School Maths Olympics 2021

1. (20 points) Phoebe the Painter and Peter the Procrastinator are hired to paint a house. Phoebe takes 4 hours to paint the house by herself, whereas Peter takes 12 hours to complete the job by himself. How long does it take to paint the house if Phoebe and Peter work together?

Answer: 3

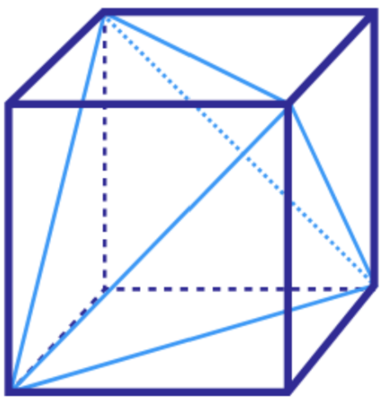
2. (20 points) Alice chooses two integers x and y with $1 \le x \le y$. She whispers the sum $x + y$ to Sam, and the product $x \times y$ to Pam, so that neither knows what the other was told.

Pam says "I don't know what x and y are."
Sam then says "I don't know what x and y are."
Pam then says "I now know x and y."

Suppose the product is 4. What is the value of $x + y$?

Answer: 5

3. (20 points) A tetrahedron fits perfectly inside a unit cube as shown.



What is the volume of this tetrahedron? Express your answer as a simplified fraction in the form a/b.

Answer: 1/3

4. (20 points) 30 knights and knaves sit around a table in chairs labelled 1 to 30. Knights always tell the truth, whereas knaves always lie. Each person has exactly one friend among the table. The friend of a knight is always a knave, and the friend of a knave is always a knight. Everyone is asked the following question: 'Is your friend sitting next to you?'. All 15 people sitting at an odd chair number answer 'Yes'. How many people sitting at even numbers will answer 'Yes'?

Answer: 0

5. (20 points) You have 16 switches on the wall, and each one is hooked up to exactly one of 16 lights. Each time, you are allowed to flip any chosen combination of any number of switches at the same time to see which lights show up. What is the minimum number of times needed to guarantee that you can determine which light is each switch connected to?

Answer: 4

6. (20 points) What is your MC Alex's favourite sport?

   A. soccer

   B. cricket

   C. aikido

   (Please enter a single capital letter.)

Answer: C

7. (20 points) You were bored during lockdown, so you decided to toss 3 fair six-sided dice. Given that the sum of the three numbers you got from the three dice was 9, what is the probability that you receive a three on each die? Express your answer as a simplified fraction in the form a/b.

Answer: 1/25

8. (20 points) Knights always tell the truth, and knaves always lie. 6 people, A, B, C, D, E and F, who are either knights, or knaves, say the following:

   A: "B is a knave."
   B: "C is a knave"
   C: "A is a knight"
   D: "E is a knave"
   F: "E is a knight"

   How many possible assignments of knight/knaves are there that are consistent with what they say?

Answer: 4

9. (20 points) Ramanujan, a famous Indian mathematician, proved in 1911 that

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\ldots}}}} = 3$$

Using the former equation, find the value of

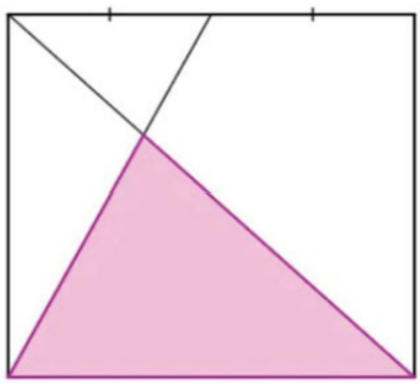
$$\sqrt{1+100\sqrt{1+101\sqrt{1+102\sqrt{1+\ldots}}}}$$

Answer: 101

10. (25 points) The Ramanujan-Hardy number, also known as the second taxicab number, is the smallest number expressible as the sum of two positive cubes in two different ways. What is the Ramanujan-Hardy number? (You are allowed to Google this one ☺)

Answer: 1729

11. (25 points) Find the area of the shaded triangle, given that the side length of the square is 1. Express your answer as a simplified fraction in the form a/b.



Answer: 1/3

12. (25 points) We can't travel the world right now but let's go on a trip on the Cartesian plane! Starting from the origin $(0,0)$, we first move one unit up to $(0,1)$. We then turn right and move $\frac{1}{2}$ unit to $(\frac{1}{2},1)$. Next, we move $\frac{1}{4}$ unit downwards ... At each move, We keep making a $90°$ clockwise turn and move for half as far as before. If we keep traveling forever, we will become closer and closer to a point (a,b). What is the value of $a+b$? Express your answer as a simplified fraction in the form x/y.

