Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society

MUMS

In This Edition of Paradox

Features

Articles

Maths in the News:

Is 42 the answer to life, the universe and everything? Apparently, it's actually 72, according to Michael Blastland (BBC, 20/07/2011). The basic idea is that $\log(2)/\log(1+100x)$ is approximately $72/x$ in the region (0,10). Therefore, if we want to know how many years it will take to double something given that it grows at $x\%$ a year, then we can just calculate $72/x$. In fact, the number 70 is even more accurate, since the difference is pretty much within 0.3 over the region (1,10). For more, see http://www.bbc.co.uk/news/magazine-14217443.

Words from the Editor

Welcome to this year's third issue of Paradox, the magazine produced by the Melbourne University Mathematics and Statistics Society (MUMS). A new committee has been elected since the last issue, and with it a new Editor.

This SMO/Open Day issue is a tribute to Stephen Muirhead, Paradox Editor of the past six issues. He was flooded with articles from eager contributors from all over the world, and published issues averaging a never-before-seen 40–50 pages in length. Yours truly has really big shoes to fill!

In this edition, you will discover why Heaven is hotter than Hell, how combinatorial game theory can be used to solve simple Go problems, and why an exodus of New Zealanders to Australia might raise the IQ of both countries. Also featured are the mathematics behind camming devices, an interview with a MUMS alumnus, and how one of the founders of statistics was also a fierce eugenicist.

The problems and solutions that are a regular feature of Paradox have been put on hold while we work on developing an exciting new way to engage you with mathematical and logical curiosities and challenges. Similarly, we hope to develop new regular features over the next few issues and we'd be glad if you could help us out! Perhaps we could have a new resident cartoonist on our books...

As a student-run magazine, Paradox relies on contributions from people like you to make it great! Articles, puzzles, reviews, jokes, or anything else even remotely related to mathematics, statistics, and logic are more than welcome. Just drop by the MUMS room and ask about Paradox or contact us at our new email address: paradox.editor@gmail.com.

— Kristijan Jovanoski

It is a mathematical fact that fifty percent of all doctors graduate in the bottom half of their class.

Words from the President

Welcome back to MUMS!

Rather than have me tell you what has been happening in MUMS over the last semester in this President's report, how about we have a. . .

Surprise test!

1. What colour was our most recently deceased couch? Yes, we have replaced it with a new couch thanks to Gil.

2. Who are you likely to find sleeping in the MUMS room in the morning?

3. If anyone, who are you most likely to find in the MUMS room at 1am?

4. What game has recently come to prominence to rival Settlers of Catan? We still play Settlers too!

5. What is the shape of our new money box? Delicious.

6. What is the answer to everything?

How did you go? Were you able to answer them all? If you don't know any of them, it is imperative that you head over to the MUMS room to find out.

In other news, a new semester has just begun and MUMS is looking as good as ever. Look out for our usual Friday seminars, where we invite guest mathematicians to share with you some interesting and fun mathematics. These talks are aimed at undergraduates so don't be afraid to see what they are all about. Look out for our (often orange) posters in the Richard Berry Building.

We will also be running our annual University Maths Olympics later in the semester, where you get to flex your mathematical muscles as well as your other muscles.

Feel free to come by the MUMS room in G24 (Richard Berry Building) to play some games, do some (fun) math, or just to meet other MUMSians. You might even find out who's sleeping all the time!

— TriThang Tran

Which is Hotter: Heaven or Hell?

The temperature of Heaven can be rather accurately computed:[1]

> The light of the Moon shall be as the light of the Sun and the light of the Sun shall be sevenfold, as the light of seven days.
> — Isaiah 30:26

Thus, Heaven receives from the Moon as much radiation as we do from the Sun, and 7×7 (49) times as much as the Earth does from the Sun, or roughly 50 times in all. The light we receive from the Moon is 1/10,000 of the light we receive from the Sun, so we can ignore that...

The radiation falling on Heaven will heat it to the point where the heat lost by radiation is just equal to the heat received by radiation, i.e. Heaven loses 50 times as much heat as the Earth by radiation. Using the Stefan-Boltzmann Law for Radiation and assuming that the temperature of the Earth is around 298 K (25 °C), the temperature of Heaven is found to be 798 K (525 °C).

While the exact temperature of Hell cannot be computed:

> But the fearful, and unbelieving... shall have their part in the lake which burneth with fire and brimstone.
> — Revelations 21:8

But a lake of molten brimstone means that its temperature must be at or below the boiling point, 444.6 °C. Therefore, Heaven at 525 °C is hotter than Hell at 445 °C.

— Kristijan Jovanoski

> Can you do Division? Divide a loaf by a knife—what's the answer to that?
> — Lewis Carroll, *Through the Looking Glass*.

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[1]From *Applied Optics*, Vol. 11, A14, 1972.

Interview with a MUMS Alumnus

Zhihong Chen graduated in 2008 with a double degree in Science (majoring in mathematics and statistics) and Commerce (taking economics, finance, and actuarial subjects). He currently works as a quantitative analyst at Goldman Sachs in Sydney.

Why did you join MUMS?

Well, I think it was because I knew a lot of people there. When I was in first year, a lot of friends were from my [International Mathematics] Olympiad days, and they were a year ahead. Given that they were always in the MUMS room, it was just easy to hang out with them there. For the first few years they were always there, and then when they left, I had stuck around long enough to make friends with the newer group.

So basically, the MUMS room was a place to hang out with friends?

Yeah, when I had nothing else to do. Towards the latter part of the degree, when people kind of stopped showing up to university, the MUMS room was always the emergency place! I still keep in touch with them, but less so now that I'm working full-time, and in another city.

What kind of things did you do in the MUMS room?

We went through phases, but games would be a part of it. Chess was in for a while, then we switched to Scrabble, and then Settlers [of Catan]. Of course we did some maths too. Sometimes, we'd discuss maths problems from class.

