Issue 1 2019

Paradox

The Magazine of the Melboourne University Mathematics and Statistics Society

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On the cover (clockwise, from top-left): 1) Sophie Germain, a Heroine of Number theory, 2) An incredible knight sacrifice by Google Deepmind's Alpha Zero takes down Stockfish in late 2018, 3) Dame Mary Cartwright was the founder of modern Chaos Theory, 4) Morphling is Nature's Magnum Opus. Can a neural network beat a team of world-class DotA players? The team at Open AI5 think so!

Words from the President

Hello everyone and welcome to a shining new year of exciting maths!

The birds are singing, the sun is shining and someone even vacuumed the MUMS room! We are in a great mood for 2019, and hope you are too! Regardless of whether you are new or old or very old to maths (we see you, PhD students), we hope you find some aspect of MUMS to engage with this semester. We are here to serve the interests of all students of maths, in all your diverse forms, so always feel free to email us your thoughts/feedback/ideas/memes at mums@unimelb.edu.au, or chat to a committee member while they're loitering in the MUMS room pretending to study.

This semester, look out for:

- Paradox, the (number) one and only maths student magazine on campus. In your hands right now! If you like what you see (or especially if you don't), please get those mathematical creative juices flowing for submissions to Issue 2.
- Keep an eye on our First Year Representative (Epsilon) Elections, the perfect way to get more involved in MUMS if you're a first year this semester. Epsilons help out with events and other MUMS activities, and is a great place to start if you think you might be interested in joining the committee in the future. Elections will be held in the first few weeks of semester, stay tuned for exact dates!
- Puzzle Hunt, a week and a half of mathematical adventure with teams competing from all over the world and awesome prizes to be won (24.04 – 03.05).
- The MUMS Room – located opposite the general office in Peter Hall (look for our banner!). It's equipped with comfy sofas, whiteboards and various mathematical flotilla, and is the perfect place to nap, socialise, or work on some maths with your mates.
- Fortnightly social events – kicking off in week 1 with our subject specific first year picnics AND our welcome back games night for all maths students. Free food AND new friends, what more could you need?
- Coffee catch ups for women and non-binary people studying maths - we care about underrepresentation and promoting diversity in maths, and are always looking for feedback about how we can do this better.
- Seminars and workshops, delivered by everyone from experienced academics to enthusiastic undergrads on a range of exciting topics – be sure to drop by for the knowledge and also the snacks.
- MUMS Study Club, where we make a space for you to study and chat to your classmates about the problems you're working on. And, it goes without saying, includes snacks.

If you're a student of any kind of maths, at any level, MUMS is the society for you! Membership is free and you can join at https://goo.gl/forms/cUdrxaJf5PQxRquy1. If you've been a member of MUMS previously, we still need you to sign up again for this year. If you can't be bothered typing out a link, here in this year of 2019, head to our Facebook page and hit the 'Join' button to take you there.

Hope to see you around!

Yours mathematically,
Madeleine Johnson, President

Words from the Editor

It is my great pleasure to kick off 2019 with an exciting edition of Paradox! We have some fantastic contributions on topology and number theory, and some fascinating biographies of two lesser-known mathematicians from the industrial and modern era. I offer some practical advice in my article addressed to new students. In my other article, I hope to clarify a number of common misunderstandings in a crucial topic that you will encounter many times as an undergraduate. This is a must-read for any aspiring physicists!

We also have some fun content in this edition of Paradox. Many new university students are uncertain about the road ahead. Maybe our Career Chooser algorithm will help you find your perfect career! Decide which Algebraic structure you are! And don't forget to participate in Semester Bingo! Read about your MUMS committee members and don't be afraid to say "Hi" when you see us around the Peter Hall building!

I'm always happy to receive content from our readership! Do you have a good idea for an article? E-mail me at mums.paradox.editor@gmail.com

Regards,
Steven Xu, Paradox Editor

President: Madeleine Johnson

Hi, I'm Madeleine! I've just started my Masters in Pure Maths, after finishing my BSci in Maths and DipLang in Mandarin last year and deciding I wasn't ready to face the real world yet, and also maths is pretty fun! So far my favourite maths subject was Metric and Hilbert spaces in third year #proofmachine My interests outside maths include: education policy (specifically how we can support students disadvantaged by/underrepresented in our current system), Eurovision, amateur theatre and drinking way too much coffee.

Secretary: Vanessa Thompkins

Hey, I'm Vanessa! I'm doing a Bachelor of Science with a Pure Maths major, with a Diploma in Informatics on the side, and am the MUMS secretary for this year. So far my favourite area of maths is group theory - I really love the symmetry and neatness of it. You may also see me around the place in Peter Hall as I help out in the mathAssist drop-in centre and run Calc 1 workshops. When I'm not mathsing, I'm into effective altruism, binge-watching Netflix dramas, and baking cakes in the shape of fractals.

Treasurer: Luke Ireland

Hi I'm Luke! I am currently studying the Bachelor of Commerce majoring in economics and the Diploma of Mathematical Sciences specializing in applied mathematics. I have always had a profound interest in mathematics because I enjoy understanding how the world works! Maths is applicable to everything, particularly economics. As the treasurer of MUMS I organize the group's finances and work alongside sponsors to help fund our events. I'm also an avid guitarist and absolutely love playing music every chance I get!

Education Officer: Anand Bharadwaj

Hi everyone, I'm Anand Bharadwaj, and I am currently a first-year student in the Bachelor of Science planning to major in Maths. I have always loved maths ever since I was young both for its intrinsic logic and beauty and its relevance to so many other disciplines. My current maths subjects are Statistics and Accelerated Maths 2 (I also took Probability in Semester 1). My favourite maths subject thus far has probably been AM2 as I have really enjoyed the rigour and depth of the subject (as well as Barry's unique sense of humour). Outside of university, my main interests include competitive Scrabble, trivia and archery.

Publicity Officer: Ivy Weng

Hello Hello MUMsters, I'm Ivy Weng and I'm a current B.Sci undergrad studying Bioengineering systems and the Publicity Officer for MUMS! I love that Maths is literally the granddaddy of everything and can both make things cleaner and more chaotic. My favourite math subject at Unimelb is literally anything with Christine M. because she is a legend. My favourite animals are dogs, alpacas and chickens. Thanks for reading! My favourite meme: https://www.facebook.com/MathematicalMemesLogarithmicallyScaled/photos/a.1605246506167805/2716479878377790/?type=3&theater

Paradox Editor: Steven Xu

Hi I'm Steven! I am currently studying a Bachelor of Science with the Applied Mathematics and Mathematical Physics majors. As your Paradox Editor, I'll be taking your submissions for our 3-monthly publication, and bundling them up into an entertaining magazine. I have always been interested in the way mathematics describes our physical world. I am interested in Lie groups and differential geometry. On the physics side, I like both quantum field theory and statistical mechanics – I can't commit yet! I'm a keen guitarist, and I write my own songs for Standup Comedy.

Undergrad Representative: Billy Price

Hi I'm Billy, I'm studying the Bachelor of Science, majoring in Pure Mathematics. I feel nice and cozy in the intersection of pure maths, computational theory and logic, so please try not to scare me with things that actually exist. Catch me on campus wearing basically the same outfit everyday, hunting for a free sosig. In my free time - when I'm not listening to the sweet serenade of last weeks lecture capture - I pretend to relate to angry Hip-Hop music.

Undergrad Representative: Dominic Yew

Hello, I'm Dominic! I'm studying a Bachelor of Science majoring in maths and specialising in Statistics & Stochastic Processes while I desperately try to convince myself I'll get a job. I got hooked on math when I realised matrix groups were locally R^n, never looked back. In my spare time I love singing showtunes in the shower and crashing South Lawn dog picnics.

Undergrad Representative: Elinor Mills

Hey, I'm Elinor! I'm currently studying a Bachelor of Science majoring in maths & stats and heading merrily down the path to unemployment with a pure maths specialisation and a desire to study as much logic as humanly possible. I'm passionate about addressing underrepresentation in maths and consistently grouchy about the misconception that you need to be a “””genius””” to enjoy or succeed in maths. When I'm not doing maths things, I'm probably thinking about dogs.

Undergrad Representative: Elizabeth Ivory

Hi, I'm Elizabeth! I'm currently studying a Bachelor of Science. Initially I wasn't sure what to major in, but (applied) maths stole my heart. I love how problems can be manipulated such that mathematics (some of which you may have thought is not applicable to anything) can solve it- eg using complex numbers or residues to calculate otherwise difficult problems with ease. During my non-mathsing time, I love playing games, binge watching Netflix, adoring my guinea pigs and attempting to cook new recipes.

Undergrad Representative: Fiona Guan

Hi my name is Fiona! I'm currently in my second year of Bachelor of Science majoring in Computing and also taking a Diploma of Mathematics, majoring in Operations Research. I love the beauty of theory and the usefulness of logic and maths in so many unexpected places. In my spare time, I enjoy reading broadly and finding solutions to issues important to me, eg. underlying reasons for underrepresentation of women in certain STEM fields.

Undergrad Representative: Hannah Perry

Hi, I'm Hannah! I'm currently studying a Bachelor of Science, majoring in Maths & Stats and intending to specialise in Pure Maths. What I like about maths is how there's a lot of creativity in solving problems and finding proofs, and also how essential maths is to so many different areas! In my opinion, the best maths meme is the list of “What to say instead of ‘trivially’“ :https://www.reddit.com/r/math/comments/7gqhlc/what_to_say_instead_of_trivially/. Specifically, my favourites of the list include “It was revealed to me in a dream that...“, “The math gods demand that...” and “Even my grandma knows that...“.

Undergrad Representative: Keshini Karunaharan

Hi I am Keshini and I am a Bachelor of Science undergrad student. My major is statistics and the reason why I choose this major is because statistics influences many of our decisions in life. I have always found stats really interesting regardless of their subject. My favourite writer at the moment is the Guardian data editor Mona Chalabi who uses statistics to write articles and create cartoons. Moreover, in my spare time I like to watch crime dramas, read novels and paint.

Undergrad Representative: Matt Walker

Hi I'm Matt, and I'm a second/(third i guess, if these are gonna be up next year) year pure maths student. Despite thinking that prime numbers are hella ugly, I'm really interested in number theory. Catch me around the MUMS room talking to Billy about how amazing Kanye West is!