Answer: 6/5

13. (25 points) What is the size of the largest subset, S, of $\{1, 2, 3, \ldots, 36\}$ such that no pair of distinct elements of S has a sum divisible by 5?

Answer: 16

14. (25 points) The POM system of numbering is a base three system, with the digits P, O and M representing +1, 0 and −1 respectively. For example, $PMOMP$ represents the number

$$P \times 3^4 + M \times 3^3 + O \times 3^2 + M \times 3 + P = 3^4 - 3^3 - 3 + 1 = 52.$$

When the number 2003 is represented in the POM system, what are the last two digits? (Please enter two capital letters with no space in between.)

Answer: MM

15. (25 points) What's my favourite multiple of 7 between 0 and 20?

Answer: 0

16. (25 points) You wish to colour the cells of a 5×5 grid red and blue such that no 2×2 sub-grid contains an odd number of red cells. Find the number of ways to colour the grid.

Answer: 512

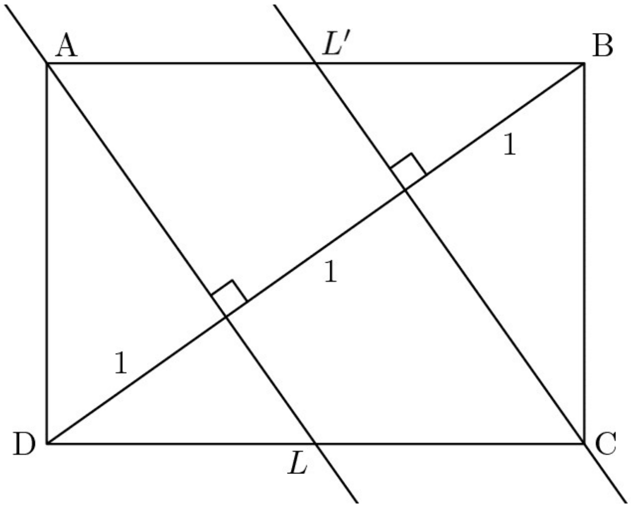
17. (25 points) A wire frame in the shape of a pyramid has vertices $PQRST$. $PQRS$ is one of its faces, $PQ = 9$, $QR = 7$, $RS = 11$, $ST = 8$ and $TP = 7$. A sphere touches each of the edges PQ, QR, RS, ST, and TP. If this sphere touches PQ at the point X, what is the length of PX?

Answer: 6

18. (25 points) A 3×3 square is divided up into nine 1×1 unit squares. Different integers 1 to 9 are written in these 9 unit squares. For each two squares sharing a common edge, the sum of the integers in them is calculated. What is the minimum possible number of different sums?

Answer: 4

19. (25 points) Let $ABCD$ be a rectangle with a diagonal BD. The two lines AL and CL' are perpendicular to BD and trisect the diagonal into three segments each of length 1. What is the value of $|ABCD|^2$, the square of the area of rectangle $ABCD$?



Answer: 18

20. (25 points) In the year 2004, there were 5 Sundays in Febuary. What is the next year in which this will occur?

Answer: 2032

21. (25 points) Which city will the 2032 Summer Olympics be held in?

Answer: Brisbane

22. (30 points) Some chess pieces are put on a 8×8 chess board, with at most 1 piece in each square. After taking all pieces on any chosen 4 rows and 4 columns, there is at least 1 piece left on the board. Find the least number of pieces originally on the board.

Answer: 13

23. (30 points) In Miles's tennis tournament of 27 people, every player plays every other player exactly once. A trilogy is a group of three players A, B, and C such that A beats B, B beats C, and C beats A. What is the maximum possible number of trilogies among these 27 players?

Answer: 819

24. (30 points) A group of travellers consists of n members. In an alternate universe without COVID-19, they travelled 6 times, each of which consisting of exactly 5 members. No two trips share more than two members. What is the minimum number of members in the group?

Answer: 10

25. (30 points) Alice and Bob are playing a game. Bob has 3 piles of stones each with p, q, r stones in them, where p, q, r are distinct integers. Each turn proceeds as follows:

1: Alice chooses a positive integer y and gives it to Bob.

2: Bob chooses a pile, and adds y stones to that pile. Bob cannot choose the same pile as he did in the previous turn.

Alice wins whenever two of Bob's piles contain the same number of stones. What is the minimum number of turns needed for Alice to guarantee a win?