What was your favourite MUMS event?

Probably SMO (School Maths Olympics). It was good fun organising a competition and raising the profile of maths for high school kids.

How else were you involved in MUMS?

I think in first year I helped out with the problem setting for UMO (University Maths Olympics). I didn't really hold any positions in MUMS. I was second year rep when I did a lot of the stuff for the SMO. I don't think you need positions to contribute. There were some reps who never showed up. Ever [laughs]!

What happened after that?

Well, I think in later years you start worrying about jobs, internships and stuff like that, so I found myself less inclined to do anything serious for MUMS in fourth and fifth year compared to the earlier years. But I still hung around and helped where needed.

So tell me about your current job at Goldman Sachs. What kind of things would you do in a typical day?

It's a bit hard to describe, but just about all investment bank graduates are called 'analysts'. This is basically the most junior rank. But a more detailed description of my role is probably a quantitative analyst. I'm the guy who uses maths to solve financial problems. In a nutshell, I structure trade ideas using financial instruments. I also analyse historical data, and sometimes work on mathematical models that we use to price up financial instruments.

Would you say that your maths major has been useful in your current role?

Yes, although I wouldn't say that any particular thing I learnt at university is that useful. It's more about developing the general skill set. Maths students are meant to be good at problem solving, and that's the most useful thing to take into the type of job that I'm in.

So you don't use any particular maths knowledge in your current role?

As a maths major, you learn a lot of things that will never be used in industry. But having learnt it preps you in other ways. Basically, it teaches you to think outside the square and to know how to understand complexity.

What about the usefulness of a maths major for finding jobs generally?

I don't really know. I know that in finance, people with strong maths backgrounds generally have many options if they're resourceful enough to look.

What do you mean by that?

I mean exactly that, Lu. There's no other way to say it [laughs].

Is this what you imagined yourself doing when you were at university? I remember you always thought you'd do a PhD in maths.

I think most people end up in places they didn't expect while they were

still back at university. I think it's good to be open-minded to opportunities and experiences that are offered to you, and not focus on a particular set path. So no, I didn't imagine myself doing my current job while I was at university. But you know, maybe in a few years I'll be doing something else that I couldn't have imagined now.

Now, I happen to know that you have rubbed shoulders with many famous people. I am referring specifically to Julian Assange and Terence Tao. Can you tell us about that?

Well, with Julian, I kind of knew him before he became the person he is today. I didn't really have much to do with him. He was a member of MUMS but never more than an acquaintance. Back in the day, he was very much into maths and problem solving. With Terence, I was his student in one of his graduate classes when I went on exchange to UCLA (University of California Los Angeles). There were around twenty of us in the class. He taught graduate level analysis, a first-year PhD subject.

What was Terence like?

I felt he was relatively shy, but a very organised lecturer. He was brilliant. And he looked very young. His teaching assistant was his PhD student, but you could be forgiven for thinking that Terence was the PhD student.

Did you talk about anything in particular with him?

I had a brief chat with him after I had finished the course. He always opened his office for students who wanted to talk to him, especially those doing his course. I asked him what it was like being an academic versus working in industry. He said that it's really up to the individual and what they want. He mentioned that his brother was working in Google and was pretty happy there. But he thought that academia was a good life as well.

You asked for his autograph—how did you work up the courage to do that?

I just asked him. He does get a few requests here and there. There were a couple of students who came into our class after the last lecture and ambushed him for a photograph. I thought the autograph was enough. I didn't feel the need to prove that I was in his class.

— Lu Li

Biography: Karl Pearson (1857–1936)

Students the world over have the English mathematician Karl Pearson to thank for founding the world's first statistics department in 1911 at University College London. His many contributions to statistics include the correlation coefficient, p-value, and chi-square test. As a fellow of the Royal Society and protégé of the polymath Francis Galton,[1] he believed that his mentor, and not Charles Darwin, would be remembered as the greatest grandson of the scientist Erasmus Darwin.

When the 23 year-old Albert Einstein started a study group, the Olympia Academy, with two of his younger friends in 1902, the first book they read was Pearson's *The Grammar of Science*, whose ideas on relativity significantly influenced the theories Einstein would later put forward. Pearson claimed that the laws of nature were relative to the observer and that natural processes merely seemed irreversible as a purely relative conception of the human mind. He also claimed that someone who travels faster than light would see a time reversal and that if they did travel at the exact same speed, they would see an eternal now, that is, an absence of motion.

Born Carl Pearson in London, he inadvertently changed the spelling of his first name to Karl when he enrolled in the University of Heidelberg but later stuck with that name. He took his mentor's ideas on eugenics much further and more aggressively, applying Social Darwinism to entire nations and openly advocating war against what he deemed to be inferior races:

> *No degenerate and feeble stock will ever be converted into healthy and sound stock by the accumulated effects of education, good laws, and sanitary surroundings. Such means may render the individual members of a stock passable if not strong members of society, but the same process will have to be gone through again and again with their offspring, and this in ever-widening circles, if the stock, owing to the conditions in which society has placed it, is able to increase its numbers.[2]*

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[1] See *Great Lives* in Paradox Issue 1, 2011 for more information on Galton's life.
[2] Introduction, *The Grammar of Science*.

To him, of course, the solution was obvious:

> *History shows me one way, and one way only, in which a high state of civilization has been produced, namely, the struggle of race with race, and the survival of the physically and mentally fitter race. If you want to know whether the lower races of man can evolve a higher type, I fear the only course is to leave them to fight it out among themselves, and even then the struggle for existence between individual and individual, between tribe and tribe, may not be supported by that physical selection due to a particular climate on which probably so much of the Aryan's success depended. . .*[3]

Yet interestingly, he was regarded in his lifetime as a freethinker and a socialist, giving lectures on Karl Marx and issues like women's suffrage. He refused a Knighthood because of his commitment to socialism and his ideals. But then again, it seems that some animals are more equal than others. . .

