2023 Integration Bee (Full Solutions)

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Contents

Chapter 1

Watch the Event

A recording of the event is up on the PSS youtube channel https://www.youtube.com/watch?v=9-pIJ15Kvq0

Chapter 2

Definitions and Miscellaneous Notation

Definitions:

$$\lfloor x \rfloor := \max \{ n \in \mathbb{Z} \mid n \leq x \} \qquad \text{the floor of x} \tag{2.1}$$

$$\lceil x \rceil := \min \{ n \in \mathbb{Z} \mid n \geq x \} \qquad \text{the ceiling of x} \tag{2.2}$$

$$\{ x \} := x - \lfloor x \rfloor \qquad \text{the fractional part of x} \tag{2.3}$$

$$\operatorname{sgn} x := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \qquad \text{the sign of x} \tag{2.4}$$

Some integrals will have the differential before the integrand rather than after

$$\int \mathrm{d}x\, f(x) := \int f(x)\, \mathrm{d}x$$

The difference is entirely cosmetic and is used to make some multidimensional integrals such as (4.2.3) easier to parse

$$\lim_{n\to\infty} \int_0^x \mathrm{d}\sigma_1 \int_0^{\sigma_1} \mathrm{d}\sigma_2 \cdots \int_0^{\sigma_{n-1}} \mathrm{d}\sigma_n\, e^{\sigma_n} \tag{2.5}$$

vs

$$\lim_{n\to\infty} \int_0^x \int_0^{\sigma_1} \cdots \int_0^{\sigma_{n-1}} e^{\sigma_n}\, \mathrm{d}\sigma_n \ldots \mathrm{d}\sigma_2\, \mathrm{d}\sigma_1 \tag{2.6}$$

Chapter 3

Quarter Finals

3.1 Match 1 (Eric Z vs Eric B) @ 18:20

3.1.1 Integral 1 — Time Out

$$\int_0^{2023} \left[\sin\left(\frac{\pi}{2}\lfloor x\rfloor\right) + \cos\left(\frac{\pi}{2}\lfloor x\rfloor\right)\right] \mathrm{d}x \tag{3.1}$$

There were quite a few integrals containing the floor function this integration bee. The trick with most of these is to convert the integral into a summation

$$\int_0^{2023} \left[\sin\left(\frac{\pi}{2}\lfloor x\rfloor\right) + \cos\left(\frac{\pi}{2}\lfloor x\rfloor\right)\right] \mathrm{d}x = \sum_{n=1}^{2023} \sin\left(\frac{\pi}{2}(n-1)\right) + \cos\left(\frac{\pi}{2}(n-1)\right)$$

Once we have the summation, we can see the terms cycle between 1,1, -1, and -1. This means the first 2020 terms cancel out and we're left with

$$\sum_{n=2021}^{2023} \sin\left(\frac{\pi}{2}(n-1)\right) + \cos\left(\frac{\pi}{2}(n-1)\right) = 1 + 1 - 1 = 1$$

3.1.2 Integral 2 — Time Out

$$\int_0^{23} \lfloor x\rfloor\{x\}\,\mathrm{d}x \tag{3.2}$$

Similarly to the previous integral, we separate it into smaller integrals to get rid of the floor and fractional functions

$$\begin{aligned}\int_0^{23} \lfloor x\rfloor\{x\}\,\mathrm{d}x &= \sum_{n=0}^{22}\int_n^{n+1} n\{x\}\,\mathrm{d}x \\ &= \sum_{n=0}^{22} n\int_0^1 x\,\mathrm{d}x \qquad = \frac{253}{2}\end{aligned}$$

3.1.3 Integral 3 — Time Out

$$\int_0^1 \left(\sqrt{2x-x^2}+\sqrt{1-x^2}-1\right)\mathrm{d}x \tag{3.3}$$

$$\begin{aligned}\int_0^1 \left(\sqrt{2x-x^2}+\sqrt{1-x^2}-1\right)\mathrm{d}x &= \int_0^1 \left(\sqrt{1-(1-x)^2}+\sqrt{1-x^2}-1\right)\mathrm{d}x \\ &= \int_0^1 \left(2\sqrt{1-x^2}-1\right)\mathrm{d}x\end{aligned}$$