Postgrad Representative: Daniel Johnston

Hiya, my name is DJ! I'm currently doing my masters in pure mathematics and am taking Quantum Computing, Category Theory and Discrete Mathematics this semester. My research is in algebraic number theory which I think is HYPER fun since whole numbers are so cool and nice. I've loved every algebra subject at uni sosososososo much! I'm also a very proud meme page admin and have run a math-related meme page since 2016. After being at unimelb for 4 years I think the real treasure in mathematics is the friends we made along the way.

Postgrad Representative: Rachael McCullough

Hi, my name is Rachael! I'm in my first year of a Master of Science in Applied Mathematics. I'm interested in mathematical biology and will be doing a research project on species distribution models. I love understanding the underlying dynamics of biological systems, and that's why I decided to pursue a Masters after doing an undergraduate degree in ecology and applied mathematics. I'm passionate about education and equity in STEM and can't wait to do some exciting things as one of your MUMS postgraduate reps!

What is a Vector? - Steven Xu

"A vector is something that transforms like a vector. A tensor is something that transforms like a tensor" – every physics professor, ever.

In this article, I hope to clarify a set of related definitions and concepts that provoke anxiety among undergraduate students. These ideas are explored in Applied Mathematical Modelling (MAST30030), Geometry (MAST30024) and Methods of Mathematical Physics (MAST30031). They also turn up implicitly in Vector Calculus (MAST20009) and Linear Algebra (MAST10007), and various physics subjects. My goals here are:
- Reconcile different definitions of a "vector"
- Clarify definition of co-vectors
- Define contravariant and covariant components of a vector – why they are different, and **why it matters!**

1 Definition of a Vector

In MAST10007 and MAST10008, a vector is formally defined as an element of a vector space V – an Abelian group under vector addition $(+)$, with identity 0, which can be compatibly scaled by some field $\mathbb{K}$ (normally, the field is the real or complex numbers):

$$\forall (a, b, c, u, v) \in \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times V \times V, \exists (-v) \in V :$$

$$v + (-v) + u = 0 + u = u \quad \text{(Existence of identity and inverse)}$$

$$(u + v) + w = (v + w) + u \quad \text{(Associative and commutative)}$$

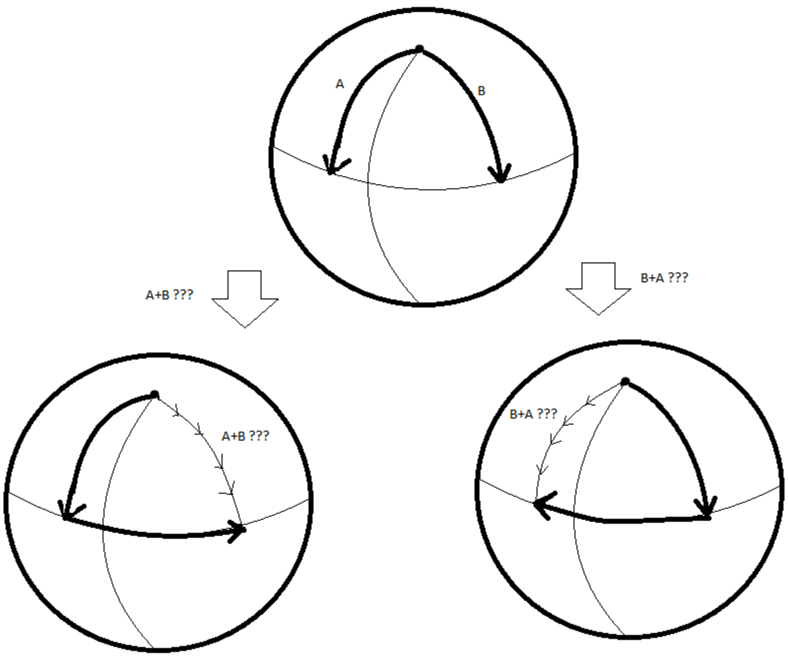
$$(ab + c)u = a(bu) + cu \in V \quad \text{(Distributive, compatible, closed under addition and scaling)}$$

$$a(u + v) = au + av \quad \text{(Distributive over vectors)}$$

Under this broader definition, vector spaces include:
- n-tuples of real numbers (defining addition componentwise)
- Functions of the real numbers (defining addition pointwise)
- Special classes of functions (e.g. polynomials, smooth functions, continuous functions, functions that share a compact support)

This article is about the special type of "vector" (more properly called a **geometric vector**) that you encountered in high-school - an arrow with a specified length (magnitude) and direction. We will make one crucial amendment to this visualisation: we "shrink" the arrow to infinitesimal length. A vector is an arrow that depends on some parameter ϵ, in the limit as ϵ tends toward zero – the vector is "how quickly" the arrow disappears at that instant. We can sidestep any question about the location of the arrow tip – it is “directly adjacent” to its tail. The set of all possible vectors at a point P is (loosely speaking) the set of all points that are infinitesimally close to P – we call this the **tangent space** of at point P, in a **manifold**. The tangent space is the vector space of geometric vectors.

The flat Euclidean space encountered in high school is a special case of a manifold. Informally, a manifold is a space that locally “looks” like a Euclidean space - but distortion or curvature is allowed over large regions. The classic example is the surface of the earth, which locally looks like a flat two-dimensional plane – which it is clearly not, when seen from outer space. If you drew two giant arrows on the earth’s surface, and tried to add them together the way you learned in high school, you would need to translate one arrow so that its tail meets the head of the other one. On a curved surface it is unclear how you might meaningfully do this. You might guess one solution: nail one giant arrow to the roof of your car, and drive in a perfectly straight path along the other arrow, until you reach the tip. Sadly, this construction violates the commutative property of vector addition:



The ambiguity is diminished when the arrows are small and the earth looks “almost flat” over a small region. Vector addition is defined in the limiting case where the points are infinitesimally close. This enables vector addition in the way you learned in high school - and confirms that the geometric vectors in a tangent space do, indeed, satisfy the aforementioned vector space axioms. This can be made mathematically rigorous by visualizing the two ”infinitesimally close” points as a differential operator on the manifold. It acts on a scalar field ϕ by sampling the field at the two points, and divide their difference by ϵ – again, take the limit as $\epsilon \to 0^+$. In this way, two ”infinitesimal arrows” can be converted to differential operators X and Y; their sum is the well-defined differential operator: $((X+Y)(\phi) := X(\phi) + Y(\phi)$. Any differential operator X satisfies the following conditions:

$$\forall a \in \mathbb{K}, \phi, \psi \in C^{\infty}(M)$$

$$X(a\phi + \psi) = aX(\phi) + X(\psi) \text{ (Linearity)}$$

$$X(\phi\psi) = \phi X(\psi) + \psi X(\phi) \text{ (Leibniz property)}$$

The converse is less obvious: any map with these properties can be converted to a differential operator, and therefore identified as some "infinitesimal arrow". The two notions are therefore equivalent; vectors *are* differential operators. Placing a vector at every point on a manifold, we have a **vector field** (not to be confused with the field $\mathbb{K}$), which acts on scalar fields, to give scalar fields. The composition of two such actions on a scalar field violates the Leibniz property (and therefore fails to define a geometric vector), but interestingly the **commutator** of two such operators, $[X, Y](\phi) := X(Y(\phi)) - Y(X(\phi))$, satisfies both differential operator axioms. This is called the **differential geometric Lie** ("Lee") **bracket** of two vector fields.

An aside: vectors cannot be meaningfully translated on a curved manifold, without additional instructions that connect adjacent tangent spaces - a **connection**. The previous picture depicts **parallel transport** with respect to the **Levi-Civita connection**. As demonstrated, the outcome is path-dependent - the connection alone does not meaningfully identify vectors at distant sites.

2 Vectors and Co-vectors

The space of linear maps from vectors to the real numbers is itself a vector space – formally define linearity as the property $f : V \to \mathbb{R} : f(av + u) = af(v) + f(u)$, and confirm that this is preserved under linear scaling and addition of two such maps $((kf + g)(v) := kf(v) + g(v))$. We call this vector space the dual space V^*. Its dimensionality matches V, so there exist many isomorphisms (bijective, linear maps) between the two. No particular choice is "naturally" favoured over the others. :)

Specifically, the dual space to the tangent space is called the **cotangent space**. Extending our previous analogy, the geometric vectors described in the previous section are infinitesimal arrows, their corresponding co-vectors are calipers that act on ("eat") arrows by measuring their length (producing a real number). Just like vectors, co-vectors have direction (i.e. the orientation of the caliper shaft), and magnitude (number of caliper graduations per unit length). It exists at a single point - the location of its stationary edge. When an "arrow" shrinks to zero, we are interested in the rate at which the caliper reading changes in the limit $\epsilon \to 0^+$.

I want to dispel a piece of "common wisdom". There exists a false dichotomy between physical quantities such as electric field which "are" co-vectors, and other quantities such velocity and acceleration, which "are" vectors.

The line is misleadingly drawn between units that have length in the denominator (e.g. V/m) vs. the numerator (e.g. m/s). *No such distinction exists*. For example, electric field, E, has equivalent units N/C, or $kgm/s^2/C$, in which length appears in the numerator. Does this make E a vector, or is it a co-vector? **It is neither**. Vectors and co-vectors are both mathematical entities. It behoves the scientist to encode a physical phenomenon (such as electric field) as a mathematical entity. There are at least two sensible encodings:
a) "If I place a voltmeter at point P, and displace it slightly, how much will the voltmeter reading change?"
b) "If I release a (classical) electron at point P, and what would be its (classical) velocity a short time afterward?"

The first question naturally encodes E into a co-vector – an entity that maps a small displacement to a scalar (δV). The second question naturally encodes E as a vector, since velocity is a small displacement (given time δt). The electric field's reputation as "a co-vector" is merely an artefact of the formulation of electric field strength as the gradient of electrostatic potential. The preference for a) over b) is not a physical law – it is a matter of taste, arising from the elegance of the potential-formulation of electromagnetism.