— Kristijan Jovanoski

'Every minute dies a man, Every minute one is born;' I need hardly point out to you that this calculation would tend to keep the sum total of the world's population in a state of perpetual equipoise, whereas it is a well-known fact that the said sum total is constantly on the increase. I would therefore take the liberty of suggesting that in the next edition of your excellent poem the erroneous calculation to which I refer should be corrected as follows: 'Every moment dies a man, And one and a sixteenth is born.' I may add that the exact figures are 1.067, but something must, of course, be conceded to the laws of metre.
— Charles Babbage, in a letter to Alfred, Lord Tennyson about a couplet in his *The Vision of Sin*.

[3]*National Life from the Standpoint of Science*, 1905.

Let's Play Some Go

'Cut first think later!'

Go is an ancient board game originating from China with a recorded history from over 3000 years ago, although it probably dates back even further to over 4000 years ago. It is a beautiful game of skill, intuition, and elegance.

Its name comes from the Japanese name *igo*, literally meaning 'surrounding game'. Similarly, the Chinese name *wei-qi* literally means 'surrounding chess'. It is not surprising then that the aim of the game is to surround things. Two players, Black and White, take alternate turns placing stones on the intersections of a 19 × 19 board, aiming to accumulate points by surrounding either empty intersections (territory) or enemy stones (prisoners).[1]

Now that we know what Go is, here is a simple Go problem:
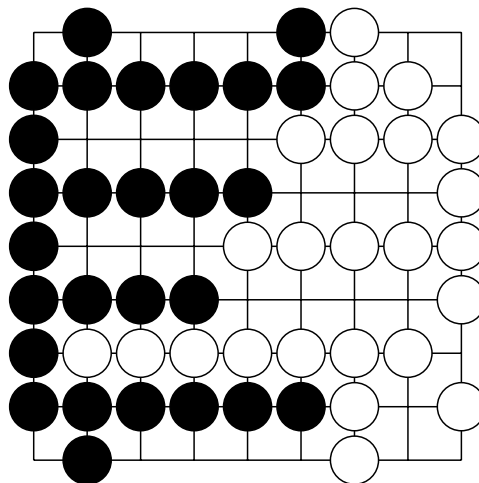
Who wins?

While we can answer this by just reading ahead and counting, let us do it another way with some combinatorial game theory. We will see that the amount of 'looking ahead' will be somewhat reduced. For larger and more complicated examples, this will be even more important.

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[1]For the rules, see http://www.gokgs.com/tutorial/. If you actually want to play the game, see http://gogameguru.com/learn-go-easy-way-go-game-1/. Alternatively, come to the MUMS room and ask me how to play!

Solving the problem

The first thing we might notice is that the problem naturally splits off into disjoint regions. We identify them by shading:

Separating the board
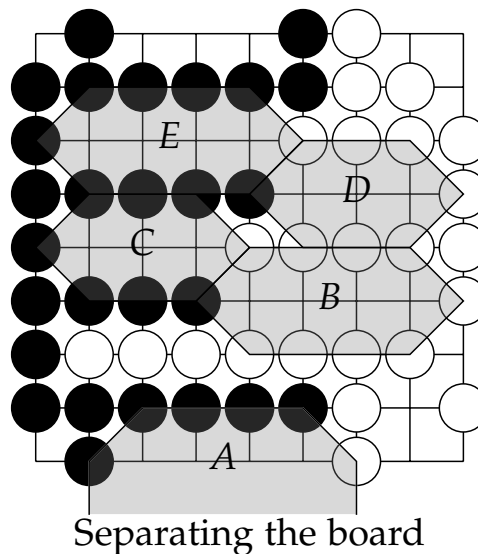
The goal now is to somehow solve each region individually and 'sum' them up afterwards. Combinatorial game theory (CGT) is the basic tool in this game. It was invented in the 1960s-70s by Elwyn Berlekampe, John Conway, and Richard Guy specifically to deal with situations where we can break up a game into simpler games. For us, a *combinatorial game G* (or just a game) is an object of the form $G = \{G^L|G^R\}$ where G^L and G^R are sets of games.

Argh, this definition is recursive! You might say that we cannot define a game in terms of games. However, there is no real problem, since we always have empty sets, which yield the initial game $\{|\}$, with which we begin the construction of other games. We shall call this game $\{|\} := 0$.

example$\{\ 0\ |\ \}\{\ |\ 0\}\{0\ |\ 0\}$ *are all games*.

Just to simplify things, we shall also insist that our game is finite and does not loop.

To interpret these objects as 'real' games, we imagine two players, Left and Right, playing a game G. On their turn, Left chooses a game in G^L with which to replace the game G. We often refer to the games in G^L as Left options. Similarly, G^R are Right options. A player is said to *lose* when it is their turn and they are unable to move. That is, their set of options is empty.

We can also add games!

The way we define additions of two games is to play them side by side. When it is their turn, each player can choose options from either board. Symbolically, if $G = \{G^L | G^R\}$, and $H = \{H^L | H^R\}$, we define their *sum* to be:

$$G + H = \{(G^L + H), (G + H^L) | (G^R + H), (G + H^R)\}.$$

A game is called a *zero game* if it is a second mover win. In other words, no one wants to move first in a zero game. We can also define the *negative* of a game. Given a game G, we can define $-G$ to be the game $\{-G^R | -G^L\}$, i.e. the game G where Left and Right swap positions. In a game of Go, this would happen if we swapped the white and black stones.

Fortunately with these definitions, $0 = \{|\}$ is a zero game—whoever moves first cannot make a move and so loses. Moreover, $G + (-G)$ is also a zero game. To see that the second player can always win, just observe that they can play the *copycat* strategy. Whatever the first player does on G $(-G)$, the second can copy on $-G$ (G). In this way, the second player always has a move, and since the game is finite the first player eventually loses.

