
- Always flexes on Moebius band
- Also ambidextrous, easily disorientated
- Can never decide whether he wants to be inside or outside
- A gym rat. Pretty buff — can't seem to fit in 3 dimensions

Ball
- Hates being mistaken for "Sphere" — especially by physics students
- All conversation points contract down to a single point
- Spends 2 hours every morning combing hair. Can never get it to lie flat
- Is a well-rounded kind of person

Sphere
- Describes self as an "idealized racehorse"
- Is sometimes a bit of a cow
- Often confused with "ball"
- Has same hair problem as Ball

Map of Mathematics

(credit to anonymous contributor to Mathematical Mathematics Memes)

The totality of mathematical statements can be hard to categorize. This map will help the uninitiated!

Definition

Alternative Definition

Axiom

Choice

(filtration)

Of mate says

Something a physics prof said about L^2 functions

Stuff written on serviettes and bar mats

Facts about the category of quasicoherent sheaves

things you read from nLab

Id

∃!

Definition formulated using category theory

Id

Intuition

Theorems and formulae with more than three eponymous names

Existential dread"

Gödel's Incompleteness Theorems / Halting problem

Hoopsiedoodle

Margin too narrow to contain truly marvellous proofs

Remark

Exercise left to reader

"it's trivial"

Things that Norman Wildberger said

Theorems that are actually definitions e.g. compact operators on a Hilbert module, pythagoras

Named Theorem

Fundamental theorem

Theorem

Proposition

Lemma

Lebesgue's Covering Lemma

???

???

Corollary Corollary Corollary Corollary

k-rollary

Theorems you've never heard of because they're false

Infinity doesn't exist

-1/12 memes

For every...

epsilon > 0

diagram of shape J

totally bounded sequence

open cover

... there exists....