3 Contravariant and Covariant Components

A bilinear map $\circ : V \times V \to \mathbb{R}$, is called a **bilinear form**; its action on basis $\{e_i\}_{i=1}^n$ forms a matrix $g_{ij} := e_i \circ e_j$. It is an **inner product** if g is symmetric and positive definite (all eigenvalues are strictly-positive reals). In choosing $\{e_i\}_{i=1}^n$, on a vector space, we pick one possible isomorphism between V and V^* - namely, to send a vector e_i to the co-vector, ε^i, that returns "1" when it "eats" e_i, and "0" for any other basis vector (i.e. $\varepsilon^i \circ e_j = \delta^i_j$), extending this map by linearity (we call $\{\varepsilon^i\}_{i=1}^n$ the **dual basis** to $\{e_i\}_{i=1}^n$). We can explicitly construct this map $\varepsilon^i : V \to \mathbb{R}$ using an inner product. Specifically, $\varepsilon^i(\cdot)$ is simply $e^i \circ \cdot$, where the vector e^i "equivalent" to co-vector ε^i is a linear combination of $\{e_k\}_{k=1}^n$, whose coefficients are precisely the i-th row of g^{-1} (confirm that, indeed, $e^i \circ e_j = \delta^i_j$).

For this reason (assuming an inner product is defined on V), physicists grant themselves artistic license to conflate a co-vector ε^i with its "equivalent" vector e^i - this is artistically called the **musical isomorphism**. Namely, the **flat isomorphism** replaces vector e^i with co-vector ε^i (written $(e^i)^\flat = \varepsilon^i$); and the opposite direction we have the **sharp isomorphism** ($(\varepsilon^i)^\sharp = e^i$). Physicists neglect to write these sharps and flats (deep down inside, they are all Jazz musicians). Under this isomorphism, we ostensibly claim a co-vector to be "the same thing" as some linear combination of $\{e^i\}_{i=1}^n$. For example $\varepsilon = \sum_i v_i e^i$ "is" a co-vector that "eats" arbitrary vector $w = \sum_j W^j e_j$ thus:

$$\varepsilon(w) := \sum_i v_i(e^i \circ w) = \sum_i v_i(e^i \circ \sum_j W^j e_j) = \sum_{ij} v_i W^j (e^i \circ e_j) = \sum_i v_i W^i$$

A manifold with a smoothly-varying g on each tangent space (called a **metric**), is called a **Riemannian manifold**. For example, the local notion of distance on the earth's surface produces a meaningful metric - define the inner product of a tangent vector with itself as the square of the physical distance travelled along the small portion of the earth's surface. Most manifolds you encounter in physics come with some metric representing some analogue of "distance".

Define a **chart**, on a manifold M as an injective map $\phi : U \to \mathbb{R}^n$, that depicts U (a neighbourhood of M) as some neighbourhood in Euclidean space. Some authors (a minority) define ϕ in the opposite direction, sending Euclidean space to the manifold. Either way, a chart endows a neighbourhood U of M with an unambiguous coordinate system - longitude and latitude on the earth's surface being the prime example. We can define scalar fields, and delineate regions using these coordinates. As a matter of taste, we describe vectors on a manifold using a "chart compatible" basis, namely the **covariant basis** to a given chart (ϕ, U): define the vector $e_i \in T_pM$ (the i-th basis vector), as the "small arrow" whose tail lies at $p = \phi^{-1}(x^1, ...x^n)$, and tip lies at $\phi^{-1}(x^1, ...x^i + \epsilon, ...x^n)$, where $\{x^i\}_{i=1}^n$ are chart coordinates. We now have tools to properly describe scalar, vector and co-vector fields on a Riemannian manifold. Vector fields are linear combination of basis vectors $\{e_i\}_{i=1}^n$ (as a function of $\{x^i\}$), whereas co-vector fields are linear combinations of $\{e^i\}_{i=1}^n$, (with e^i defined above).

The terminology, "covariant", refers to the "co-varying" transformation behaviour under a coordinate change. Specifically, if a second chart, ϕ', spaced out coordinate grid markings on M to be twice as far apart, the e_i' basis vectors would be twice as long as e_i. To describe the same vector on such a manifold, the components in front of the basis vectors would be half as large - the components would vary **contravariantly** ("oppositely-varying") with respect to a change in chart. Furthermore, we defined $\{e^i\}_{i=1}^n$ to satisfy $e^i \circ e_j = \delta^i_j$. Since the Kronecker delta and inner product are both independent of chart choice, it follows that $\{e^i\}_{i=1}^n$ must be contravariant too - more precisely, suppose linear transformation $[A]$ converts basis vectors in chart (ϕ, U) to the basis vectors in chart (ϕ', U') via $e_j' = \sum_k [A]^k_j e_k$, then the corresponding dual basis elements become $\sum_k e^{i\prime} = [A^{-1}]^i_k e^k$. The aforementioned identification, $e^i \leftrightarrow \varepsilon^i$ turns our contravariant set of basis vectors into a basis comprising of co-vectors. Co-vectors are conveniently written with covariant components and contravariant basis elements.

Throughout this article, I have adopted the standard convention that contravariant indices are written "upstairs", and covariant indices go "downstairs". Physicists often abbreviate a vector to a list of components with "upstairs" indices. It is equally correct to express the same vector as a linear combination of contravariant basis vectors ($\{e^i\}_{i=1}^n$) - each of which is a linear combination of $\{e_i\}_{i=1}^n$ vectors. When the index is "lowered" in this way, the numbers are different (indeed it amounts to multiplication by matrix g_{ij}), but crucially, the **numbers represent the same mathematical entity**. Similarly, a covector (a linear combination of $\{e^i\}_{i=1}^n$ vectors) can be expressed in terms of $\{e_i\}_{i=1}^n$ vectors, thereby "lifting" the index. The "lifted" components are found by multiplying by matrix $(g^{-1})^{ij}$.

Differential geometry is a beautiful topic - and also a core component of any physics degree. I hope that this article gives undergraduate students some helpful insight into the machinery that occurs "behind the scenes" when we talk about vectors and co-vectors

Steven Xu

An Introduction to Homotopy Theory - Blake Sims

1 Introduction

This article will introduce the ideas behind a fundamental tool used in algebraic topology, that of homotopy and homotopy groups. Topology is the study of the properties of spaces which remain unchanged under continuous deformation or morphing. For instance, what are the properties shared by both the surface of the Earth and a balloon, or a coffee mug and the surface of a doughnut? (I promise there are some!)

In topology one is naturally interested in questions of the sort: what are all the possible surfaces one can have. This question seeks to classify all the possible spaces up to topological equivalence, meaning if two spaces have all the same topological properties then they are the same. In general, it can be difficult to work out if two spaces are the same as one another, so the idea is to associate algebraic objects to surfaces in a particular way which can help to answer this question.

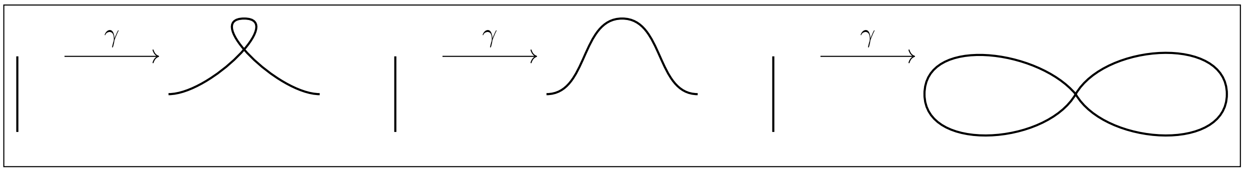
Given two spaces, if we can detect a hole in one, but not the other, then they are fundamentally and topologically different. This article will seek to give an understanding of how one might go about finding holes in a space. To find a hole in a space, we will map loops into the space and see if we can pull these loops tight to a point. If we cannot, then there must be a hole somewhere in the space (we don't really care where it is, just that it is there). Before we can formalize this 'pulling tight' of loops we will need to talk about pulling or morphing one path into another path.

2 How to morph one path into another path

Although the intuitive idea of a path is clear, we will have to make it mathematically precise so that we can talk about morphing one path into another.

A **path** in the Euclidean plane is a continuous function that takes real numbers in the interval between 0 and 1 into the Euclidean plane, which we denote $\mathbb{R}^2$ (loosely speaking the word continuous here means we can draw the path without lifting our pen from the page). More precisely, a path in the Euclidean plane is a continuous function $\gamma : I \to \mathbb{R}^2$ (γ is the Greek letter pronounced *gamma*), where $I = [0, 1]$ is the interval containing all the real numbers between 0 and 1, so that γ takes a number $t \in [0, 1]$ to a point in the plane.

A useful way to think about a path is as a *mapping* of the unit interval (like a piece of string of length 1) into the plane. It can take the string and stretch and curve it as much as it likes; as if the string is made of a stretchy kind of rubber. Some pictorial examples include the following:



Another example is a constant path, which starts at one point and stays there (this is the most boring path, but is very important). To define this exactly, we can pick a point in the plane $(x, y) \in \mathbb{R}^2$ and we will denote the constant path $c_{(x,y)} : I \to \mathbb{R}^2$ which is defined by $c_{(x,y)}(t) = (x, y)$ for all $t \in I$ (we use c for constant and put (x, y) as a subscript to remind ourselves of where the path is in the plane).

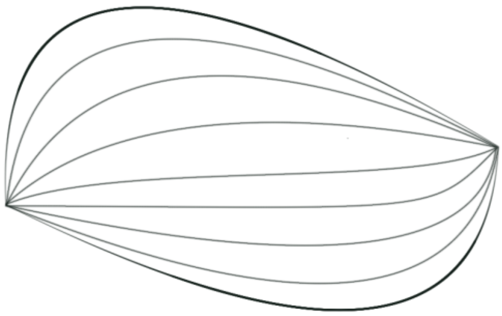
Another example is the path that traces out a circle. This can be defined in different ways, one such way is

$$I \ni t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix} \in \mathbb{R}^2.$$

So the path starts at the point $(x, y) = (\cos(0), \sin(0)) = (1, 0)$ and ends at $(x, y) = (\cos(2\pi), \sin(2\pi)) = (1, 0)$, and half way at $t = 1/2$ the path is at $(x, y) = (\cos(\pi), \sin(\pi)) = (-1, 0)$. All the other values will give us the unit circle. Notice that this path starts and ends at the same point. These form an important type of path called a **loop**.