We shall say that two games H and G are equal if $H - G$ is a zero game. In particular, this lets us conveniently write $G - G = G + (-G) = 0$.

For those of you who have read Mark's previous Paradox article on *The monster*,[2] you will realise that what we are defining is a group! Once one checks that the addition is associative, then the set of games under the equivalence relation = forms an abelian group.

The study of combinatorial games has also been used to solve mathematical games such as Nim and Hackenbush. There has even been progress made on more popular games such as Go and Chess, although there are some difficulties, particularly with the latter since it is not readily seen as a combinatorial game. Furthermore, in his book *On Numbers and Games*, Conway studies a subclass of games called *Surreal Numbers*. This subclass has some amazing properties: it is a well-ordered field that contains, for example, real numbers, ordinals, as well as infinitesimals.

[2]See Mark Kowarsky's *Group theory could save your life* in Paradox Issue 1, 2010.

Solving our original problem

Let us get back to the problem at hand. In order to view Go as a combinatorial game, we need to think of it as a last mover win game. One can do this by realising that the territory one surrounds essentially gives one extra move per point of territory. If we adopt Chinese rules for counting, i.e. we count stones as well as empty intersections, then Go essentially becomes a combinatorial game.[3]

While the board is initially far too open to break up nicely into individual games, once we reach the late endgame of Go, we often get many separate disjoint regions. It is now that one might hope to apply the power of CGT, although even now it is often difficult and an area of ongoing research.

Our problem is actually a very simple endgame problem that nicely yields to CGT. In fact, in our case, we do not even need most of the language that we developed earlier.

Let us have another look at the board:
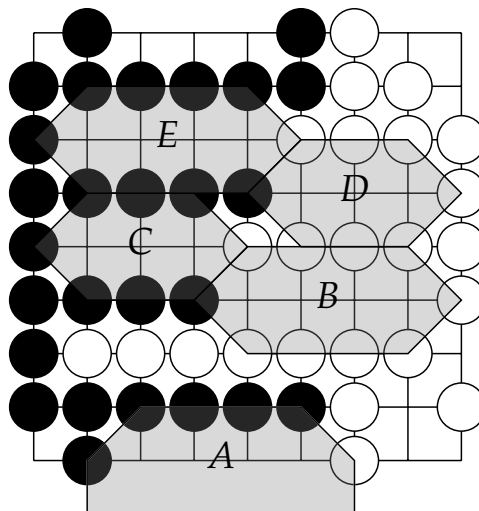
We can now think of each region as its own game. The unshaded regions are already secure territory. In particular, Black has 5 secured points of territory, while White has 7. We now need to work out what happens in the open corridors and this is where CGT comes in. Observe $D = -C$ as a game, since it is essentially the same position with Black and White reversed. Similarly,

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[3]For more on how this works, read the book *Mathematical Go: Chilling Gets the Last Point* by Berlekampe and Wolfe.

$B = -A$. Therefore, the game simplifies as:

$$A + B + C + D + E = 0 + 0 + E = E.$$

Since we do not need to worry about what happens on 0 (no player wants to move first in a zero game), what is left is to decide what happens on E. If Black moves first, then Black closes the corridor and secures an additional 3 points. If White moves first, then White is able to make one approach move before Black closes the corridor, meaning that Black only secures an additional 2 points.

Thus, if White moves first, White can draw. But if Black moves first, then Black can win by one point!

Beyond this example...

For a more rigorous introduction to mathematical Go, a good way to start is to read *Mathematical Go: Chilling Gets the Last Point* by Berlekampe and Wolfe, where they solve many more Go endgame problems and situations.

— TriThang Tran

> Three recent graduates are invited for an interview: one has a degree in pure mathematics, another one in applied mathematics, and the third one in statistics. All three are asked the same question: 'What is one third plus two thirds?'
> The pure mathematician: 'It's one.'
> The applied mathematician takes out his pocket calculator, punches in the numbers, and replies: 'It's 0.999999999.'
> The statistician: 'What do you want it to be?'

Mathematics in Climbing

I recently visited www.totemcams.com for camming devices. Impressively, this site provides the mathematics behind their devices, which I shall now explain.

Camming devices

A camming device is an item of active protection gear for trad climbing. Unlike traditional pitons or nuts, it can be placed on parallel cracks with one hand. Introduced in the mid-1970s, it opened many traditionally unprotected routes to new generations of climbers and it is now a must-have on any climber's rack.
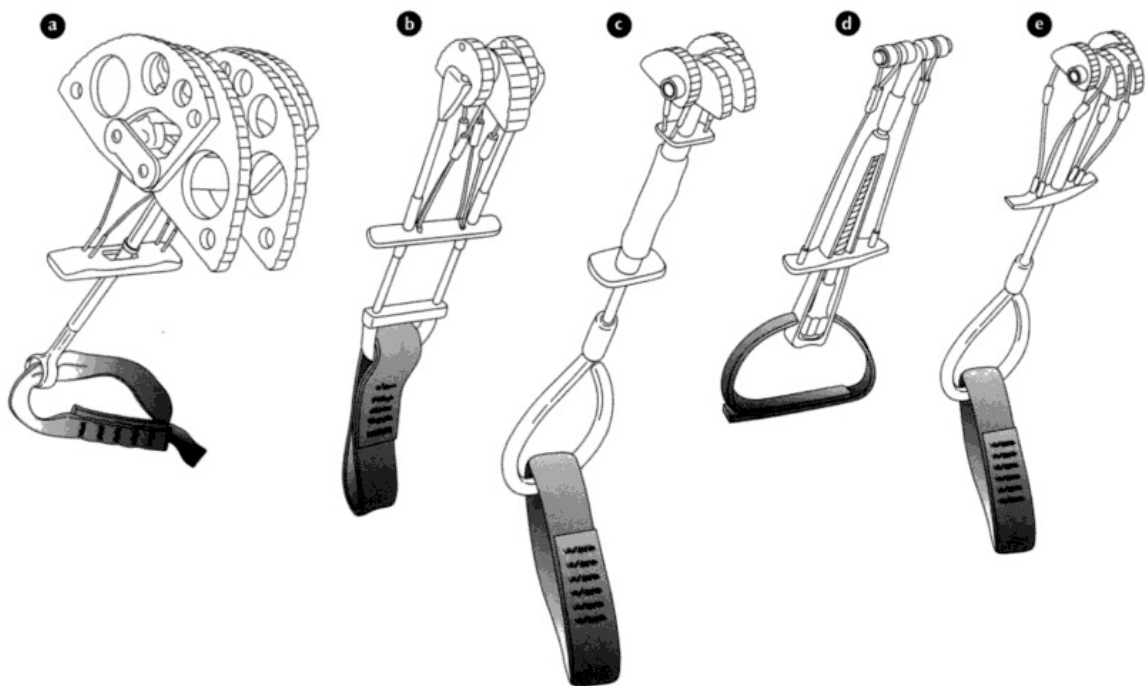
Figure 1: Some traditional camming device designs.