At this point, it is possible to solve this with a trigonometric substitution. However, it's faster to think of this geometrically as $y=\sqrt{1-x^2}, x\in(0,1)$ forms a quarter circle with radius 1. This quickly gives us the answer of

$$\int_0^1 \left(2\sqrt{1-x^2}-1\right)\mathrm{d}x = \frac{\pi}{2}-1$$

3.1.4 Integral 4 (tiebreaker) — Eric B

$$\int_0^{3\pi} \left(\sin x+\frac{1}{2}\operatorname{sgn}\sin x\right)\mathrm{d}x \tag{3.4}$$

$$\begin{aligned}\int_0^{3\pi} \left(\sin x+\frac{1}{2}\operatorname{sgn}\sin x\right)\mathrm{d}x &= \int_0^{\pi} \left(\sin x+\frac{1}{2}\operatorname{sgn}\sin x\right)\mathrm{d}x \\ &= \int_0^{\pi} \left(\sin x+\frac{1}{2}\right)\mathrm{d}x \\ &= 2+\frac{\pi}{2}\end{aligned}$$

3.2 Match 2 (Nat vs Alex G) @ 37:25

3.2.1 Integral 1 — Alex G

$$\int_{-2}^1 \sqrt{8-2x^2}\,\mathrm{d}x \tag{3.5}$$

It's possible to solve this with a trigonometric substitution, however, we can also solve this geometrically

$$\int_{-2}^1 \sqrt{8-2x^2}\,\mathrm{d}x = \sqrt{2}\int_{-2}^1 \sqrt{4-x^2}\,\mathrm{d}x = \sqrt{2}\left(\frac{4\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

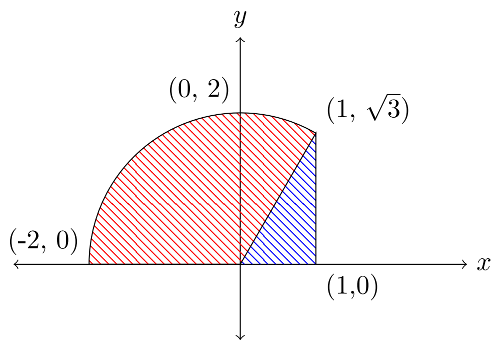


Figure 3.1: $y = \sqrt{4 - x^2}$ forms a circle of radius 2

3.2.2 Integral 2 — Alex G

$$\int_0^{2023} e^{x - \lfloor x \rfloor} \, \mathrm{d}x \tag{3.6}$$

We can see that the integrand repeats itself over and over again, 2023 times

$$\therefore \int_0^{2023} e^{x - \lfloor x \rfloor} \, \mathrm{d}x = 2023 \int_0^1 e^x \, \mathrm{d}x = 2023 \, (e - 1)$$

3.3 Match 3 (Mugilan vs Ilia) @ 46:55

3.3.1 Integral 1 — Ilia

$$\int \frac{x^2 \sec^2(\arctan x) \cos(\arcsin x)}{(1 + x^2) \sqrt{1 - x^2}} \, \mathrm{d}x \tag{3.7}$$

By making use of the identities

$$\sec^2 x = 1 + \tan^2 x$$
$$\cos x = \sqrt{1 - \sin^2 x}$$

everything ends up cancelling nicely,

$$\int \frac{x^2 \sec^2(\arctan x) \cos(\arcsin x)}{(1 + x^2) \sqrt{1 - x^2}} \, \mathrm{d}x = \int x^2 \, \mathrm{d}x = \frac{1}{3} x^3 + c$$

3.3.2 Integral 2 — Time Out

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left\{ \sqrt{2 - x^2} \right\} \mathrm{d}x \tag{3.8}$$

Similarly to before, the integrand (without the $\{\}$) forms a semicircle of radius $\sqrt{2}$. $\sqrt{2 - x^2} \geq 1$ for $x \in [-1, 1]$. Therefore, the integral is equal to $\pi - 2$.