Now that we have a solid mathematical formalism for a path we will now introduce the concept of morphing one path into another path. We will only be interested in paths which have the same beginning and end points in space. Imagine we have two such paths, as seen in the thick lower and upper paths in the figure below, then we want to see if we can morph the lower one into the upper one. To make this more formal we will use a parameter t, between 0 and 1, to keep track of which path we have, that is, for each value of t between 0 and 1 we will have a path which is a morphed version of the two original paths. In some more detail, when $t = 0$ we will have the lower path in the figure below, and when $t = 1$ we will be finished with the upper path in the figure below. For the values of t in-between 0 and 1 we have a path which is somewhat distorted from both the paths of interest. For instance, for the value of $t = 1/2$, we should see a path which is half-way between the two paths we are interested in. Several other values of t are shown in the figure below.

Such a morphing of one path into another is called a **homotopy of two paths**. If we can find a homotopy which starts at one path γ_0 and ends at another path γ_1, then we say that γ_0 is homotopic to γ_1, or that the two paths are **homotopic** (roughly translates from Greek to same or similar place). A natural mathematical question we might want to ask is if every pair of paths that have the same beginning and end points are homotopic. What do you think? Given any two paths with the same beginning and end, can we continuously morph one into the other?



To answer this, we will need to formalize the notion of a homotopy of paths. Recall we said we would use the parameter $t \in [0,1]$ to denote how far along the homotopy we are, so that for each such t we have a path which is somewhere between the two paths we started with. This means, if someone gives us a particular number t between 0 and 1 then the homotopy should give us a deformed path. Let's give this path a symbol to refer to it, say H_t (H for homotopy and t tells us which path we have). Since this is a path then it is a continuous function that takes another real number $s \in [0,1]$ to a point in the plane, i.e. $H_t : I \to \mathbb{R}^2$ is a mapping of the interval into the plane. So there are two steps; the first parameter t tells us which path we have, this step is $t \mapsto H_t$, and then we have a path which takes a number between 0 and 1 to a point in space, this step is $s \mapsto H_t(s)$ which is a point in space.

Since there are two inputs, t and s, then there are two unit intervals (so instead of mapping from just the unit *interval*, we are mapping from the unit *square*). This means the homotopy is a continuous function $H : I \times I \to \mathbb{R}^2$ which maps the square into the plane. The first 'copy' of I will tell us which curve we have, for instance when the first parameter is 0 then we have the first path, and when it is 1 we have the second path. These are denoted by $H_0(s) = H(0,s)$ and $H_1(s) = H(1,s)$, respectively.

Say we have two paths, γ_0 and γ_1 that both start at the point p and end at the point q (see the right-hand side of Figure 1 for a picture of these). Then, if we want the movie to show a morphing of γ_0 into γ_1 then the homotopy should be such that $H_0 = \gamma_0$ and $H_1 = \gamma_1$. We also need the technical condition that the homotopy keeps the end points fixed, so we need $H(t,0) = \gamma_0(0) = \gamma_1(0) = p$ and $H(t,1) = \gamma_0(1) = \gamma_1(1) = q$. If all of these symbols are a bit much, then you might wish to consider the following picture instead (it has all the same information in it!)

Now that we have a mathematically precise idea of what it means to morph one path into another path, we can return to the question posed earlier: are any two paths with the same beginning and end points in the Euclidean plane homotopic?

Let us start with the two loops we had earlier: the constant loop and the circle loop. Can you imagine a morphing of the circle loop into the constant loop? This amounts to asking if you could take the circle and shrink it down to a point.

It is easy to visualize, now let's make sure we can write it mathematically. So we want to say that the loop $\gamma : I \to \mathbb{R}^2$ defined $s \mapsto (\cos(2\pi s), \sin(2\pi s))$ is homotopic to the constant loop $c_{(1,0)} : I \to \mathbb{R}^2$ which takes all s to the point $(1,0)$, i.e. $s \mapsto (1,0)$ for all s. Recall, to say these are homotopic means to say there is a continuous function $H : I \times I \to \mathbb{R}^2$ which starts at the circle loop γ and ends at the constant loop that stays at $(1,0)$. In symbols this means $H(0,s) = \gamma(s)$ and $H(1,s) = c_{(1,0)}(s) = (1,0)$ for all s. As always, we want the technical condition that the beginning and end points remain fixed throughout the movie, so $H(t,0) = H(t,1) = (1,0)$ for all t.

Now we need to do the hard part, which is coming up with a formula telling one what to do when one has been given two inputs, $(t,s) \in I \times I$. We need $H(t,s)$ to be a point in the plane so that the start of the homotopy $H(0,s)$ is $\gamma(s)$, tracing out the circle as we vary s, and the end $H(1,s)$ is the constant loop $c_{(1,0)}(s) = (1,0)$. The required

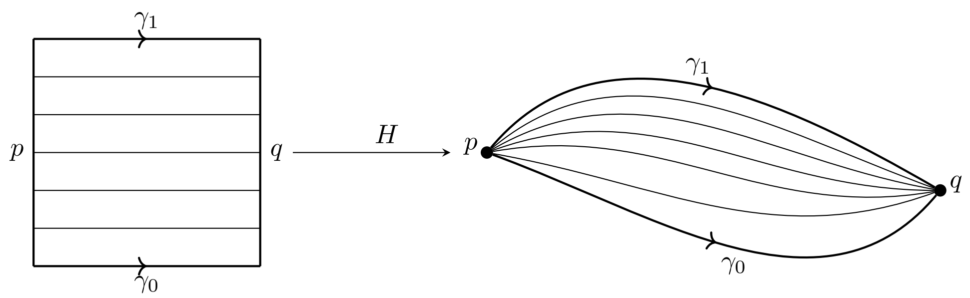


Figure 1: A diagram of a homotopy $H : I \times I \to \mathbb{R}^2$ which starts at the bottom of the square $I \times I$ with γ_0, the first path, and ends at the top of the square with γ_1. Several other values of t have been marked in the square and the resulting morphed path is shown on the right.

formula is

$$H(t, s) = (1 - t)\gamma(s) + tc_{(1,0)}(s).$$

Let's check this works correctly: it starts at $H(0, s) = (1 - 0)\gamma(s) = \gamma(s)$, so this is tracing out the circle, and ends at $H(1, s) = (1 - 1)\gamma(s) = 0\gamma(s) + c_{(1,0)}(s) = (1, 0)$ as required. To see what the homotopy is doing at the times in-between consider Figure 2

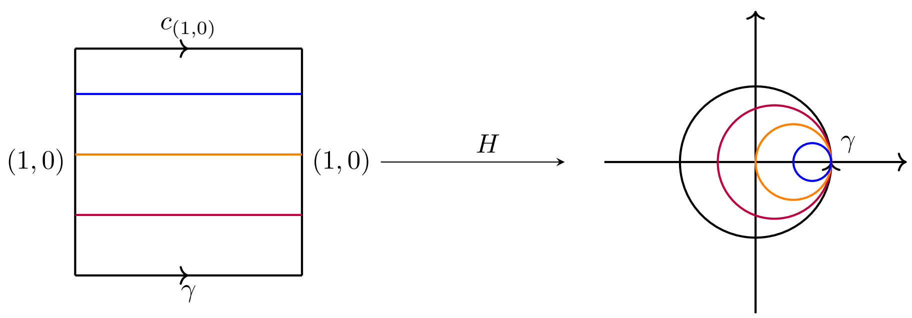


Figure 2: A diagram of the homotopy defined above showing the unit circle γ morphing/shrinking into the constant loop. The homotopy starts by mapping the bottom of the square to the unit circle loop, and then continuously morphs this loop into the constant loop by shrinking it. The loops $H_{1/4}$, $H_{1/2}$ and $H_{3/4}$ are shown in red, orange and blue, respectively.

We have shown that the circle loop is homotopic to the constant loop, and actually we can use the same method to show that any two paths are homotopic. Lets say we have two paths γ_0 and γ_1 (again with the same starting and ending points as each other), then we want to morph γ_0 into γ_1, that is, we want a continuous function $H : I \times I \to \mathbb{R}^2$ such that $H(0, s) = \gamma_0(s)$ and $H(1, s) = \gamma_1(s)$. Since there are no holes in the Euclidean plane, then intuitively, what we can do is take the straight line from each point on γ_0 directly to the corresponding point on γ_1. This is called the *straight line homotopy* and has the formula $H(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s)$, it is called the straight line homotopy because we can draw a straight line from all the points on the first path to all the points on the second path. One can check that the homotopy starts at $H(0,s) = (1-0)\gamma_0(s) + 0\gamma_1(s) = \gamma_0(s)$, and ends at $H(1,s) = (1-1)\gamma_0(s) + 1\gamma_1(s) = \gamma_1(s)$, as we wanted.

Hence, we have shown that any two paths in the Euclidean plane are homotopic. So now the question is: what kind of space can we imagine where we don't have this property? What stops one path from being morphed into another path? What we need is a space that is fundamentally different (in some way) to the ordinary Euclidean plane we have been using.

It doesn't take much to fundamentally change a space, sometimes just a single point. Let's return to our earlier example of the two homotopic loops, namely the circle loop and the constant loop. What would stop us from pulling the circle loop tight to a point?

3 Holes in a space

If we are plotting a trip to the other side of the galaxy, and our trip includes passing a black hole, we would need to consider completely avoiding that black hole (at least if we want to avoid spaghettification). This means, at least according to our navigation software, that the coordinates of the black hole in 3-D space do not exist to us; we cannot pass through that point. Returning to our flat 2-D Euclidean plane, we can imagine a 2-D black hole: a point where residents in the Euclidean plane simply do not have access to. This would be the same as the Euclidean space but without the origin, in set notation this is $\mathbb{R}^2 - \{(0,0)\}$.

Now we can return to our homotopy that we had earlier. In our new space this homotopy would no longer work because we cannot pull the circle over the origin since it no longer exists as a point we can use to get from one loop to another. Then, it seems as if we have two loops which are not homotopic. Unfortunately, the technical details for showing they truly are not homotopic are a bit out of our reach (the tools required are sitting above us in a space called the covering space which we wont be able to reach from here). So for now, you will have to take my word that they are not homotopic: we cannot morph the circle loop into the constant loop when we are living in the Euclidean plane without the origin, $\mathbb{R}^2 - \{(0,0)\}$.

So far we have two different loops that are not homotopic. Can you think of any other loops starting and ending at $(1,0)$ which are also not homotopic to either the constant loop or the circle loop?