Most designs have four blades or lobes called cams that are mounted on one or two axles so that a pull on an axle spreads the cams apart.[1] Each cam can be retracted by pulling on a trigger on a single stem that is attached to the axle as in Figure **??**. By pulling on a trigger, the profile of the device

[1]Cox SM, Fulsaas K. *Mountaineering: Freedom of the Hills*. Seattle: Mountaineers Books, 2003.

becomes more narrow, allowing one to place it in a crack. Once released, each cam expands to fill the gap. Therefore, when a climber falls on the device, the force pulls on the central stem and spreads the four cams apart. This increases the friction force on the rock and prevents a ground fall.

The cam shape

During such a fall, the force is transferred to the axle of each cam. By ensuring that the position of the axle is above its contact point with the wall as in Figure **??**, there is enough frictional force to hold the fall. This depends only on the shape of each cam. The angle α shown in Figure **??** is called the *cam angle*.

Figure 2: Parametrising the distance between the contact point and the centre of the axle.

It turns out that α remains constant, regardless of the angle θ between the cam's actual contact point and the minimal contact point. By using polar coordinates, we may parametrise the distance between the contact point and the centre of the axle as a function of θ by $r(\theta)$. We shall demonstrate that $r(\theta) = C \exp(\tan(\alpha)\theta)$.

Consider increasing θ by a very small $d\theta$. The triangle ABA' can be approximated to a right-angle triangle, where $|\overline{OB}| = |\overline{OA}|$. We have:

$$|\overline{A'B}| = dr = dr/d\theta \, d\theta \quad \text{and} \quad |\overline{AB}| = r(\theta) \, d\theta$$

So,

$$\tan(\alpha) = \frac{1}{r}\frac{dr}{d\theta}.$$

Since α is fixed, solving this differential equation gives a general solution of $C \exp(\tan(\alpha)\theta)$. This shape is known as a logarithmic spiral and is now used by most major manufacturers.
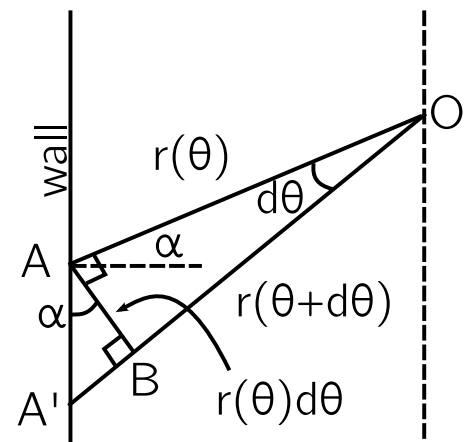
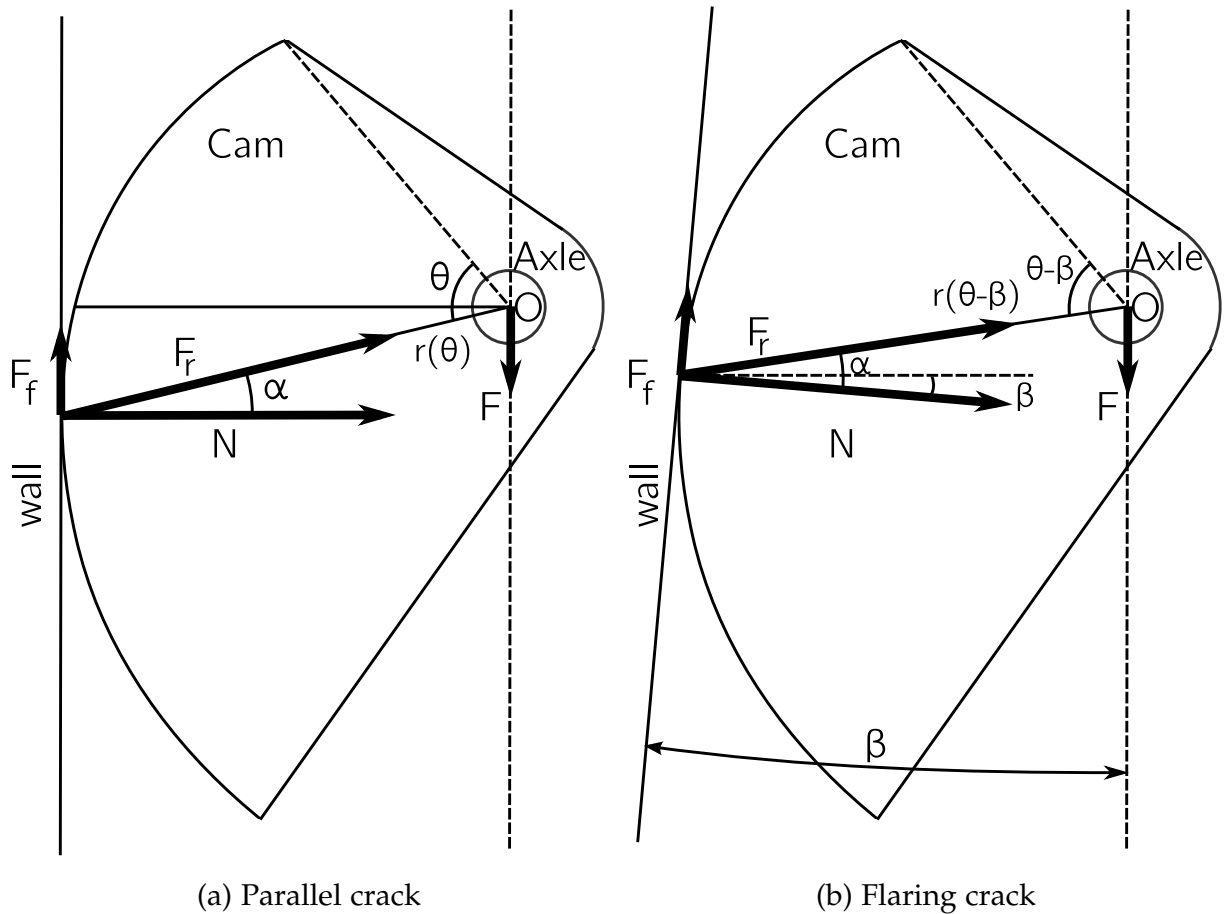(a) Parallel crack   (b) Flaring crack