3.3.3 Integral 3 — Ilia

$$\int_{\sqrt[3]{\log 3}}^{\sqrt[3]{\log 4}} \frac{x^2 \sin\left(x^3\right)}{\sin\left(x^3\right) + \sin\left(\log 12 - x^3\right)} \, \mathrm{d}x \tag{3.9}$$

Substitute $y = x^3$, then $z = \log 12 - y$

$$\begin{aligned} &\int_{\sqrt[3]{\log 3}}^{\sqrt[3]{\log 4}} \frac{x^2 \sin\left(x^3\right)}{\sin\left(x^3\right) + \sin\left(\log 12 - x^3\right)} \, \mathrm{d}x \\ &= \frac{1}{3} \int_{\log 3}^{\log 4} \frac{\sin y}{\sin y + \sin\left(\log 12 - y\right)} \, \mathrm{d}y \\ &= \frac{1}{3} \int_{\log 3}^{\log 4} \frac{\sin\left(\log 12 - z\right)}{\sin z + \sin\left(\log 12 - z\right)} \, \mathrm{d}z \end{aligned}$$

If we average the previous two lines together, we get

$$\begin{aligned} \frac{1}{6} \int_{\log 3}^{\log 4} \frac{\sin y + \sin\left(\log 12 - y\right)}{\sin y + \sin\left(\log 12 - y\right)} \, \mathrm{d}y &= \frac{1}{6} \int_{\log 3}^{\log 4} \mathrm{d}y \\ &= \frac{1}{6} \log \frac{4}{3} \end{aligned}$$

3.4 Match 4 (Jingying vs Alex H) @ 58:05

3.4.1 Integral 1 — Alex H

$$\int_0^{2023} (2023 - x)^{2023} \, \mathrm{d}x \tag{3.10}$$

$$\int_0^{2023} (2023 - x)^{2023} \, \mathrm{d}x = \int_0^{2023} x^{2023} \, \mathrm{d}x = \frac{2023^{2024}}{2024}$$

3.4.2 Integral 2 — Alex H

$$\int_0^2 \max(1, x, x^2, \sin x, \log x, e^x - 1) \, \mathrm{d}x \tag{3.11}$$

We can immediately eliminated $\sin x$, $\log x$ and x as they are less than $\max(1, x^2)$. Additionally, we can eliminate x^2 since $\frac{d^n}{dx^n} x^2 < \frac{d^n}{dx^n}(e^x - 1) \; \forall \, n \geq 0$ at $x = 1$. All that leaves is 1 and $e^x - 1$, which intersect at $x = \log 2$

$$\begin{aligned} \therefore \int_0^2 \max(1, x, x^2, \sin x, \log x, e^x - 1) \, \mathrm{d}x &= \int_0^{\log 2} \mathrm{d}x + \int_{\log 2}^2 (e^x - 1) \, \mathrm{d}x \\ &= 2 \log 2 + e^2 - 4 \end{aligned}$$

Chapter 4

Semi Finals

4.1 Match 1 (Eric B vs Alex G) @ 1:05:35

4.1.1 Integral 1 — Time Out

$$\int_0^\infty \{x\}e^{-\lfloor x \rfloor}\,\mathrm{d}x \tag{4.1}$$

$$\begin{aligned}\int_0^\infty \{x\}e^{-\lfloor x \rfloor}\,\mathrm{d}x &= \sum_{n=0}^{\infty} e^{-n}\int_0^1 x dx \\ &= \frac{1}{1-e^{-1}} \times \frac{1}{2} \qquad = \frac{e}{2e-2}\end{aligned}$$

4.1.2 Integral 2 — Eric B

$$\int_{-\pi/3}^{\pi/3} \sin(\tan x)\cos x\,\mathrm{d}x \tag{4.2}$$

sin and tan are both odd functions and so their composite is also odd. Since cos is also even, the integrand is odd and the resulting integral is 0.