If we went around the hole in the origin twice and then returned to our starting point, then we would obtain a new loop which is also not homotopic to any of the other loops so far. This is shown as the loop γ_0 in Figure 3 below. In fact, we could go around the origin as many times as we like. This means, given any positive integer n we can draw a loop that goes around the hole in the origin n times in an anti-clockwise direction. Lastly, what about going around the origin in a clock-wise direction? These are also not homotopic to the other loops we have so far.

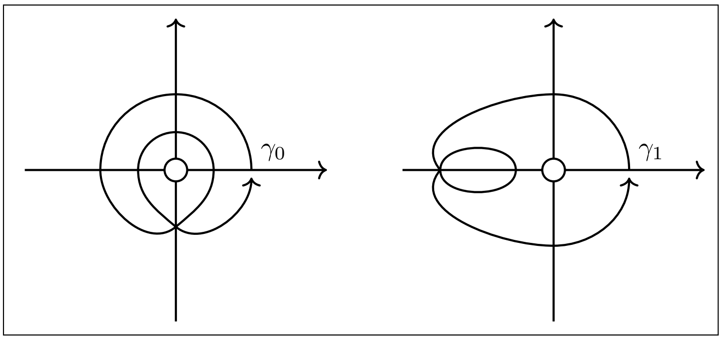


Figure 3: In our new space, the Euclidean plane without the origin, we have two loops γ_0 and γ_1. The loop γ_0 is not homotopic to either the circle loop nor the constant loop, since the hole at the origin prevents us from pulling it tight. However, the loop γ_1 is homotopic to the circle loop that goes around the origin once. Can you see why?

This means that given any integer number, we can associate a unique homotopy class of loops (a collection of loops which are homotopic to one another). More formally, we can associate the integers, denoted $\mathbb{Z}$, with the space obtained by removing the origin from the Euclidean plane, $\mathbb{R}^2 - \{(0,0)\}$. When we do this association we say that the space $\mathbb{R}^2 - \{(0,0)\}$ has **fundamental group** equal to the integers. Another classical question one might ask at this stage: is there another space out there where one can associate the integers in the same way? That is, what other spaces also have fundamental group the group of integers? The answer is the circle. If you think about how one might send an interval into a circle, we can either just wrap the interval around once, twice, three times, etc., or once, twice, three times, etc. in the opposite direction, or of course, not at all. It turns out this is not a coincidence: these two spaces are actually homotopic in the sense that you can morph one into the other, analogous to morphing one path into another path. You should try and visualize this!

4 Conclusion

We started talking about how topology is the study of the properties of spaces which do not change when we deform the space continuously. One of these properties is how many holes the space has in it, and in order to find these holes we need a formal way to send loops into the space and try and pull them tight. This process of pulling one loop into another loop was called homotopy. We found that any loop we put into the standard Euclidean space was homotopic to a point, revealing there must not be any holes in it. Then we considered the Euclidean space but with the origin removed. All the different types of loops possible in this space (up to homotopy equivalence) gave us the integers. This process of finding holes generalizes to higher dimensions, so that one might find holes in surfaces that live in three dimensions, or higher dimensional holes (like a three dimensional hole - think of black holes) contained in higher dimensional spaces. In the three dimensional case, we are interested in mapping *squares* into space instead of *lines* like we were here.

Blake Sims

Prime Races - Daniel Johnston

The other week I had the pleasure of attending a talk by Greg Martin from the University of British Columbia. The talk was titled "Prime Races" and featured some fascinating results that are deeply related to one of the most famous unsolved problems in mathematics.

The $4n + 1$ vs. $4n + 3$ race

Every odd number can be expressed in the form $4n + 1$ or $4n + 3$, where n is some other integer. For instance, 11=4(2)+3 and 17=4(4)+1. Now, every prime number except for 2 is an odd number, so a reasonable question to ask is "how many prime numbers are there of the form $4n + 1$ and how many of the form $4n + 3$?"

A first approach to this question would be to simply list all of the primes up to some point and check which form they satisfy. We have done this in the table below for the first ten odd primes.

	$4n + 1$**?**	$4n + 3$**?**
Primes	5, 13, 17, 29	3, 7, 11, 19, 23, 31
Total	4	6

From this small sample of prime numbers it seems as if the distribution of primes is quite even with perhaps a slight bias towards primes of the form $4n + 3$. However, we fortunately live in the age of computers so why not experiment on a much larger set of data?

Figure 1 compares how many $4n + 1$ primes or $4n + 3$ primes there are less than some given number x. The bias towards the $4n + 3$ primes (shown in blue) now becomes very clear. In fact, for all the values of x in figure 1, there is no point in which there are more $4n + 1$ primes than $4n + 3$ primes.

Now, the first question one might ask is "are there ever more $4n + 1$ primes than $4n + 3$ primes?" This question does have a definitive answer. If we continued the plot in Figure 1 up to $x = 26,861$, we would find that the $4n + 1$ primes finally take over. However, the next prime at $x = 26,863$ is of the form $4n + 3$, so it is a very short-lived victory for the $4n + 1$ primes. In his talk, Martin claimed that the $4n + 3$ primes are in the lead about 99.59% of the time.

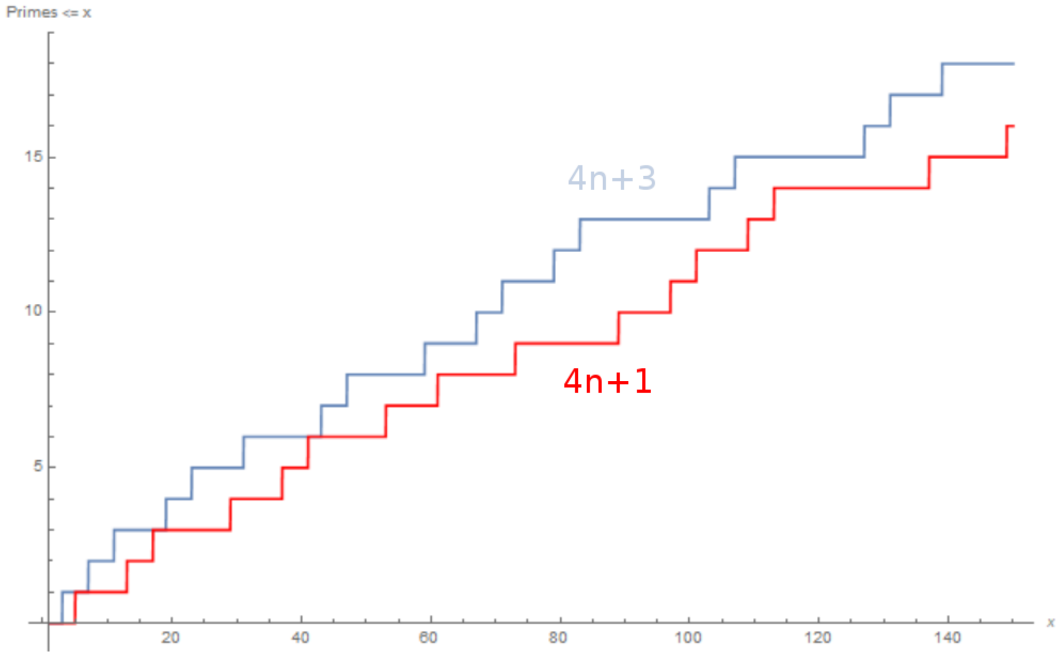


Figure 4: Number of 4n+1 and 4n+3 primes less than or equal to some integer x (for $0 \leq x \leq 150$)

An even more interesting question is "why does this bias occur?" For the answer to this question we have to explore the theory behind the famous Riemann Hypothesis.

The Riemann Hypothesis

The Riemann Hypothesis was first conjectured by Bernhard Riemann in 1859. It is a statement about a function $\zeta(s)$, called the Riemann zeta function. The conjecture is concerned with the values of s for which $\zeta(s) = 0$. Such s are called the zeros of the zeta function.

It turns out to not be too difficult to show that every negative even number is a zero of the zeta function. However, every other zero found has been a complex number of the form $s = 1/2 + ib$, for some real number b. The Riemann Hypothesis states that every zero (except for the negative even numbers) is of this form.

To this day, no one is sure as to whether the Riemann Hypothesis is actually true or not. Since the zeta function has proven to be so important in modern mathematics, there is currently a $1,000,000 prize on offer for anyone who can prove the Riemann Hypothesis. However, this problem is exceedingly difficult (it has been around for over 150 years) that many mathematicians joke that proving the Riemann Hypothesis is the hardest way to earn a million dollars.

It turns out that the Riemann Hypothesis is deeply related to the distribution of primes and thus our problem at hand. Martin showed that if we assume a generalised form of the Riemann Hypothesis (in which we look at the zeros of functions called "Dirichlet L-functions") we get a rather complicated looking formula:

$$\frac{\pi(x;4,3)-\pi(x;4,1)}{\sqrt{x}/\ln(x)} \sim 1+2 \sum_{\substack{\gamma>0 \\ L(1/2+iy,\chi)=0}} \left(\frac{\gamma\sin(\gamma\ln x)}{1/4+\gamma^2}+\frac{\cos(\gamma\ln x)}{1/2+2\gamma^2}\right)$$

where $\pi(x;4,3)$ and $\pi(x;4,1)$ are respectively the number of $4n+3$ and $4n+1$ primes less than some number x. Therefore, assuming one of the most difficult problems in maths we can get an expression for the difference between the number of $4n+3$ and $4n+1$ primes ... yay!

A simple reason for the bias

All of this zeta function stuff is quite complicated. Fortunately Martin was able to extract the exact reason for the $4n+3$ bias we saw earlier. He stated in an almost philosophical manner:

Humans count primes; Nature counts primes and their powers.

To see what this means, we will now look at odd numbers of the form $4n+1$ and $4n+3$ which are either primes or powers of primes such as 27 = 3^3 or 49=7^2.

	$4n+1$**?**	$4n+3$**?**
Odd primes (≤ 100**)**	5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97	3, 7, 11, 19, 23, 31, 43, 47, 59, 67, 71, 79, 83
Total	11	13
Odd prime powers (≤ 100**)**	9, 25, 49, 81	27
Total	4	1
Grand total	15	14

That is, if we include prime powers into our calculations then the race is closer and the $4n+3$ bias vanishes.