Figure 3: Schematic diagram of a cam.

Restrictions on cam angle

The cam angle is constant regardless of θ. So, we may choose α such that the frictional force F_f can hold a climber during a fall. Let F be the force loading on the axle of each cam. We shall assume that the load spreads evenly across the four cams and that F is parallel to the gravitational force. Let F_f be the frictional force exerted by the wall and N be the normal force. By the torque balance with the centre on O, we have

$$F_f r(\theta) \cos(\alpha) = N r(\theta) \sin(\alpha)$$

And thus, $F_f = N \tan(\alpha)$. If μ is the coefficient of friction between the rock and the cam, then to avoid sliding, $F_f \leq \mu N$. This implies that the cam angle must be chosen such that $\tan(\alpha) \leq \mu$. Generally, the cam angle is chosen to be around $13.75°$. Now, the reaction force F_r from the wall is the sum of F_f

and N. Furthermore, by balancing the vertical forces, $F = F_f = F_r \sin(\alpha)$, so $F_r = F / \sin(\alpha)$. Notice that the cam angle is small, meaning that N / F_f is quite large. As a result, a little force on the camming device can generate a strong reaction force from the wall.

Flaring cracks

Suppose that the crack is not quite parallel, but flares out a little at the bottom as in Figure **??**. Let β be the angle between the wall and the vertical direction. By balancing torque, $F_f = N \tan(\alpha)$. The difference between the two scenarios is in the reaction forces. First, balancing forces on the whole cam gives:

$$F + N \sin(\beta) = F_f \cos(\beta). \tag{1}$$

After substituting $F_f = F_r \sin(\alpha)$ and $N = F_r \cos(\alpha)$ in (**??**), we get:

$$F = F_r \cos(\beta) \sin(\alpha) - F_r \cos(\alpha) \sin(\beta) = F_r \sin(\alpha - \beta).$$

Therefore, as the angle β approaches α, the reaction force F_r approaches infinity. In practice, this means that either the camming device or the rock breaks, which is not great news for the falling climber either way. So, is there an alternative design that allows one to place the cam in a highly-flaring crack?

Totem cams

Recently, a manufacturer produced a new design which they claim to be more effective against highly flared cracks. Instead of having a central stem taking the force of each fall, they have a wire connected to each cam to take the load as in Figure **??**. A wire follows the contour of a cam before exiting downwards. By designing the shape of the back, they can change the reaction force from the wall.

There are two active areas in each cam. Their shapes are logarithmic spirals, where one can be obtained from the other by a $180°$ rotation around the centre of the axle and then applying an appropriate scaling. Let each active region be parametrised by

$$B(\theta_B) = b \cdot \exp(\tan(\alpha)\theta_B) \quad \text{and} \quad C(\theta_C) = c \cdot \exp(\tan(\alpha)\theta_C). \tag{2}$$

We shall assume the following:

1. The load is perfectly aligned and distributed evenly to all four cams.
2. The force loaded on each cam is in a downward direction.
3. The angle θ_B is equal to the angle θ_C.

First, by balancing force on the whole cam, $F_f = F$. By then balancing torque centred on the axle, we get:

$$F_f \cos(\alpha)B(\theta_B) + F\cos(\alpha)C(\theta_C) = N\sin(\alpha)B(\theta_B)$$

$$F_f \cos(\alpha)(B(\theta_B) + C(\theta_B)) = N\sin(\alpha)B(\theta_B)$$

$$F_f \cos(\alpha) \cdot (b+c) \cdot \underline{\exp(\tan(\alpha)\theta_B)} = N\sin(\alpha) \cdot b \cdot \underline{\exp(\tan(\alpha)\theta_B)}$$

$$F_f = N\frac{b\tan(\alpha)}{b+c} = N\frac{\tan(\alpha)}{1+c/b}.$$

The second step is just a substitution with equation (**??**). Again, to avoid sliding, $F_f \leq \mu N$, where μ is the friction coefficient. This requires that $\tan(\alpha)/(1+c/b) \leq \mu$. For simplicity, we may interpret this as a new cam angle. We shall define an equivalent cam angle by:

$$\tan(\alpha_e) = \frac{\tan(\alpha)}{1+c/b}.$$

Note that since tan is an increasing function between 0 and $\pi/2$, $\alpha_e < \alpha$. Now, the reaction force is just $F_r = F/\sin(\alpha_e)$.