Answer: 3

School Maths Olympics 2021

Answer Table

Question	Points	Answer
1	20	3
2	20	5
3	20	1/3
4	20	0
5	20	4
6	20	C
7	20	1/25
8	20	4
9	20	101
10	25	1729
11	25	1/3
12	25	6/5
13	25	16
14	25	MM
15	25	0
16	25	512
17	25	6
18	25	4
19	30	18
20	30	2032
21	30	Brisbane
22	30	13
23	30	819
24	30	10
25	30	3

School Maths Olympics 2021

Editorials

1.

$$\frac{1}{\frac{1}{4} + \frac{1}{12}} = 3$$

2. You can cheese this question with the 3 attempts, (x,y) is either (1,4) or (2,2), so you can just guess 4 and 5 and get the answer correct. However, here is how you're meant to actually solve it. Pam gets 4, and so (x, y) could either be (1,4) or (2,2). She has no idea which one is possible, so she says she doesn't know.

   Then Sam receives this information, that Pam doesn't know what (x, y) are. Suppose Sam received the sum to be 4 (which corresponds to (2,2)). Then Sam knows that (x,y) is either (1,3) or (2,2). However, if (x,y) was actually (1,3), then Pam would have unambiguously known the value of (x,y) contradicting what Pam said, ruling out (1,3) as a possibility. Then Sam would have been able to deduce that (x,y) would be (2,2), but Sam said he didn't know, contradicting the fact that Sam received 4. Therefore Sam must have received 5.

3. It is easiest to do this problem by taking the volume of the unit cube and subtracting out 4 triangle-based pyramids to obtain the tetrahedron. Each pyramid has volume $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ so the total volume is $1 - 4(\frac{1}{6}) = \frac{1}{3}$

4. Consider a person at an odd position that answered "yes". If that person was a knight, then his friend, a knave would be sitting next to him. The knave (that is at an even position) would correspondingly answer "no".

   If that person was a knave, then his friend, a knight, would be sitting not next to him. The knight cannot be at an odd position as everyone at an odd position answered "yes", hence the knight would be at an even position (and answer "no").

   As everyone at an even position has a friend at an odd position, we conclude that everyone at an even position answered "no", and so the answer is 0.

   Alternatively, we can observe that in each pair of friends, the knight and the knave would always give different responses, regardless of whether they were sitting next to each other or not. Hence, if we asked all 30 people around the table, we would always obtain 15 number of "Yes" and 15 number of "No". Since all 15 people sitting at an odd chair number answered "Yes", the remaining people would all answer "No".

5. The idea is to test a different group of 8 switches each time. We will use 1 to represent flipping a switch on and 0 to represent not flipping a switch.

   ```
   1st: 1111111100000000
   2nd: 1111000011110000
   3rd: 1100110011001100
   4th: 1010101010101010
   ```

   Notice how each column is unique; i.e. the inputs to each switch unique. Hence we are able to distinguish each connection by observing the sequence of flashes each individual light performs, and matching that to the corresponding input. Since $2^4 = 16$, 4 times is the smallest number to allow all columns to be distinct.

6. Alex Ghitza's favourite sport is Akido. You can just use all three of your attempts to guess each option.

7. The easiest way to list all the number of possible ways to sum to 9 with three integers is by listing each combination in non-increasing order, and counting the number of permutations for each combination. Here are all the possible combinations to sum to 9, with the number of permutations in brackets

$$6, 2, 1\ (6)$$
$$5, 3, 1\ (6)$$
$$5, 2, 2\ (3)$$
$$4, 4, 1\ (3)$$
$$4, 3, 2\ (6)$$
$$3, 3, 3\ (1)$$

There are 25 ways to sum to 9, and hence our answer is 1/25.

8. You'll quickly find that if you guess whether A is a knight or not, you'll unambiguously deduce B and C. Determining D also determines E and F without contradiction. Hence there are two ways to assign (A,B,C), and two ways to assign (D,E,F), and the answer is hence $2 \times 2 = 4$.

9. Square both sides of the upper equation, subtract 1, and dividing by the factor outside the square root (in this case, the factor is 2) results in

$$\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \dots}}}} = 4$$

As you can see, we have successfully stripped off a square root, and the right hand side just increments by one. If we repeat this process, we get

$$\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{1 + \dots}}}} = 5$$

Miraculously enough, every time you strip off a square root the right hand side increments by one. One can prove this by induction and deduce that the answer is 101.