But why are so many of the prime powers of the form $4n+1$? Well, this actually has a simple answer. Most of the prime powers we are concerned with will be the squares of odd numbers. Now, the definition of an odd number m is any number of the form $m = 2k + 1$, for some integer k. Hence for any odd number we have that

$$m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1.$$

Letting $n = k^2 + k$, we get that $m^2 = 4n + 1$. That is, the square of an odd number is always of the form $4n + 1$!

So, although it is common to want to count prime numbers up to a certain point, the most balanced distributions occur when we consider prime powers. This is exactly what Martin meant by nature counting prime powers.

I found Martin's talk on prime races to be very engaging and there were many more amazing results and technicalities which I haven't mentioned here. For a bit of fun to finish off, here is a plot of a different race between primes of the form $3n + 1$ and $3n + 2$. By considering prime powers, can you also explain the bias in this race?

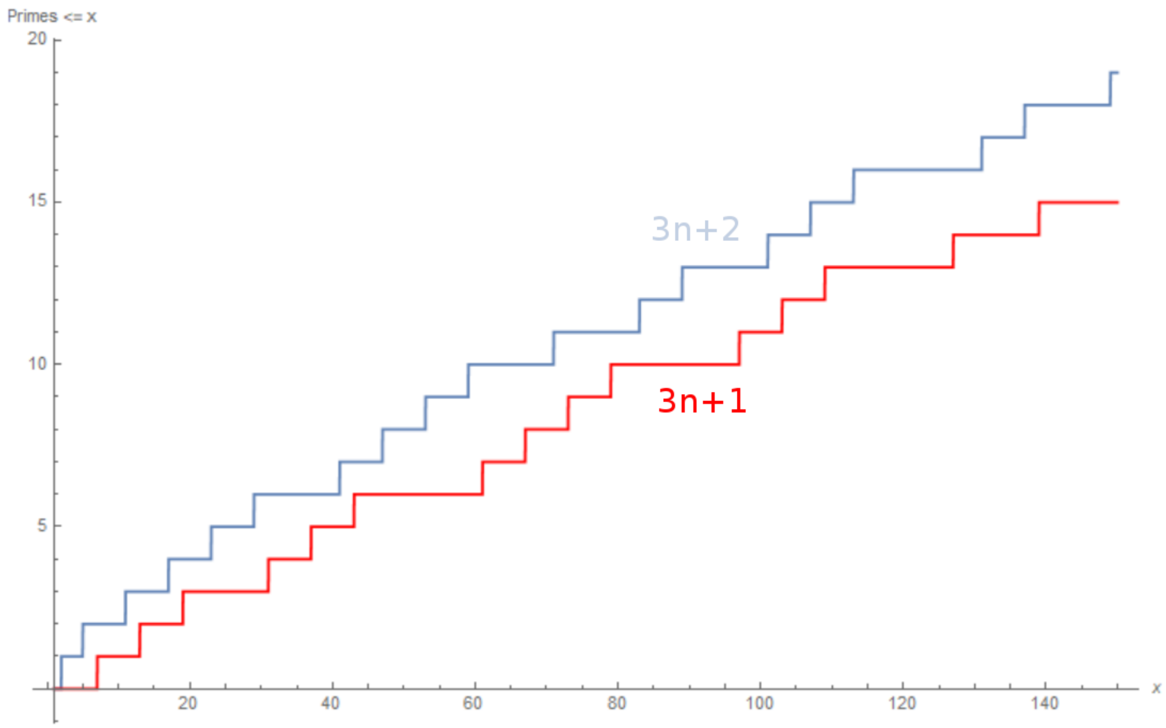


Figure 5: Number of 3n+1 and 3n+2 primes less than or equal to some integer x (for $0 \leq x \leq 150$)

Daniel Johnston

Things to do when you start your Math/Physics/Engineering degree

Disclaimer – the following advice is the opinion of Steven Xu

1 Decide on your major

Many students enter the B.Sc programme without a major in mind, intending to defer this decision until they have taken a good mix of first-year elective subjects from different disciplines. Keep your options open like this if you are truly uncertain, but I strongly advise you to collect as much information as possible, early on, to help you make a commitment sooner rather than later. Many crucial prerequisite subjects are offered only in the first or second semester each year. Once you know your major, you know which subjects you must complete in order to graduate on time. Deciding on a major at a late stage may limit your subject choices in third year.

When confronted with a bewildering menu at a fancy restaurant, one can regret a choice and later feel envy. Mitigate your risk by staying well informed about your options and subject matter you intend to study! Socialize with second and third-year students. Ask for copies of lecture notes. Attend higher-level lectures, if you have the opportunity. You may also like to Google keywords from the subject handbook, and browse through Wikipedia articles. There are plenty of fantastic educational Youtube videos (I highly recommend Fredrick Schuller's lectures if you might like Differential Geometry, Fibre Bundles, or Quantum Mechanics). Keep posted on Careers nights, and remember ask to industry representatives about the type of degree or major that is required for each job.

A major in the University of Melbourne B.Sc programme comprises four third-year (300-level) subjects. To "complete" the major requires that you complete these subjects and their prerequisites. Often there is choice involved – for example, the Applied Mathematics major comprises Complex Analysis, and Numerical and Scientific Computing, together with any two of three options, of which Methods of Mathematical Physics is one. By contrast the Mathematical Physics major requires Complex Analysis and Quantum Physics, and two other subjects, of which Methods of Mathematical Physics can be chosen. Taken together, there is potentially a two-subject overlap between the two majors (i.e. Complex Analysis and Methods of Mathematical Physics). If you are sitting on the fence, it may be wise to prioritise these overlapping subjects on your study plan.

2 Decide whether you want to do the concurrent diploma

The Melbourne Model was phased in at the beginning of this decade emphasising breadth of learning. Between 50 and 75 points of the standard 300-point undergraduate degree (B.Sc, B.A, B.Comm) are mandated to come from outside the respective departments. Additionally, 175 points of the B.Sc programme are mandated to be elective (i.e. within Science, but outside your intended major).

These constraints made it difficult to complete a B.Sc with a double major, except in cases where two intended majors have significant overlap. As such, the university has stopped formally acknowledging double-major degrees

– students who have completed all requirements for two majors may only have one printed on their certificate. In lieu of this, the University of Melbourne introduced the Concurrent diploma, which is awarded upon completion of all the requirements of a second major – thereby placing the student on equal footing with another student who completed this major. The diploma comprises 100 points, of which up to 50 points may be cross-credited with the main degree. Effectively, the double-major degree is now 350 points, split into two qualifications.

Departments are responsible for their own Concurrent Diploma programme. The Mathematics diploma allows any of the four mathematics majors – Applied, Pure, Statistics, and Discrete mathematics. Crucially, your main degree must not be within this set, nor Mathematical Physics or Data Science. Therefore, you may complete at most one of these six "math-related" majors as an undergraduate. If you are mathematically inclined, you might consider doing a Concurrent Diploma while completing an engineering major (especially electrical systems), a physics or chemistry major, or a B.Comm major. Carefully consider how these 100 points of mathematics subjects can overlap with your main degree – i.e. as elective subjects or breadth subjects. Make sure you select your other elective or breadth subjects to achieve the mandated spread of first, second and third-year subjects, to ensure that you can cross-credit the maximum 50 points. To create a study plan that meets the aforementioned quotas, and adheres to all the subject prerequisite rules, is an extremely difficult exercise (probably involving K-theory in some way – exercise left to the reader).

You must carefully weigh whether you actually want to do this. The HECS fees are inconsequential in the larger scheme of things (approximately $4.4k). The real consideration here are the six additional months of your life spent on the Parkville campus, which are deducted from your future life as a quantitative analyst, or chemical engineer. If you complete your bachelor degree with a single major – and it's one you love – you can either start gaining industry experience, or start a Masters degree six months sooner, when your mind is still young. Many students who complete the Concurrent Diploma en route to the Master of Science or Engineering programme end up underloading and stretching their undergraduate study to 4 years due limitations placed on midyear intake students – Physics Masters subjects are highly specialized and run either in semester 1 or semester 2, with the former being prerequisites to the latter. Luckily, the Math department is more flexible. If you are destined for postgraduate study in the Northern Hemisphere, mid-year graduation may not be so bad.

3 Put the Core subjects on your Study plan

As soon as you have decided on your major, or a putative list of potential majors, choose the third-year subjects from the list of available options. Read each subject description closely, and remember that "or" does not preclude you from completing both subjects, if you have room on your study plan. Stochastic Modelling has a very different flavour to Methods of Mathematical Physics, and aspiring Applied Mathematicians may want a taste of both worlds.

List the second year pre-requisites for each choice, and lock them in. Carefully check their prerequisites in turn, and make sure they are on your study plan. If not, act immediately! Once you have your major and core prerequisites written down on a timeline, recheck subject availabilities (i.e. Semester 1 vs. Semester 2) and the rules around pre-requisites, before adding them to your study plan. Take a screenshot of your study plan at this stage, for future reference.

You might need to ask a professor for a prerequisite waiver in the future. I have personally done it three times during this degree. You are more likely to get a green light if you have good marks on your academic record, and also if the waiver in question is a downgrade from a pre-requisite to a co-requisite.

4 Seriously think about your Elective and Breadth subjects

The Melbourne Model breath requirement has been controversial at times. You must play with the hand you are dealt.

For some students, breadth subjects serve as welcome interludes on their study plan, freeing up time for more "serious" subjects. World Music Choir has been a popular choice in this regard. As a classically-trained guitarist (and John Mayer's biggest fan!) I had seriously considered enrolling in Guitar Ensemble or Guitar Group, both of which are assessment-free (except some multiple choice tests on basic music theory).

For other students, breadth subjects are a way to gain experience in a serious subject that may serve them well in their future career. This may be a wiser expenditure of time and money. Finance subjects, and business law are common choices.

Elective (science) subjects should also be selected in this vein – aim for a healthy balance between broadening your horizons, and strengthening your core knowledge base. As a math or physics major, you should seriously consider the mechatronics or mechanical systems subjects, to hedge for the possibility that you might switch to engineering as a professional degree. Conversely, if you have already committed to engineering, a strong command of math and physics subjects can give you a competitive edge over your classmates. I strongly recommend all engineering majors to take Real Analysis (MAST20026) as an elective. The subject matter is intuitive, and students who think logically and construct sound mathematical arguments will be rewarded with a healthy boost to their WAM.