4.1.3 Integral 3 — Alex G

$$\int_0^{10}\left(x + x^2 + |x^2 - x| + \frac{22}{30}\right)\mathrm{d}x \tag{4.3}$$

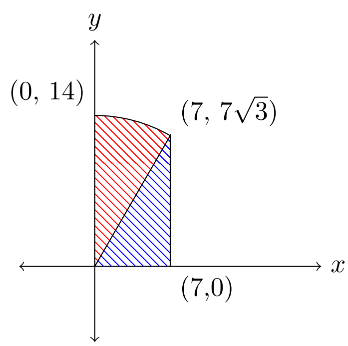
$$\begin{aligned}\int_0^{10}\left(x + x^2 + |x^2 - x| + \frac{22}{30}\right)\mathrm{d}x &= \frac{22}{3} + \int_0^1 2x\,\mathrm{d}x + \int_1^{10} 2x^2\,\mathrm{d}x \\ &= \frac{22}{3} + 1 + \frac{2}{3}(10^3 - 1) \\ &= \frac{2023}{3}\end{aligned}$$

4.1.4 Integral 4 — Alex G

$$\int_0^7 \sqrt{196 - x^2}\,\mathrm{d}x \tag{4.4}$$

This is nearly identical to integral 3.2.1

$$\int_0^7 \sqrt{196 - x^2}\,\mathrm{d}x = \frac{196\pi}{12} + \frac{49\sqrt{3}}{2} = \frac{49\pi}{3} + \frac{49\sqrt{3}}{2}$$



4.1.5 Integral 5 — Alex G

$$\int_0^1 x^2(1 - x)^4\,\mathrm{d}x \tag{4.5}$$

$$\begin{aligned}\int_0^1 x^2(1-x)^4\,\mathrm{d}x &= \int_0^1 (1-x)^2 x^4\,\mathrm{d}x \\ &= \int_0^1 \left(x^4 - 2x^5 + x^6\right)\mathrm{d}x \\ &= \frac{1}{5} - \frac{2}{6} + \frac{1}{7} &= \frac{1}{105}\end{aligned}$$

Note: this is also equal to $B(3, 5) = \frac{\Gamma(3)\Gamma(5)}{\Gamma(8)} = \frac{2!4!}{7!} = \frac{1}{105}$ where B is the beta function

4.2 Match 2 (Alex H vs Ilia) @ 1:27:25

4.2.1 Integral 1 — Alex H

$$\int_0^{\pi/2} \min(\sin x, \cos x, \tan x, \sec x, \csc x)\,\mathrm{d}x \tag{4.6}$$

Over the range $\left[0, \frac{\pi}{2}\right]$ we have $\sin x < \tan x < \sec x$ and $\cos x < \csc x$. sin and cos intersect at $\frac{\pi}{4}$ and so our integral reduces to

$$\int_0^{\pi/4} \sin x \, \mathrm{d}x + \int_{\pi/4}^{\pi/2} \cos x \, \mathrm{d}x = 2 \int_0^{\pi/4} \sin x \, \mathrm{d}x = 2 - \sqrt{2}$$

4.2.2 Integral 2 — Ilia

$$\int \frac{1}{x(x^{2023}+1)} \, \mathrm{d}x \tag{4.7}$$

Substitute $y = x^{2023}$

$$\begin{aligned}
\int \frac{1}{x(x^{2023}+1)} \, \mathrm{d}x &= \frac{1}{2023} \int \frac{1}{y(y+1)} \, \mathrm{d}y \\
&= \frac{1}{2023} \int \left(\frac{1}{y} - \frac{1}{y+1} \right) \mathrm{d}y \\
&= \frac{1}{2023} \left[\log y - \log (y+1) \right] \\
&= -\frac{1}{2023} \log \left(1 + \frac{1}{x^{2023}} \right)
\end{aligned}$$

4.2.3 Integral 3 — Alex H

$$\lim_{n \to \infty} \int_0^x \mathrm{d}\sigma_1 \int_0^{\sigma_1} \mathrm{d}\sigma_2 \cdots \int_0^{\sigma_{n-1}} \mathrm{d}\sigma_n \, e^{\sigma_n} \tag{4.8}$$

$$\begin{aligned}
&\int_0^x \mathrm{d}\sigma_1 \int_0^{\sigma_1} \mathrm{d}\sigma_2 \cdots \int_0^{\sigma_{n-1}} \mathrm{d}\sigma_n \, e^{\sigma_n} \\
&= \int_0^x \mathrm{d}\sigma_1 \int_0^{\sigma_1} \mathrm{d}\sigma_2 \cdots \int_0^{\sigma_{n-2}} \mathrm{d}\sigma_{n-1} \, (e^{\sigma_{n-1}} - 1) \\
&= \int_0^x \mathrm{d}\sigma_1 \int_0^{\sigma_1} \mathrm{d}\sigma_2 \cdots \int_0^{\sigma_{n-3}} \mathrm{d}\sigma_{n-2} \, (e^{\sigma_{n-2}} - [1 + x]) \\
&= e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!}
\end{aligned}$$

