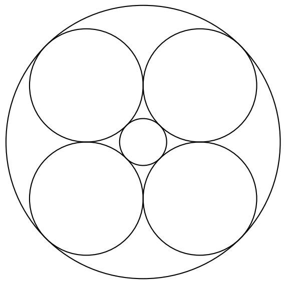
University Maths Olympics 2020

Questions & Answers

1. (20 points) In this diagram, the ratio between the radius of the largest circle and the radius of the smallest circle is $a + b\sqrt{2}$, where a and b are positive integers. What is the sum of a and b?



Answer: 5

2. (20 points) The numbers $3, 5, 7, a$, and b have an average (arithmetic mean) of 15. What is the average of a and b?

Answer: 30

3. (20 points) Bob: Alice, what is your favourite question from the UMO?

Alice: The question number of my favourite UMO problem is a multiple of 6.

Bob: I don't think that gives me enough information.

Alice: This question number has the same number of factors as the number of maths memes we looked at today!

Bob: Hmm... That's still not enough information.

Alice: The ages of my dogs are all primes and the product of their ages happens to be this question number. Oh wait, I think my oldest dog just ate my homework.

Bob: Oops! But at least now I know which UMO question is your favourite!

What is the question number of Alice's favourite UMO problem?

Answer: 12

4. (20 points) How many 4-digit positive digits (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?

Answer: 100

5. (20 points) There is a 99% chance that my dog would eat meat on Monday and 1% chance that my dog would eat vegetables on Monday. Whatever she eats on Monday, there's 90% chance that she will have the same thing on Tuesday. Given that she ate vegetables on Tuesday, the probability that she also had vegetables on Monday is $\frac{1}{a}$, where a is an integer. What is the value of a?

Answer: 12

6. (20 points) Po writes down five consecutive integers and then erases one of them. The four remaining integers sum to 153. Compute the integer that Po erased.

Answer: 37

7. (20 points) The area between the parabolas with equations

$$y = x^2 + 2ax + a$$

and

$$y = a – x^2 = 9.$$

What is the absolute value of a?

Answer: 3

8. (20 points) An equilateral triangle has circumcentre O and side length 1. A straight line through O intersects the triangle at two distinct points P and Q. What is the minimum possible length of PQ?

Answer: $\frac{2}{3}$

9. (20 points) Compute the smallest positive integer n such that there do not exist integers x and y satisfying $n = x^3 + 3y^3$.

Answer: 6

10. (25 points) Compute the sum of all positive integers n such that the median of the n smallest prime numbers is n.

Answer: 25

11. (25 points) Compute the number of ordered pairs (m, n) of positive integers that satisfy the equation $\text{lcm}(m, n)+ \gcd(m, n) = m + n + 30$.

Answer: 6

12. (25 points) What is Kerry's favourite letter in the Greek alphabet? ϕ, χ or π?

Answer: π

13. (25 points) Robin, Sam, Terry, Una and Viv competed with each other in a contest. When the results were finalised, there were no ties. Same was twice as many places ahead of Robin as Una was behind Terry. Viv's place was an odd number. The only true statement is

    (a) Viv came first
    (b) Sam came second
    (c) Terry came fourth
    (d) Una came third
    (e) Robin came fifth

Answer: e

14. (25 points) Let $a > b$ be positive integers. Compute the smallest possible integer value of:

$$\frac{a! + 1}{b! + 1}$$

Answer: 103

15. (25 points) I want to number pages of a book but unfortunately the key "1" on my keyboard is broken. As a result, I resort to number the pages without using numbers containing the digit 1, as follows:

    2, 3, 4, 5, 6, 7, 8, 9, 20, 22, 23, ... , 29, 30, 32, ...

    I numbered the last page of this book as 2020. How many pages does this book actually contain?

Answer: 738

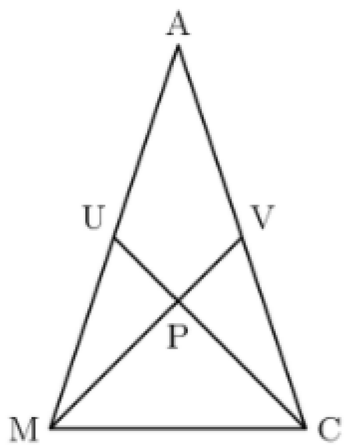
16. (25 points) Let $ABCD$ be a square with side length 4. Consider points P and Q on segments AB and BC, respectively, with $BP = 3$ and $BQ = 1$. Let R be the intersection of AQ and DP. If BR^2 can be expressed in the form mn for co-prime positive integers m, n, compute $m + n$.