Totem cams on flaring cracks

Let β be defined as before. Note that angle θ_B is not the same as θ_C anymore. In fact, $\theta_B = \theta_C - \beta$ (exercise). The force balance on the cam gives $F_f \cos(\beta) = F + N\sin(\beta)$. By the torque balance,

$$F_f \cos(\alpha)B(\theta_B) + F\cos(\alpha)C(\theta_C) = N\sin(\alpha)B(\theta_B)$$

$$F_f B(\theta_B) + (F_f \cos(\beta) - N\sin(\beta))C(\theta_C) = N\tan(\alpha)B(\theta_B)$$

$$F_f(B(\theta_B) + \cos(\beta)C(\theta_C)) = N(\tan(\alpha)B(\theta_B) + \sin(\beta)C(\theta_C))$$

Again, to avoid sliding,

$$\frac{\tan(\alpha)B(\theta_B) + \sin(\beta)C(\theta_C)}{B(\theta_B) + \cos(\beta)C(\theta_C)} \leq \mu.$$

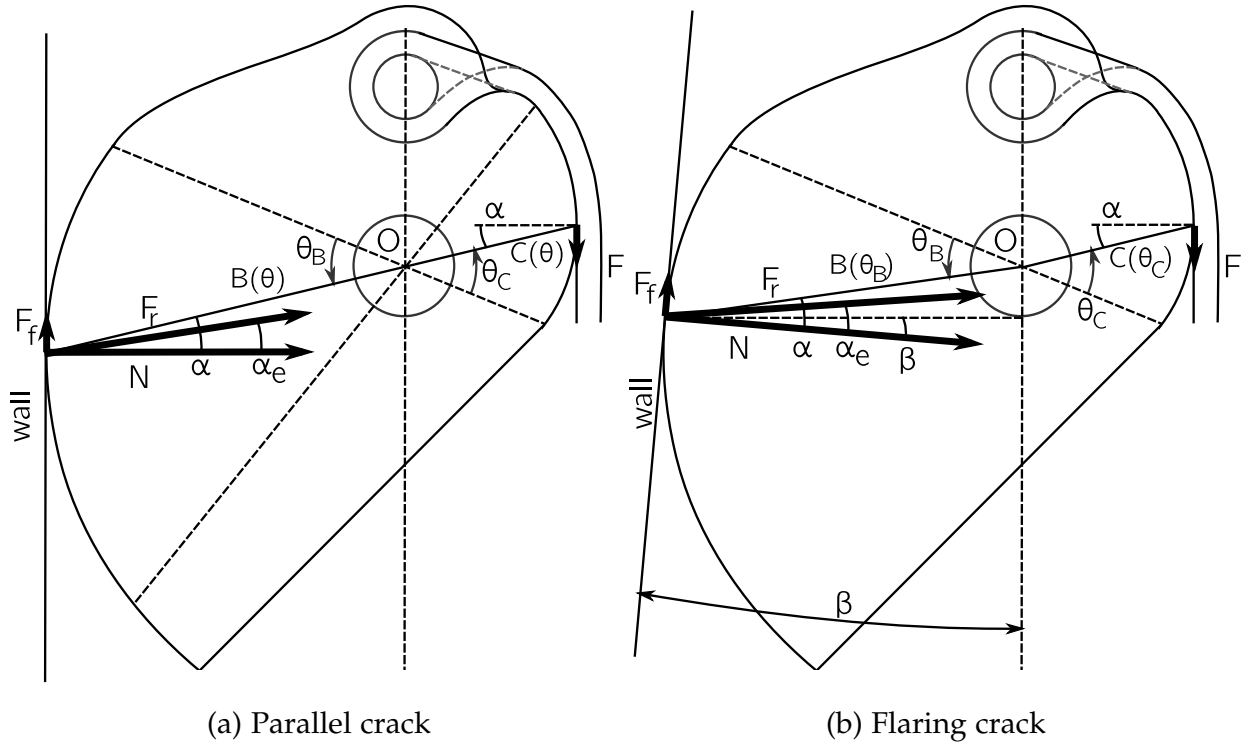(a) Parallel crack　　　　　　　　　　　　(b) Flaring crack

Figure 4: Schematic diagram of a totem cam.

Then substituting equation (**??**) to get:

$$\frac{\tan(\alpha)be^{\tan(\alpha)\theta_C-\beta}+\sin(\beta)ce^{\tan(\alpha)\theta_C}}{be^{\tan(\alpha)(\theta_C-\beta)}+\cos(\beta)ce^{\tan(\alpha)\theta_C}}=\frac{\tan(\alpha)be^{-\tan(\alpha)\beta}+\sin(\beta)c}{be^{-\tan(\alpha)\beta}+\cos(\beta)c} \quad (3)$$

At $\beta=0$ (parallel wall), equation (**??**) gives $\tan(\alpha)/(1+c/b)\leq\mu$. At $\beta=\alpha$, equation (**??**) gives:

$$\frac{\tan(\alpha)be^{-\tan(\alpha)\alpha}+\sin(\alpha)c}{be^{-\tan(\alpha)\alpha}+\cos(\alpha)c}=\tan(\alpha)\frac{be^{-\tan(\alpha)\alpha}+\cos(\alpha)c}{be^{-\tan(\alpha)\alpha}+\cos(\alpha)c}$$

$$=\tan(\alpha)\leq\mu$$

A careful differentiation will yield the fact that the expression (**??**) is an increasing function between $\beta=0$ and $\beta=\alpha$. So, the minimum required friction coefficient to avoid sliding increases from $\tan(\alpha)/(1+c/b)$ to $\tan(\alpha)$ as β increases. The reaction force is:

$$F_r=\frac{F}{\sin(\alpha_e-\beta)} \quad\text{where}\quad \alpha_e=\arctan\left(\frac{\tan(\alpha)be^{-\tan(\alpha)\beta}+\sin(\beta)c}{be^{-\tan(\alpha)\beta}+\cos(\beta)c}\right).$$