As we take the limit as $n \to \infty$, the summation becomes the taylor series for e^x leaving us with 0.

4.2.4 Integral 4 — Time Out

$$\int_0^{2023} \left(x^3 - \lfloor x \rfloor x \lceil x \rceil \right) \mathrm{d}x \tag{4.9}$$

$\lfloor x \rfloor + 1$ and $\lceil x \rceil$ only differ when x is an integer (and we aren't taking any other limits), so we can interchange them inside the integral

$$\begin{aligned}\int_0^{2023} \left(x^3 - \lfloor x \rfloor x \lceil x \rceil\right) \mathrm{d}x &= \int_0^{2023} \left[\left(\lfloor x \rfloor + \{x\}\right)^3 - \lfloor x \rfloor \left(\lfloor x \rfloor + \{x\}\right)\left(\lfloor x \rfloor + 1\right)\right] \mathrm{d}x \\ &= \int_0^{2023} \left[\lfloor x \rfloor^2 \left(2\{x\} - 1\right) + \lfloor x \rfloor \left(3\{x\}^2 - \{x\}\right) + \{x\}^3\right] \mathrm{d}x \\ &= \sum_{n=0}^{2022} \frac{n}{2} + \frac{1}{4} \\ &= \frac{2022 \times 2023}{4} + \frac{2023}{4} \\ &= \frac{2023^2}{4}\end{aligned}$$

4.2.5 Integral 5 — Ilia

$$\int_0^{\infty} x^{2023} e^{-x} \, \mathrm{d}x \tag{4.10}$$

Every time we integrate by parts, we get the power of x out front and its power drops by one

$$\begin{aligned}\int_0^{\infty} x^{2023} e^{-x} \, \mathrm{d}x &= \left[-x^{2023} e^{-x}\right]_0^{\infty} + 2023 \int_0^{\infty} x^{2022} e^{-x} \, \mathrm{d}x \\ &= 0 + 2023 \int_0^{\infty} x^{2022} e^{-x} \, \mathrm{d}x \\ &= 2023! \int_0^{\infty} e^{-x} \, \mathrm{d}x \\ &= 2023!\end{aligned}$$

Note: this is the definition for the gamma function $\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} \, \mathrm{d}x$

4.2.6 Integral 6 — Alex H

$$\lim_{n \to \infty} \frac{1}{n!} \int_0^x (x - t)^n e^t \, \mathrm{d}t \tag{4.11}$$

$$\left| \frac{1}{n!} \int_0^x (x - t)^n e^t \, \mathrm{d}t \right| \leq \left| \frac{x^n}{n!} \int_0^x e^t \, \mathrm{d}t \right| \to 0 \text{ as } n \to \infty$$

Therefore, by the sandwich theorem, our answer is 0.

Chapter 5

Finals

5.1 3rd and 4th (Eric B vs Ilia) @ 1:50:00

5.1.1 Integral 1 — Time Out

$$\int_0^\pi \sin 1x \,\mathrm{d}x + \int_0^\pi \sin 2x \,\mathrm{d}x - \int_0^\pi \sin 3x \,\mathrm{d}x - \int_0^\pi \sin 4x \,\mathrm{d}x + \int_0^\pi \sin 5x \,\mathrm{d}x + \int_0^\pi \sin 6x \,\mathrm{d}x - \dots \tag{5.1}$$

The integrals with even coefficients are all 0, leaving

$$\begin{aligned}\int_0^\pi \sin x \,\mathrm{d}x - \int_0^\pi \sin 3x \,\mathrm{d}x + \int_0^\pi \sin 5x \,\mathrm{d}x - \dots &= \frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \dots \\ &= 2\arctan 1 \\ &= \frac{\pi}{2}\end{aligned}$$