Given the sheer volume of Breadth and Elective options, there is availability to study over the summer or winter. By packing your core subjects efficiently onto your study plan, you can potentially overload and take summer/winter classes to complete your degree in 2.5 years, or possibly faster. For other students, the credits from summer/winter subjects enables a reduced course load during the Spring/Fall semesters (particularly if they are difficult subjects – e.g., Algebra, Metric and Hilbert, Electrodynamics).

5 Get Good Study Habits

Tertiary study is a very expensive endeavour for yourself, and the Australian taxpayer. It behoves you to make the most of this experience. Get into the pattern of studying at particular times of the day, schedule for it. Lectures are all recorded, but missed tutorials are effectively money lost down the drain. Tutorials are staffed by the very top Masters and PhD students in the department; they are paid top dollar to answer all of your questions, so you must make sure you come prepared with good questions to make the most of this opportunity. To have good questions, you must do all the exercises. So do all the exercises.

Having said this, you should never feel embarrassed to ask questions that might seem "basic" on the surface – your classmates may well have the same question, stemming from some oversight in lecture materials! Form friendships within your class, as talking through a problem can be helpful. Search YouTube and other online resources for a fresh perspective. Remember that the friendly community on Math Stack Exchange can help if you formulate your question in a clear, concise way (by contrast, Math Overflow is intended exclusively for professional mathematicians).

Richard Feynman famously advised that the best way to learn something is to work out how to explain it in the simplest possible way. I strongly advise that you follow this philosophy in structuring your notes. Create an online file storage account (Google Drive) before O-week finishes. This account will serve as reference material – available to you for the rest of your life, wherever you are in the world. Each concept that you encounter needs to be summarized in a document, and as you revisit topics in your later-year and Masters studies, you need to review and update these reference documents accordingly.

Start a repl.it account, and Overleaf account. Start getting used to programming in Python, and writing Latex code. These are crucial skills in every STEM field.

6 Get the Influenza Vaccine

The Baillieu Library is a breeding ground for influenza and upper-respiratory viruses. University assessment is a high-stakes situation, and it is unfair when students get sick during the semester - or worse, during Swotvac or exam period. Make sure you get the flu vaccine as soon as possible every Autumn. Avoid close contact with classmates when you are sick; please wear a face mask (covering nose and mouth) if you must enter a crowded library.

During stressful exam periods, immune systems are weakened. The University of Melbourne and Monash University student cohorts become two giant reservoirs of respiratory viruses.

If you are beginning your tertiary studies in 2019, I want to congratulate you for making it to a critical juncture in your life. For many students this is a chaotic and confusing time, but I hope that the advice here is helpful.

Steven Xu

Dame Mary Cartwright and the Importance of Grit - Vanessa Thompkins

Mary Cartwright was an English mathematician born in 1900. As a teenager, she intended to study history. It wasn't until her final year of high school that she began considering a future in mathematics. Recognising mathematics as a field where (unlike history) you could succeed without spending long hours memorising facts, she changed her plans, deciding to study maths.

Cartwright started at Oxford in 1919, as one woman of only five studying mathematics in the entire university. After two years of study at Oxford she suffered a setback, not achieving the results that she had hoped for. Cartwright seriously considered leaving maths and returning to her original plan of studying history.

Luckily for mathematics, she didn't. Instead, she persisted with her course, ultimately graduating with first class honours in 1923. Cartwright went on to complete her doctoral studies at Oxford. In 1961, she became the first female president of the London Mathematical Society (and the only female president until Frances Kirwan in 2003). She was also the first woman to receive the Sylvester Medal.

Mary Cartwright's story highlights the importance of grit: a personality trait associated with persisting in the face of difficulty, or continuing to try even when a task is boring or tough. Many students beginning their university journey struggle substantially, when faced with the increased workload and difficulty of university courses compared to high school. However, recent psychological research tells us that keeping on trying at things you want to achieve, even when faced with failure, can get you through. Learning better study habits is really difficult. Reattempting failed courses, and sticking at a proof that you just can't seem to figure out, can feel impossible. However, developing the ability to keep going, in spite of setbacks and feeling like you're not good enough, will stand you in good stead.

Mary Cartwright was one of the first mathematicians to research what would later be known as chaos theory. We can be thankful that she didn't make the decision, back in 1921, to leave mathematics. Reminiscent of the butterfly effect that she studied, things would look very different now if she had.

Her story tells us that it's okay to feel like you're not good enough, and as though what you're struggling to achieve comes easily to others. You don't have to be a mathematical genius to succeed: sometimes working hard and persisting is enough.

Note: it's okay to ask for help! If you're struggling with your workload, your university experience in general, or difficulties outside of university, you can contact the University of Melbourne counselling service on 8344 6927.

Vanessa Thompkins

Sophie Germain: A Heroine of Number Theory - Daniel Johnston

On the 1st of April 1776 in Paris, the highly talented and dedicated mathematician Sophie Germain was born. Modern biographers describe her as being filled with "limitless passion and devotion". I find the life and work of Sophie Germain truly inspiring and it is an absolute shame that so few people around the maths department have heard of her. Reading about her life and passion for number theory as a teenager had a great impact on me and is one of the reasons I now study mathematics and research number theory as a graduate student.

Early Life

Sophie Germain was born into a wealthy family, her father being a successful silk merchant. Despite this, she was still a middle class citizen and at the time it was completely unheard of for any woman outside of the aristocracy to study mathematics. Even for women in the upper class, mathematics was uncommon and some of the literature aimed at teaching women science and mathematics in the 18th century is absurd and almost humorous. For instance, in Algarotti's 1739 textbook *Sir Isaac Newton's Philosophy Explain'd for the Use of Ladies* the basic principles of physics are described via overly romantic dialogue. One such quote is "I cannot help thinking ... that this proportion in the squares of the distances of places ... is observed even in love. Thus after eight days absence love becomes sixty four times less than it was on the first day, and according to this proportion it must soon be entirely obliterated".

Germain however, struck upon some unusual inspiration to study mathematics. In 1789, the Bastille fell in Paris and 13 year old Germain was forced to stay safe indoors. Upon perusing her father's library, she came across Montucla's *L'Histoire des Mathématiques*. Within this book, the story of Archimedes' death particularly took Germain's interest. It is said that around 212 BC, after the city of Syracuse had been sieged, a Roman soldier found Archimedes contemplating a geometric problem. Being too engrossed in the problem, Archimedes refused to cooperate with the soldier and was killed on the spot. The last words attributed to Archimedes are "Noli turbare circulos meos" or "Do not disturb my circles". After reading this story, a desire to study mathematics awoke in Germain. If mathematics could captivate Archimedes to such an extent then she thought it must be an amazing field of study with the power to immerse an individual in its beauty.

As a consequence, Germain began reading mathematical literature during the late hours of the night. She even taught herself Latin and Greek so that she could understand the works of great mathematicians such as Newton and Euler. Her parents at the time were very troubled with her newfound interest, viewing it to be unfit for a woman to involve herself in such academic matters. It is alleged that her parents would deny her warm clothes and lighting to prevent her from studying at night. Despite this, Germain kept a secret stash of candles and would continue to deprive herself of warmth and sleep to be able to study mathematics without being caught. After realising that they could not control her passion, Germain's parents eventually came around to supporting her. In fact, Germain never married and was financially supported by her father throughout her life, allowing her to be very productive in her work.

Because of her gender, one of the most unfortunate problems with Germain's studies was that she could never receive any formal tuition. In 1794, when Germain was 18, the École Polytechnique opened. Being unable to attend, Germain assumed the pseudonym M. LeBlanc, which was the name of a former student of the university. Unaware that LeBlanc was no longer enrolled, the academy continued to print lecture notes and problems which were retrieved and worked on by Germain. After some time, the course supervisor Joseph-Louis Lagrange, considered to be one of the greatest mathematicians of all time, couldn't help but notice the brilliance of M. LeBlanc's solutions to the problem sheets and asked for a meeting with the old student. Unable to hide her identity anymore, Germain revealed herself to Lagrange. To her surprise, Lagrange supported Germain's studies and gave her guidance in continuing her work.

Mathematical career

As a young adult, Germain developed a great interest in number theory inspired by the work of Legendre and Gauss. In 1804, after analysing and expanding on Gauss' work in great detail she began to write letters detailing her mathematical discoveries to Gauss himself, again using the pseudonym M. LeBlanc. Although Gauss didn't often respond to such letters, he was very impressed by her work and would often give her great praise as evident in his letters to other colleagues.

In 1806, during the Napoleonic wars, the French occupied Gauss' hometown of Braunschweig. Fearing that Gauss would share the same fate as Archimedes, Germain wrote to General Pernety, a family friend, asking to personally ensure Gauss' safety. Pernety agreed with the request and after telling Gauss that he was sent by a woman named Sophie Germain, Gauss was greatly confused. Being unable to hide her secret any longer, Germain revealed her true identity to Gauss. Upon learning the truth, Gauss was even more impressed by her work stating that for a woman to overcome all obstacles and study high level mathematics must require "the noblest courage, quite extraordinary talents and superior genius".

Despite the fruitful collaboration between Germain and Gauss, in 1809 Gauss was appointed Professor of Astronomy at the University of Göttingen and ceased communication with Germain. As a result, Germain briefly left number theory and turned her interest towards a challenge set by the Paris Academy of Sciences. The goal was to "*formulate a mathematical theory of elastic surfaces and indicate just how it agrees with empirical evidence*". After many years of hard work and multiple attempts, Germain finally submitted a successful paper in 1816. Although the paper was far from perfect, it paved the way for many more modern theories of elasticity. Despite her success in solving the problem, she still felt a lack of respect from the scientific community and did not appear at the award ceremony.

In 1815, the Academy of Sciences posed another challenge asking for a proof of Fermat's last theorem. This challenge rekindled Germain's passion for number theory and as a consequence she started writing letters to Gauss again. Fermat's last theorem is considered to be one of the most significant problems in all of mathematics and Germain arguably made one of the most important contributions to solving the problem until it was finally proved completely in 1995.