Answer: 177

17. (25 points) How many 5-digit number plates using the digits from 0 to 9 will read the same when turned upside down, where an upside down 9 is a 6 and vice versa? Examples are 01810 and 91016.

Answer: 75

18. (25 points) Triangle AMC is isosceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?



Answer: 96

19. (30 points) Let (a, b, c, d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $0, 1, 2, 3$. For how many such quadruples is it true that $a \cdot d - b \cdot c$ is odd? (For example, $(0, 3, 1, 1)$ is one such quadruple, because $0 \cdot 1 - 3 \cdot 1 = -3$ is odd.)

Answer: 96

20. (30 points) A sequence (a_n) has the property that

$$a_{n+1} = \frac{a_n}{a_{n-1}}$$

for every $n \geq 2$. Given that $a_1 = 2$ and $a_2 = 6$, what is a_{2017}?

Answer: 2

21. (30 points) A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0), (0, 4), (4, 4)$, and $(4, 0)$. The probability that the sequence of jumps ends on a vertical side of the square is $\frac{a}{b}$ where a and b are co-prime. What is $a + b$?

Answer: 13

22. (30 points) Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. The probability that he chooses to reroll exactly two of the dice is $\frac{a}{b}$ where a and b are coprime. What is $a + b$?

Answer: 43

23. (30 points) Compute

$$\int_0^{\frac{\pi}{4}} e^{2x} \sin x \, dx$$

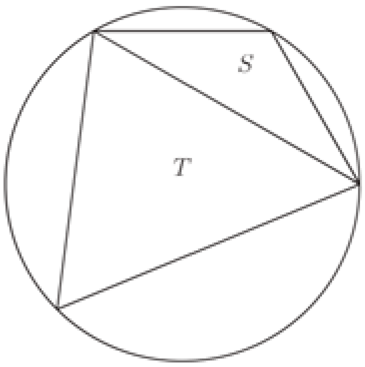
Given a, b, c are integers, your answer should be of the form:

$$\frac{\sqrt{a} e^{\frac{\pi}{2}}}{b} + \frac{1}{c}$$

What is $a + b + c$?

Answer: 17

24. (30 points) Two triangles S and T are inscribed in a circle, as shown in the diagram below.



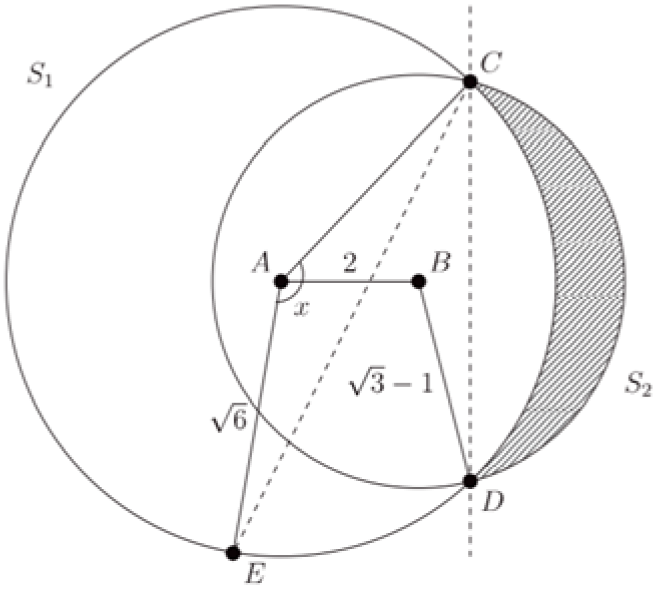
The triangles have respective areas s and t and S is the smaller triangle so that $s < t$.
The smallest values that

$$\frac{4s^2 + t^2}{5st}$$

can equal is?

Answer: $\frac{4}{5}$

25. (30 points) Consider two circles S_1 and S_2 centred at A and B and with radii $\sqrt{6}$ and $\sqrt{3} - 1$ respectively. Suppose that teh two circles intersect at two distinct points C and D. Suppose further that the two centres A and B are of distance 2 apart. The sketch below is not to scale.



What is $\angle CAD$?

Answer: 30 degrees

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Answer Table

Question	Points	Answer
1	20	5
2	20	30
3	20	12
4	20	100
5	20	12
6	20	37
7	20	3
8	20	$\frac{2}{3}$
9	20	6
10	25	25
11	25	6
12	25	π
13	25	e
14	25	103
15	25	738
16	25	177
17	25	75
18	25	96
19	30	96
20	30	2
21	30	13
22	30	43
23	30	17
24	30	$\frac{4}{5}$
25	30	30 degrees