To prove it a bit more rigorously, one can solve the recurrence relation

$$f(x) = \sqrt{1 + xf(x + 1)}$$

observing $f(2) = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \dots}}}}$ (to see this, recursively sub in the definition of f(x) a couple times) and $f(100)$ is our desired sum. We guess $f(x)$ is a linear polynomial and obtain $f(x) = x + 1$, and $f(100) = 101$

10. You can google it just like the question says you're allowed to do, and obtain 1729 as the answer.

11. Observe that the two triangles in the diagram are similar. Hence, the ratio of their heights equals to the ratio of their sides, which is $1 : 2$. Therefore, the height of the shaded triangle is $\frac{2}{3}$, leading to an area of $\frac{1}{3}$.

    Alternative, we can use coordinate geometry. We can find the intersection of lines $2x$ and $1 - x$ to get the height of the triangle at $\frac{2}{3}$, and the area is therefore $\frac{1}{3}$.

12. Consider each coordinate separately and use geometric series. You should get (a,b) = (2/5, 4/5) so the answer is 6/5

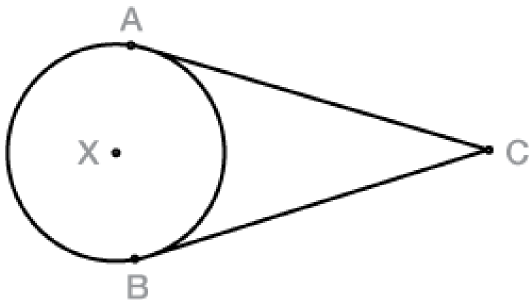
13. Consider things modulo 5. If any pair in S summed to 0 mod 5, then the condition fails. We can let S be all the numbers modulo 1 or 2, and hence it is impossible for any pair to sum to 0 mod 5. This gives an answer of 15, but we can actually include a single number that is 0 mod 5 and still not violate the condition, so the answer is 16.

14. 2003 is 5 modulo 9. The only terms that are nonzero modulo 9 are the last two digits of the POM system, and the only way to to attain something 5 modulo 9 is to subtract 3, then 1. Hence the last two digits are MM.

15. There are 3 possible multiples of 7, $7, 14$ and 0. You can guess each one and find that 0 is correct.

16. If you colour all the cells on the top border and the left border, you'll notice that it uniquely defines a colouring for the whole grid. Therefore there is a bijection between the number of colourings of the border, and the number of colourings of the whole board. The answer is therefore 2^{x+y-1} where x,y are the dimensions of the grid, and hence the answer is $2^9 = 512$.

17. First, draw a diagram so you can understand this explanation. You need to repeatedly apply the ice cream cone theorem; that is:



Whenever BC and AC are tangent to the same circle, they have the same length.

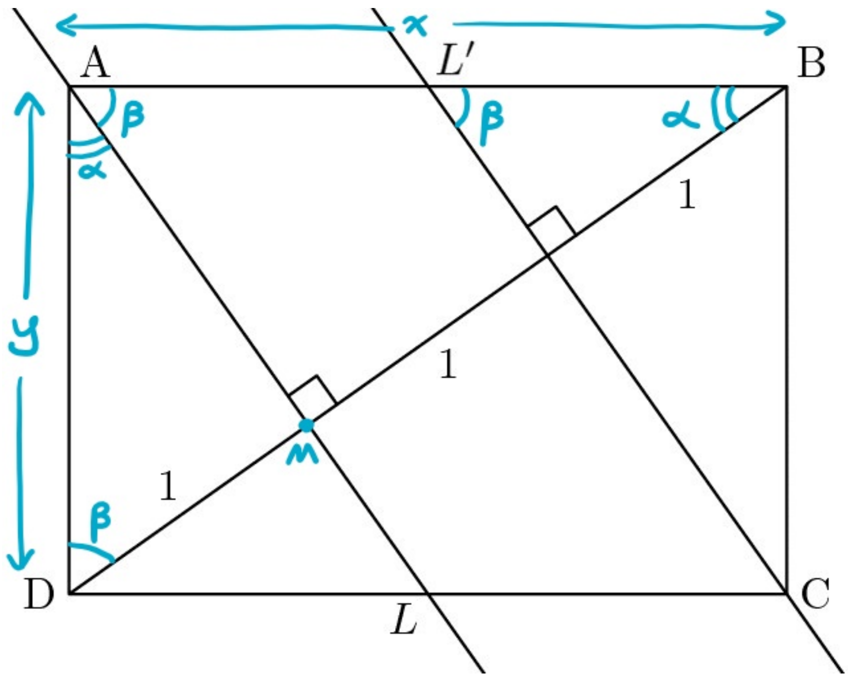
Label the points where the circle touches QR, RS, ST, TP B, C, D, E respectively. Observe that the intersection of a sphere with any plane will always be a circle, so we can consider the plane with the points X, Q, B and apply the ice cream cone theorem to determine that $XQ = QB$.