5.1.2 Integral 2 — Ilia

$$\int \frac{(x-1)e^x}{x^2} \,\mathrm{d}x \tag{5.2}$$

$$\begin{aligned}\int \frac{(x-1)e^x}{x^2} \,\mathrm{d}x &= \int \frac{e^x}{x} \,\mathrm{d}x - \int \frac{e^x}{x^2} \,\mathrm{d}x \\ &= \int \frac{e^x}{x} \,\mathrm{d}x - \left(-\frac{e^x}{x} + \int \frac{e^x}{x} \,\mathrm{d}x\right) \\ &= \frac{e^x}{x}\end{aligned}$$

5.2 1st and 2nd (Alex H vs Alex G) @ 1:57:25

5.2.1 Integral 1 — Time Out

$$\underbrace{\int\cdots\int}_{n}\log x\,\mathrm{d}x^n \tag{5.3}$$

By repeatedly applying integrating by parts, we get

$$\begin{aligned}
\underbrace{\int\cdots\int}_{n}\log x\,\mathrm{d}x^n &= \underbrace{\int\cdots\int}_{n-1}(x\log x - x)\,\mathrm{d}x^{n-1} \\
&= \underbrace{\int\cdots\int}_{n-2}\left(\frac{x^2}{2}\log x - \frac{x^2}{4} - \frac{x^2}{2}\right)\mathrm{d}x^{n-2} \\
&= \underbrace{\int\cdots\int}_{n-3}\left(\frac{x^3}{3!}\log x - \frac{x^3}{3!\times 3} - \frac{x^3}{3!\times 2} - \frac{x^3}{3!}\right)\mathrm{d}x^{n-3} \\
&= \frac{x^n}{n!}\log x - \sum_{k=1}^{n}\frac{x^n}{n!\times k} \\
&= \frac{x^n}{n!}\left(\log x - \sum_{k=1}^{n}\frac{1}{k}\right)
\end{aligned}$$

5.2.2 Integral 2 — Alex H

$$\int_{-2023}^{2023}\frac{\mathrm{d}x}{1+e^{\pi x}} \tag{5.4}$$

$$\begin{aligned}
\int_{-2023}^{2023}\frac{\mathrm{d}x}{1+e^{\pi x}} &= \int_{0}^{2023}\left(\frac{1}{1+e^{\pi x}} + \frac{1}{1+e^{-\pi x}}\right)\mathrm{d}x \\
&= \int_{0}^{2023}\mathrm{d}x \\
&= 2023
\end{aligned}$$

5.2.3 Integral 3 — Alex G

$$\int_{-\infty}^{\infty} e^{x-e^{2x}}\,\mathrm{d}x \tag{5.5}$$

Substituting $y = e^x$

$$\int_{-\infty}^{\infty} e^{x-e^{2x}}\,\mathrm{d}x = \int_{0}^{\infty} e^{-y^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

The Gaussian integral $\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y$ is fairly well known, however, the proof is quite nice so I'll include it here.

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$$

By switching to polar coordinates, this becomes

$$\int_{0}^{2\pi}\int_{0}^{\infty} re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \pi$$

$$\therefore \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

5.2.4 Integral 4 — Time Out

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\sin(nx)}{x}e^{-x^2}\,\mathrm{d}x \tag{5.6}$$

The intuitive explanation is that $\lim_{n\to\infty}\frac{\sin(nx)}{\pi x}$ is a nascent delta function. That is,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\sin(nx)}{\pi x}f(x)\,\mathrm{d}x = f(0)$$

for sufficiently nice f. You can see that outside the origin, the sin oscillates an infinite number of times in any interval, and so the integrand cancels with itself (as long as f is sufficiently nice). However, $\int_{-\infty}^{\infty}\frac{\sin(nx)}{\pi x} = 1$. This extra area of 1 becomes concentrated at the origin where $\frac{\sin(nx)}{\pi x} \to n$ which diverges to infinity. See Wikipedia for a further information.