Fermat's Last Theorem

Fermat's Last Theorem is given by the simple claim that

$$a^n + b^n = c^n$$

has no solutions for positive integers a, b, c and n with $n > 2$. We require $n > 2$ since if $n = 1$ or $n = 2$ then there are an abundance of solutions such as $3^2 + 4^2 = 5^2$ which many may recognise from their high school studies of Pythagoras' theorem.

Prior to the work of Germain, the theorem was only shown to be true for $n = 3$ and $n = 4$. The problem is renowned for inspiring new mathematics, with Euler's attempt at the $n = 3$ case being one of the first times complex numbers were ever used seriously in mathematics.

One may notice that we are most interested in the case where n is a prime number. For instance, if we know that the theorem holds for $n = 3$ (a prime), then it also holds for any multiple of 3. To see this, consider the case when $n = 15$. We are looking for solutions to

$$a^{15} + b^{15} = c^{15}$$

which can be re-expressed as

$$(a^5)^3 + (b^5)^3 = (c^5)^3.$$

But we know this has no solutions since a^5, b^5 and c^5 are positive integers and we are already assuming that there are no solutions for $n = 3$.

Germain was interested in the special case of the problem when n does not divide a, b or c. In this situation, she was able to show that if n was an odd prime such that $2n + 1$ is also prime, then no solutions to $a^n + b^n = c^n$ exist.

Primes satisfying this property are now known as "Sophie Germain primes". One such example is the prime $n = 5$ since $2n+1 = 2(5)+1 = 11$ is also prime. Using similar methods Germain was able to similarly show that Fermat's Last Theorem (in the case where n does not divide a, b or c) holds when n is an odd prime with $n < 100$.

This was the first ever "systematic" way to start analysing Fermat's Last Theorem for different values of n and influenced future progress towards a solution.

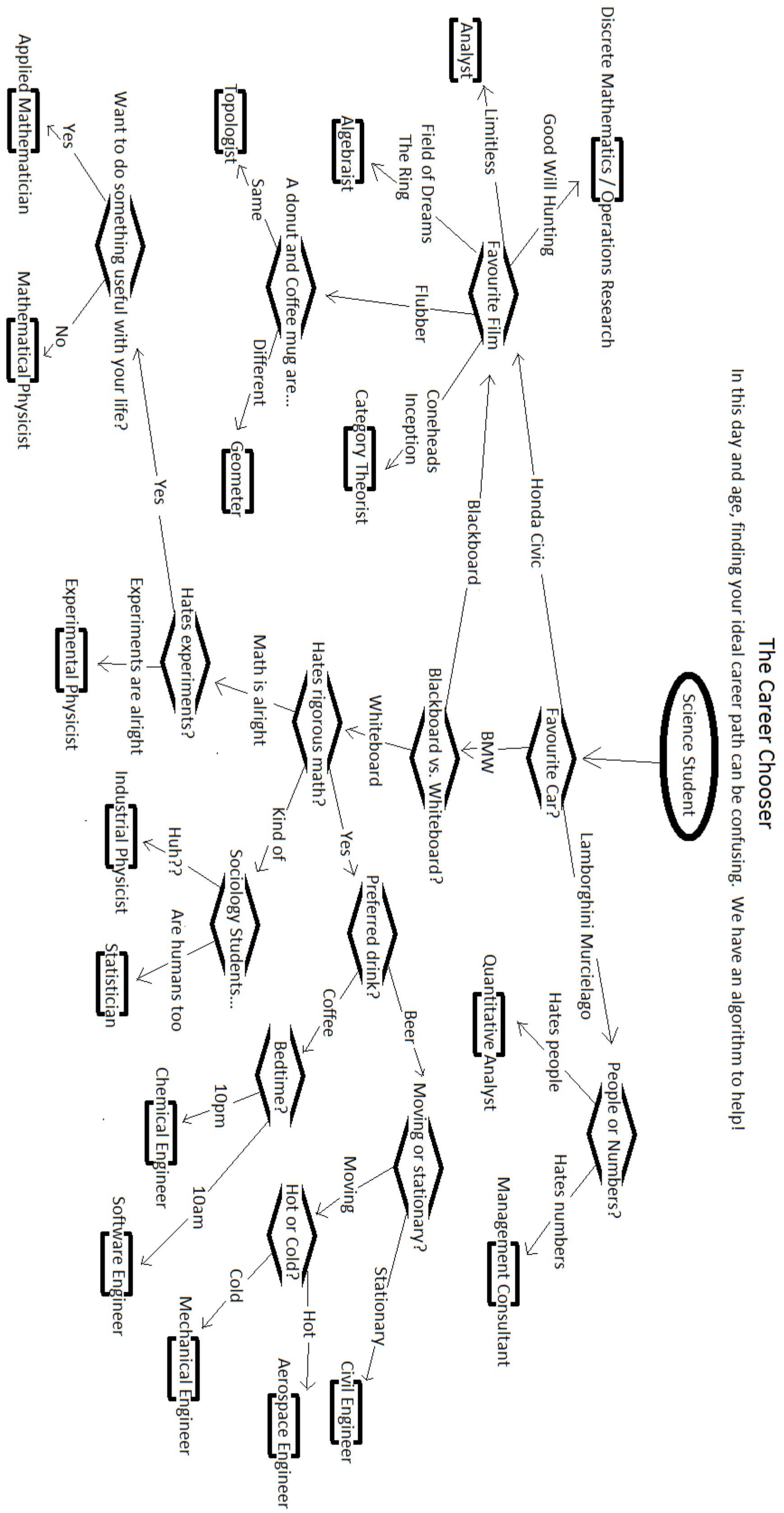
Later years

Despite the ongoing prejudice against Germain throughout her life, several renowned mathematicians came to respect her and in 1823 she was notably the first woman, who was not a wife of a member, to attend lectures at the Academy of Sciences.

Unfortunately, in 1829 Germain developed breast cancer. However, with a great deal of determination she eventually released an influential paper on the curvature of elastic surfaces in 1831 before dying later that year. In 1837, under pressure from Gauss, she was awarded an honorary degree from the University of Göttingen. Despite her huge success and influence as a mathematician her death certificate only describes her as a *rentière* which means "property owner" and implies that she was a single woman with no profession.

The life of Sophie Germain was truly remarkable, highlighting how a high level of dedication and passion can allow one to thrive and push against any societal barriers imposed on them. It is truly unfortunate that Germain's talent for mathematics wasn't fully respected or nutured in her time and I hope that her story can spark an interest in many other potential mathematicians and also emphasise the importance of inclusivity in academia.

Daniel Johnston


The Career Chooser
In this day and age, finding your ideal career path can be confusing. We have an algorithm to help!
Science Student
Favourite Car?
Honda Civic
BMW
Lamborghini Murcielago
Favourite Film
Good Will Hunting
Discrete Mathematics / Operations Research
Limitless
Analyst
Field of Dreams
The Ring
Algebraist
Flubber
Coneheads
Inception
Category Theorist
Blackboard
Blackboard vs. Whiteboard?
Whiteboard
People or Numbers?
Hates people
Quantitative Analyst
Hates numbers
Management Consultant
Hates rigorous math?
Math is alright
Hates experiments?
Yes
Experiments are alright
Experimental Physicist
A donut and Coffee mug are...
Same
Topologist
Different
Geometer
Want to do something useful with your life?
Yes
Applied Mathematician
No
Mathematical Physicist
Kind of
Sociology Students...
Huh??
Industrial Physicist
Are humans too
Statistician
Yes
Preferred drink?
Coffee
Bedtime?
10pm
Chemical Engineer
10am
Software Engineer
Beer
Moving or stationary?
Moving
Hot or Cold?
Hot
Aerospace Engineer
Cold
Mechanical Engineer
Stationary
Civil Engineer

Tag Yourself – Algebra Edition (Full credit goes to anonymous contributor from Mathematical Mathematics Memes)

Magma
- Kind of dumb
- Unsure of own identity
- Omnipresent
- Jack of all trades. Flexible

Monoid
- Straight-edged, and decisive
- Constantly flexing on Group
- Never in the same place at same time as Monad
- Friends with Ring

Group
- Constantly goes back on plans
- Loves Symmetry
- Rearranges the house furniture all the time
- "Nerdy Jock"
- Friends with Ring
- Normally looks composed

Abelian
- Tries to be the centre of attention
- Master of remaking plans
- Only ever got good at one thing
- Falls apart in many situations
- The strongest, if you can get him to engage at all

Ring
- Idealistic
- Underappreciated
- Has lots of problems with no solution
- Least popular – but somehow the mascot
- Friends with Monoid and Group

Field
- Analytic, open book
- Has his shit together
- Dating Vector Space
- Best friends with Abelian
- Commutes to work every day

Monad
- Loves endofunctors
- Suspiciously similar voice and mannerism to Monoid
- Hangs out with Category
- Talks about how cool Monoid is
- CS major. Will repeatedly tell you he can write Haskell code

Category
- Reads Kant, thinks abstractly
- Has solutions for every problem in theory, but no idea how to use them
- Unclear if he's speaking the truth, or abstract nonsense
- Very high or low energy

Vector Space
- Most popular girl by far
- Looks scary, but actually very nice
- Regularly gets people together
- Can rearrange to fit the needs of the moment
- "I'm nothing without Field"

Module
- Vector Space's less attractive sister.
- Wishes she was Vector Space
- Doesn't get invited to parties. Vaguely annoying
- Seemingly one-way friendship with Ring

MATHS BINGO

Please feel free to loudly shout ' BINGO' wherever and whenever you complete one!

signed up for MUMS!	Awkward silence in consultation w lecturer	Heated "discussion" about maths	Realise you were wrong about a key fact the whole time	3 assignments due in one week oh no
take a nap in the MUMS room	Do an assignment on the day it's due	Start an assignment on the day it's released	Make a friend who doesn't even go here join MUMS for the free food	Cry because of maths
9am lecture :'(	Write a thing for paradox	FREE FOOD	Lecturer says "obviously…" and it's not	Euphoric joy because of maths
7+ Wikipedia tabs about maths open at once	Do every problem in the problem booklet	Fall asleep in a lecture	Stay awake for a whole lecture	Destroy your enemies via crushing boardgame defeat at MUMS games night
Make eye contact with the uncomfortably large poster of Gauss in the MUMS room	Solve the Riemann hypothesis	Make a Maths Friend!	Recreational maths	This square is left as an exercise to the reader