UMO 2022 questions

Questions

1. (20 points) Consider the set of $(x, y) \in \mathbb{R}^2$ satisfying the following equation:

$$1 - ||x| - 1| = |y|$$

How many right angles are formed when this set is plotted on the cartesian plane?

Source: Quang Original

2. (20 points) Quang is organising a luxurious dinner to celebrate the new MUMS Committee! The circular tables have 12 numbered seats. Unfortunately, due to UMSU covid protocols, at most 3 guests may sit at each table, and between any two guests must be three empty seats. How many ways are there to seat *exactly* three guests at a table compliant with UMSU protocol? Two ways are different if there is a guest that sits at a different chair.

Source: Quang Original

3. (20 points) Suppose

$$\frac{a}{b} + \frac{b}{a} = 2023$$

Find the value of

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}$$

Source: Quang Original

4. (20 points) Alex saw some MUMS stickers in the MUMS room and decided to use them somewhere. The stickers are circles with a radius of 1. He's trying to stick as many stickers as possible on a square shaped paper with a side length of 5. At most how many stickers can Alex place fully inside the square without any two stickers overlapping?

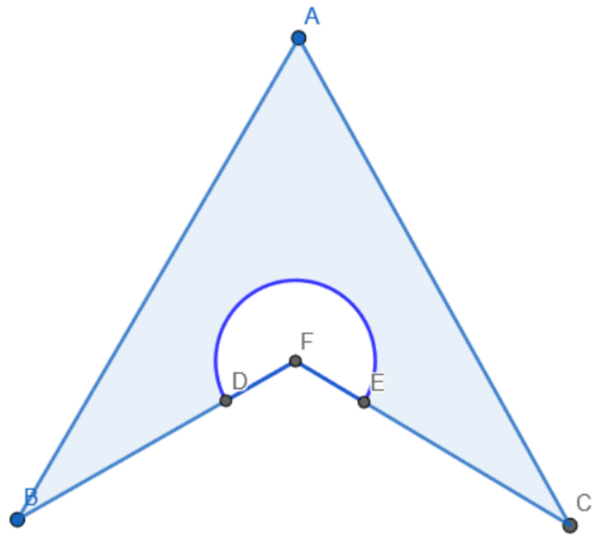
Source: Harry Original

5. (20 points) Among the positive integers (x, y) that satisfy $x^2 - 5y^2 = 1$, find values (x, y) with the smallest x that solves the equation.

Source: Harry Original

6. (20 points) A, B, and C are points on an equilateral triangle with side length 2. What is the area of the blue region bounded by the quadrilateral $ABFC$ and the circle with centre F if the length of the long arc DE is $\pi/3$ and the length of line segment BF is $2\frac{\sqrt{3}}{3}$?

   Give your answer in the form $a\sqrt{b} + c\pi$ where a, c are rational and b integer.



Source: Bon and Andres

7. (20 points) Calculate the 2022-nd derivative of x^2e^x evaluated at $x = 0$

Source: Quang Original

8. (20 points) What is the smallest n such that there is a knight's tour on a $4 \times n$ chessboard? Knights can move two squares in any direction vertically followed by one square horizontally, or two squares in any direction horizontally followed by one square vertically.

   A knight's tour is a sequence of moves a knight can make on a chessboard such that the knight visits every square exactly once.

Source: Harry (copied from https://yufeizhao.com/olympiad/comb1.pdf)

9. (25 points) You are at your friend's house, and you wish to connect to their wifi. Unfortunately, this is the password you see on the back of their router. Evaluate this integral, so you can connect to your friend's wifi.

$$\int_{\frac{\int_{\log_5 3}^{\log_2 \sqrt{3}} \frac{x^2 \sin x + x \cos x}{\cosh x} dx}{\log_{1/25} 16}}^{\int_{\log_2 5 \log_5 9}^{\log_2 18} \lim_{N\to\infty} \prod_{k=0}^{2N+1} e^{\frac{2k\pi i}{2N+1}} dx} \left(\sum_{n=0}^{2022} \binom{2022}{n} x^n\right) dx$$

   Give your answer in the form $\frac{a^b - c}{d}$ for integers a, b, c, d

Source: Harry Original

10. (25 points) It's been a few years since you've seen your friend, and you're at their house again! However, you don't seem to be automatically connecting to their wifi... you peek on the back of their router and this is what you see...

$$\int_{\int_0^{2022} \cos 2\pi x \cos 3\pi x dx}^{\lim_{n\to\infty} \int_{(1-\frac{1}{n})^n}^{(1+\frac{1}{n})^n} e \log x dx} \sqrt{16 - 2x^2} dx$$

Give your answer in the form $\sqrt{a}(b\pi + c)$ for integers a, b, c

Source: Harry Original

11. (25 points) Barry has 5 identical red gummy bears and 12 identical green gummy bears. He wishes to arrange them in a straight line such that between any two red bears, there is at least one green bear. How many different ways can Barry arrange the gummy bears?