A different way of solving the integral is to convert $\frac{\sin(nx)}{x}$ into the integral $\int_{-n}^{n}\frac{1}{2}e^{izx}\,\mathrm{d}z$ and swapping the order of integration

$$\begin{aligned}
\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\sin(nx)}{x}e^{-x^2}\,\mathrm{d}x &= \lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-x^2}\frac{e^{inx}-e^{-inx}}{2ix}\,\mathrm{d}x \\
&= \lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-x^2}\int_{-n}^{n}\frac{1}{2}e^{izx}\,\mathrm{d}z\,\mathrm{d}x \\
&= \lim_{n\to\infty}\frac{1}{2}\int_{-n}^{n} e^{-\frac{z^2}{4}}\int_{-\infty}^{\infty} e^{-(x-\frac{iz}{2})^2}\,\mathrm{d}x\,\mathrm{d}z \\
&= \lim_{n\to\infty}\frac{1}{2}\int_{-n}^{n} e^{-\frac{z^2}{4}}\sqrt{\pi}\,\mathrm{d}z \\
&= \pi
\end{aligned}$$

It is also possible to solve the integral using Fourier transforms or Laplace transforms although this is left as an exercise for the reader.

5.2.5 Integral 5 — Time Out

$$\int_{-3}^{3} \frac{\sqrt{9-x^2}}{1+e^{\sin x}}\,\mathrm{d}x \tag{5.7}$$

$$\begin{aligned}\int_{-3}^{3} \frac{\sqrt{9-x^2}}{1+e^{\sin x}}\,\mathrm{d}x &= \int_{0}^{3} \left(\frac{\sqrt{9-x^2}}{1+e^{\sin x}} + \frac{\sqrt{9-x^2}}{1+e^{\sin -x}}\right)\mathrm{d}x \\ &= \int_0^3 \sqrt{9-x^2}\,\mathrm{d}x\end{aligned}$$

Now we can solve this geometrically as this forms a quarter circle with radius 3, giving us a final answer of $\frac{9\pi}{4}$

5.2.6 Integral 6 — Alex H

$$\int_0^1 \left(\frac{1}{1-\log x} + e^{1-1/x}\right)\mathrm{d}x \tag{5.8}$$

The two terms in the integrand have four important properties.

1. $\frac{1}{1-\log x} \to 0$ and $e^{1-1/x} \to 0$ as $x \to 0$
2. $\frac{1}{1-\log x} = 1 = e^{1-1/x}$ at $x = 1$
3. $\frac{d}{dx}e^{1-1/x} = \frac{e^{1-1/x}}{x^2} > 0 \quad \forall x \in (0,1]$
4. $\frac{1}{1-\log x}$ and $e^{1-1/x}$ are inverses of each other.

This means that the curves $\left(x, e^{1-1/x}\right)$ and $\left(\frac{1}{1-\log y}, y\right)$ form the exact same strictly increasing curve from (0,0) to (1,1). The integrals $\int_0^1 e^{1-1/x}\,\mathrm{d}x$ and $\int_0^1 \frac{1}{1-\log y}\,\mathrm{d}y$ are then the areas between the curve and the x-axis and y-axis respectively. Combined this covers the entire unit square. Therefore, our answer is 1.

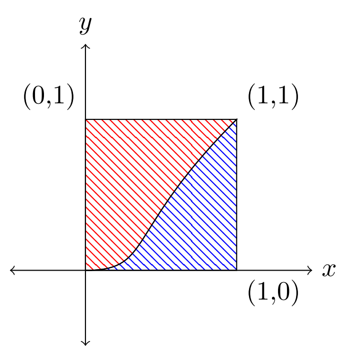


Figure 5.1: the red area represents $\int_0^1 \frac{1}{1-\log y}\,\mathrm{d}y$, while the blue area represents $\int_0^1 e^{1-1/x}\,\mathrm{d}x$

Chapter 6

Final Words

A thanks to Jamie Papworth-Dent for organising problem-setting and writing integrals. A special thank you also goes to Ziheng (Harry) Zhou and Kieran Murphy for writing the vast majority of the 30 integrals. And finally, there's the rest of the MUMS and PSS committee to thank for managing the logistics of the event. We hope to see you at the next integration bee!