School Maths Olympics 2021

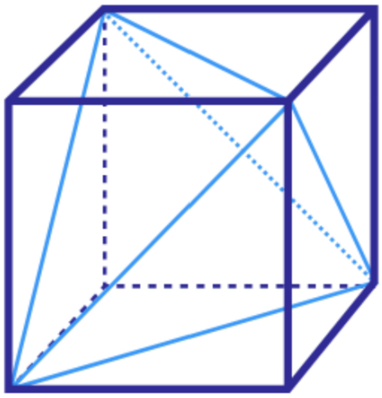
1. (20 points) Phoebe the Painter and Peter the Procrastinator are hired to paint a house. Phoebe takes 4 hours to paint the house by herself, whereas Peter takes 12 hours to complete the job by himself. How long does it take to paint the house if Phoebe and Peter work together?

2. (20 points) Alice chooses two integers x and y with $1 \leq x \leq y$. She whispers the sum $x + y$ to Sam, and the product $x \times y$ to Pam, so that neither knows what the other was told.

   Pam says "I don't know what x and y are."
   Sam then says "I don't know what x and y are."
   Pam then says "I now know x and y."

   Suppose the product is 4. What is the value of $x + y$?

3. (20 points) A tetrahedron fits perfectly inside a unit cube as shown.



What is the volume of this tetrahedron? Express your answer as a simplified fraction in the form a/b.

4. (20 points) 30 knights and knaves sit around a table in chairs labelled 1 to 30. Knights always tell the truth, whereas knaves always lie. Each person has exactly one friend among the table. The friend of a knight is always a knave, and the friend of a knave is always a knight. Everyone is asked the following question: 'Is your friend sitting next to you?'. All 15 people sitting at an odd chair number answer 'Yes'. How many people sitting at even numbers will answer 'Yes'?

5. (20 points) You have 16 switches on the wall, and each one is hooked up to exactly one of 16 lights. Each time, you are allowed to flip any chosen combination of any number of switches at the same time to see which lights show up. What is the minimum number of times needed to guarantee that you can determine which light is each switch connected to?

6. (20 points) What is your MC Alex's favourite sport?

   A. soccer

   B. cricket

   C. aikido

   (Please enter a single capital letter.)

7. (20 points) You were bored during lockdown, so you decided to toss 3 fair six-sided dice. Given that the sum of the three numbers you got from the three dice was 9, what is the probability that you receive a three on each die? Express your answer as a simplified fraction in the form a/b.

8. (20 points) Knights always tell the truth, and knaves always lie. 6 people, A, B, C, D, E and F, who are either knights, or knaves, say the following:

A: "B is a knave."
B: "C is a knave"
C: "A is a knight"
D: "E is a knave"
F: "E is a knight"

How many possible assignments of knight/knaves are there that are consistent with what they say?

9. (20 points) Ramanujan, a famous Indian mathematician, proved in 1911 that

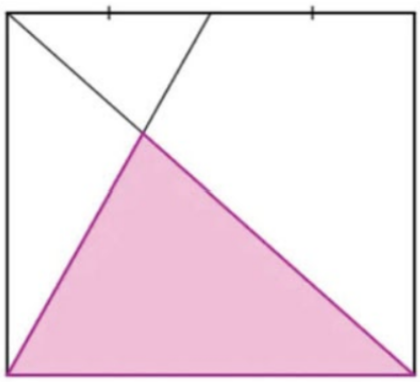
$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\dots}}}} = 3$$

Using the former equation, find the value of

$$\sqrt{1+100\sqrt{1+101\sqrt{1+102\sqrt{1+\dots}}}}$$

10. (25 points) The Ramanujan-Hardy number, also known as the second taxicab number, is the smallest number expressible as the sum of two positive cubes in two different ways. What is the Ramanujan-Hardy number? (You are allowed to Google this one ☺)

11. (25 points) Find the area of the shaded triangle, given that the side length of the square is 1. Express your answer as a simplified fraction in the form a/b.



12. (25 points) We can't travel the world right now but let's go on a trip on the Cartesian plane! Starting from the origin $(0, 0)$, we first move one unit up to $(0, 1)$. We then turn right and move $\frac{1}{2}$ unit to $(\frac{1}{2}, 1)$. Next, we move $\frac{1}{4}$ unit downwards . . . At each move, We keep making a $90°$ clockwise turn and move for half as far as before. If we keep traveling forever, we will become closer and closer to a point (a, b). What is the value of $a + b$? Express your answer as a simplified fraction in the form x/y.