Source: Ravon

12. (25 points) Barry is a maths genius. He can calculate the square of a 2×2 matrix by squaring each element. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1^2 & 0^2 \\ 0^2 & 1^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. How many different matrices can Barry correctly square like this? (given all elements in the matrices are integers from 0 to 9)

Source: Harry Original

13. (25 points) Quang the Mysterious offers to play a game with you. He secretly generates three numbers A, B and C which are guaranteed to be uniformly distributed between 0 and 1.

    First he shows you the number A. If you decide to take it, then congratulations! You win A points. Otherwise he will show you B. If you decide to take it, then you win B points. Otherwise, you win C points.

    Given you play the optimal strategy to maximise the points you win, how many points will you win on average? Give your answer as a simplified fraction.

Source: Quang Original

14. (25 points) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be concentric circles, with $\mathcal{C}_2$ in the interior of $\mathcal{C}_1$. From a point A on $\mathcal{C}_1$ one draws the tangent AB to $\mathcal{C}_2$ ($B \in \mathcal{C}_2$). Let C be the second point of intersection of AB and $\mathcal{C}_1$, and let D be the midpoint of AB. A line passing through A intersects $\mathcal{C}_2$ at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AC. Find the ratio AM/MC given as a simplified fraction.

Source: USAMO 1998

15. (25 points) A team of Unimelb biologists discovered a new creature in the MUMS room called 'Quang Jr.' that clones itself following an interesting pattern.

- On the day it is born, it will produces 0 clones.
- Throughout the first day after it is born, it will still produce 0 clones.
- Throughout the second day, it will produce 3 identical copies of itself.
- Starting from the third day, it will produce 4 further copies of itself every day.

The biologists decide to kidnap a 'Quang Jr.' on the day it is born and place it into an isolated enclosure. Let $P(t)$ be the number of 'Quang Jr.'s t days after the kidnapping. Find:

$$\lim_{t \to \infty} \frac{P(t+1)}{P(t)}$$

Give your answer in the form $a + b\sqrt{c}$ for rational a, b and integer c

(Hint: P(1)=1, P(2)=1, P(3)=4, P(4)=8)

Source: Harry Original

16. (25 points) Evaluate:

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n^2} \ln(1 + \frac{k}{n})$$

Give your answer as a simplified fraction.

Source: Harry (Zhihu)

17. (30 points) I ate at a huge feast trying 121 different types of food, and among them, exactly one unknown food x gave me a stomachache. I am left with the leftovers (consisting of a sizeable serving of each type of food), and I want to finish them! Each day, I can try a subset of the food, and I will get a stomachache iff the subset I eat contains food x (No matter how little of food x I eat!). However, stomachaches are unpleasant, so I will tolerate at most two stomachaches. What is the minimum amount of days I need to determine which food is food x?

Source: Quang Original

18. (30 points) Let S be the set of 9×9 invertible matrices whose entries are either -1 or 1. What is the greatest common divisor of all elements in the set

$$\{|\det(A)|, A \in S\}$$

Source: Quang (Inspired by a 2020 MAST10018 assignment)

19. (30 points) Let $f(n)$ be the expected value of the number of unique integers in a sequence of n integers selected uniformly randomly from $[1, n]$. Find the limit

$$\lim_{n \to \infty} \frac{f(n)}{n}$$

Give your answer in the form $a + be^c$ for rationals a, b, c

Source: Quang (inspired by Brilliant)

20. (30 points) Evaluate:

$$\int_0^{\frac{\pi}{4}} \frac{x^2+2}{(x\sin x + 2\cos x)^2} dx$$

Give your answer in the form

$$\frac{a+b\pi}{c+d\pi}$$

for integer a, b, c, d

Source: Harry (Zhihu)

21. (30 points) Let

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 1 + f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) & \text{otherwise} \end{cases}$$

Find

$$\int_{2^4}^{2^5} f(x) dx$$

Source: Dougal Original

22. (30 points) Given

$$\int_{-\infty}^{\infty} e^{-(nx)^2} dx = \frac{\sqrt{\pi}}{n}$$

Find

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} n e^{-(nx)^2-(1-x)^4} dx$$

after substituting the well-known identity $\pi = e = 3$. Give your answer as $a\sqrt{b}$ for rational a, integer b.

Source: Dougal Original

23. (30 points) For any given set S of 4 distinct, non-cyclic points in the plane with no three points collinear, let $m(S)$ be the number of distinct circles C such that C contains exactly 2 points and the distance from C to every point in S is the same. What is the maximum value of $m(S)$?

Source: Quang (inspired by a level in Euclidea)

24. (∞ points) Evaluate

$$\tan(1)$$

to as many significant digits as possible

Scoring: The first three significant figures you calculate correctly are worth 10 points each. Subsequent significant figures are worth 9 points, then 8, then 7, so on, until they are worth 2 points each.

We will not mark your submissions, nor give you any feed back on correctness until the end of the round.

Source: Dougal and Jamie