You'll also find that with similar arguments, $BR = RC, CS = SD, DT = TE, EP = PX$. With additional information like $PX + XQ = PQ = 9$, you will be able to determine that $PX = 6$.

18. The center element must create 4 different sums as it borders 4 different elements, so that is the bare minimum. In order to show that the minimum possible is actually 4, here's a construction that I made via divine inspiration.

$$\begin{bmatrix} 6 & 4 & 5 \\ 2 & 7 & 3 \\ 9 & 1 & 8 \end{bmatrix}$$

19. We do some angle chasing, and for brevity denote $\beta = 90° - \alpha$, and label edges x, y and the point M

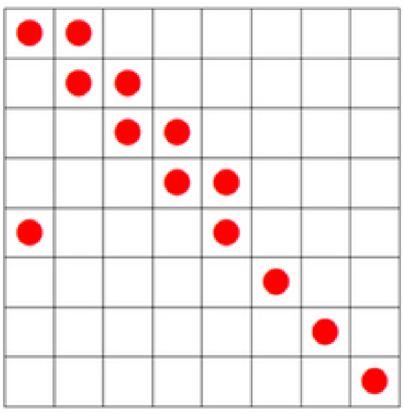


Observe that ADM and ADB are similar, and so $\frac{y}{3} = \frac{1}{y} \implies y = \sqrt{3}$. We then use pythagoras on ADB to deduce $x^2 + y^2 = 3^2 \implies x = \sqrt{6}$, so the answer is $(xy)^2 = 18$

20. Between two consecutively leap years, the first of February moves five days. To have Sunday occurring as the first again, it will have to move a multiple of seven days, so it will take another seven leap years for this to occur, and that is not until the year 2032.

21. Brisbane will be holding the Olympics in 2032! We included this question because 2032 was also the answer to the previous question.

22. The answer cannot be ≤ 12 as you can choose the 4 rows with the most counters, which will contain at least 8 elements, then choose 4 columns with the remaining 4 elements.
Here's a construction with 13 counters, which is therefore optimal.



23. Consider the nontrilogies; Each non-trilogy has exactly one "vee", which we define as a vertex with two out-edges. So let the vertices be $v_1, \ldots, v_{27}$, and let $d_1, \ldots, d_{27}$ be the out-edges. The total number of vees (and thus nontrilogies) is $\binom{d_1}{2} + \cdots + \binom{d_{27}}{2}$.

    Now $d_1 + \cdots + d_{27} = \binom{27}{2}$, because that's the number of victories in the tournament. Then using your favourite method (Jensen's is easiest but there's probably a less esoteric way), the total number of vees is minimised when $d_1 = \cdots = d_{27} = 13$, which gives a min total of $27 \times \binom{13}{2}$ nontrilogies and thus a max of $\binom{27}{3} - 27 \times \binom{13}{2} = 819$ trilogies.

24. Consider the sum of the number of people a pair of trips share for every single pair of trips, and call it L. From the problem statement the maximum amount this L is allowed to be is $\binom{6}{2} \times 2 = 30$.

    Since 6 trips were made with 5 people each, people went on a trip a total of 30 times. Now suppose there were 9 people. If you 'spread out' the trips as evenly as possible, then that means that 6 people went on 3 trips and 3 on 4 trips. However that means the L will be at least $6 \times \binom{3}{2} + 3 \times \binom{4}{2} = 36$, so it is impossible for there to be only 9 people.

    We find that for 10 people, L will be at least $10 \times \binom{3}{2} = 30$ (each person participates in 3 trips). which suggests it may be possible, and here is the construction that shows that it is. Knowing that each column must have exactly 3 elements aids in finding this construction.

A	B	C	D	E	F	G	H	I	J
○	○	○	○	○					
○	○				○	○	○		
○		○			○			○	○
	○		○			○		○	○
		○		○		○	○	○	
			○	○	○		○		○

    Hence the answer is 10.

25. WLOG let $p < q < r$. Alice chooses $2r - p - q$. If Bob chooses either pile p or q (let's say, p for the time being), then his piles will have $2r - q, q, r$ stones, when ordered is $q, r, 2r - q$. Bob now has an arithmetic sequence with difference $r - q$, and cannot pick $2r - q$ again, so when Alice gives Bob $r - q$ Bob will lose. If Bob originally picked q, he will also lose in a similar fashion.

    Hence initially picking p or q guarantees a win for Alice in 2 moves. However if Bob picks r, then Alice will have to give Bob $2l - p - q$ again (where k is now the size of the largest pile), and now Bob cannot pick the largest element, so the strategy continues as before. This guarantees Alice a win in 3 moves in total.