13. (25 points) What is the size of the largest subset, S, of $\{1, 2, 3, \ldots, 36\}$ such that no pair of distinct elements of S has a sum divisible by 5?

14. (25 points) The POM system of numbering is a base three system, with the digits P, O and M representing +1, 0 and −1 respectively. For example, $PMOMP$ represents the number

$$P \times 3^4 + M \times 3^3 + O \times 3^2 + M \times 3 + P = 3^4 - 3^3 - 3 + 1 = 52.$$

When the number 2003 is represented in the POM system, what are the last two digits? (Please enter two capital letters with no space in between.)

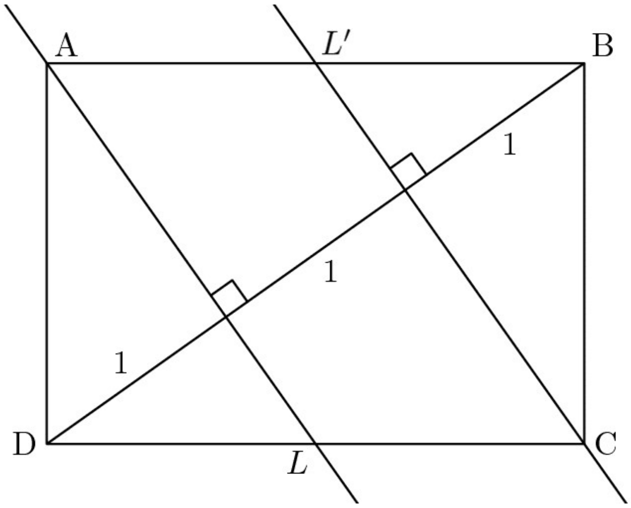
15. (25 points) What's my favourite multiple of 7 between 0 and 20?

16. (25 points) You wish to colour the cells of a 5×5 grid red and blue such that no 2×2 sub-grid contains an odd number of red cells. Find the number of ways to colour the grid.

17. (25 points) A wire frame in the shape of a pyramid has vertices $PQRST$. $PQRS$ is one of its faces, $PQ = 9$, $QR = 7$, $RS = 11$, $ST = 8$ and $TP = 7$. A sphere touches each of the edges PQ, QR, RS, ST, and TP. If this sphere touches PQ at the point X, what is the length of PX?

18. (25 points) A 3×3 square is divided up into nine 1×1 unit squares. Different integers 1 to 9 are written in these 9 unit squares. For each two squares sharing a common edge, the sum of the integers in them is calculated. What is the minimum possible number of different sums?

19. (25 points) Let $ABCD$ be a rectangle with a diagonal BD. The two lines AL and CL' are perpendicular to BD and trisect the diagonal into three segments each of length 1. What is the value of $|ABCD|^2$, the square of the area of rectangle $ABCD$?



20. (25 points) In the year 2004, there were 5 Sundays in Febuary. What is the next year in which this will occur?

21. (25 points) Which city will the 2032 Summer Olympics be held in?

22. (30 points) Some chess pieces are put on a 8×8 chess board, with at most 1 piece in each square. After taking all pieces on any chosen 4 rows and 4 columns, there is at least 1 piece left on the board. Find the least number of pieces originally on the board.

23. (30 points) In Miles's tennis tournament of 27 people, every player plays every other player exactly once. A trilogy is a group of three players A, B, and C such that A beats B, B beats C, and C beats A. What is the maximum possible number of trilogies among these 27 players?

24. (30 points) A group of travellers consists of n members. In an alternate universe without COVID-19, they travelled 6 times, each of which consisting of exactly 5 members. No two trips share more than two members. What is the minimum number of members in the group?

25. (30 points) Alice and Bob are playing a game. Bob has 3 piles of stones each with p, q, r stones in them, where p, q, r are distinct integers. Each turn proceeds as follows:

1: Alice chooses a positive integer y and gives it to Bob.

2: Bob chooses a pile, and adds y stones to that pile. Bob cannot choose the same pile as he did in the previous turn.

Alice wins whenever two of Bob's piles contain the same number of stones. What is the minimum number of turns needed for Alice to guarantee a win?