Final remarks

While the totem design is great for lowering the cam angle at low β, its cam angle increases to that of the traditional design as flaring angle increases. The manufacturer chose $\alpha = 20.35°$, and the ratio c/b so that in a parallel crack, $12.52° < \alpha_e < 13.13°$. This design allows for a theoretical placement in a $40.7°$ flared crack. Note that in this crack, the cam angle is $20.35°$, which is considered to be at the maximal end (Black Diamond Camalot's cam angle is about $14.7°$).

The benefit of this design is clear when we compare it against a traditional camming device with a cam angle of $20.35°$. At a lower flared angle, Totem cams require a lower friction coefficient than those of other competitors. However, if one compares it with, say, Black Diamond's Camalot C4, which has a cam angle of $14.7°$, we note that it sacrifices its required minimum friction coefficient at higher flaring angles for a larger maximum flaring angle. In practice, one should avoid placing cams in a flaring angle for obvious reasons anyway. Thus, it depends on what one values more in a camming device—a higher maximum flaring angle or a higher minimum friction coefficient requirement.

Final notes on the calculation:

1. We assume that the force loaded on each cam is downward. This angle depends on θ_B.
2. The angle θ_B is unlikely to equal θ_C. The means that the ratio $C(\theta_C)/B(\theta_B)$ will vary. The manufacturer suggests that it can vary between 0.67 (fully closed) and 0.59 (fully open).
3. The load never spreads evenly to each cam. The force F is the force on that cam. It is not the same as T in their calculation.

— Tharatorn Supasiti

'Students nowadays are so clueless', a mathematics professor complained to a colleague. 'Yesterday, a student came during my office hours and wanted to know if General Calculus was a Roman war hero...'

Statistical Quirks: How to Avoid Becoming Simpson's Donkey

There are three kinds of lies: lies, damned lies and statistics.

— Benjamin Disraeli

The former Prime Minister of New Zealand, Robert Muldoon, once famously quipped that the exodus of New Zealanders moving to Australia in search of work didn't concern him, since such immigration 'raises the average IQ in both countries.'

This joke is funny, but it is also remarkably subtle;[1] to arrive at the implied punchline that 'New Zealanders are smarter than Australians,' the non-statistician needs an unconscious appreciation of a statistical quirk known as the Will Rogers phenomenon as well as an intuitive sense of the conditions that are necessary for the phenomenon to occur.

So what is the Will Rogers phenomenon? Put simply, it is the effect by which migration of data between data sets can, in certain circumstances, increase the average of the data in both sets. Consider the following:

$$A = \{1, 2, 3, 4, 5\}$$

$$B = \{3, 4, 5, 6, 7\}$$

If the data point 4 migrates from set B to set A, then the average of set A will increase from 3 to 3.166 while the average of set B will also increase from 5 to 5.25. Hence, the migration has increased the average in both sets.

This phenomenon will occur whenever the migrating data point lies below the average of set B (so that its removal raises the average of set B) but above the average of set A (so that its inclusion raises the average of set A). So a necessary condition for the phenomenon to occur is that the average of set B (the intelligence of New Zealanders) is greater than the average of set A (the intelligence of Australians). Hence the implied punchline of the joke.

The fact that so many people find this joke funny speaks volumes about humans' innate ability to correctly process statistical information, even when on first glance such a conclusion may seem counterintuitive.

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[1]Especially for a politician.

Yet the Will Rogers phenomenon is but one of many statistical quirks that appear with regularity in the presentation of data, some of which can be highly misleading if the quirk is misunderstood.[2] This article will look at one such notable quirk—the Simpson reversal—whose appearances in data continues to confound laymen and statisticians alike.[3]

The Simpson reversal

Consider the following scenario.[4] You are the Vice-Chancellor of the University of Melbourne, and are being confronted by an angry group of women's rights campaigners who have just discovered that the university's acceptance rate for prospective students is 55% for males but only 45% for females. With the threat of a gender discrimination lawsuit ringing in your ears, you phone up the various faculties of the university to find out who is responsible for such a discrepency.

First, the Law Faculty, whose Dean assures you that within their Faculty the acceptance rate for females is actually higher (55%) than that for males (45%). Next, the Science Faculty, whose Dean makes the same assertion (33.2% to 28.6%). Finally, the Arts Faculty, whose Dean, in turn, claims that their Faculty is accepting comparatively more females than males (66.7% to 61.9%).

So what's going on? Is it possible that each of the faculties is accepting, as a percentage of applicants, more females than males, but that the overall acceptance rate favours males?

Yes, it is. Consider the following table, showing the percentage acceptance rates alongside hypothetical raw acceptance/total applications data:

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[2]Another common quirk involves the fact that the addition of data points to a set can produce contrasting effects on the median and the mean of the data set: for instance, it can simultaneously raise the mean of the data while lowering the median. Interestingly, the misleading effect of this quirk is also often demonstrated by another joke: 'Bill Gates walks into a bar. The average income of the people in the bar goes up 100,000%.'

[3]The Simpson reversal is often called 'Simpson's paradox' but, for reasons that will hopefully become apparent, this name is less appropriate.

[4]Based loosely on a real life situation that occured at the University of California, Berkeley: see, e.g. the references at http://en.wikipedia.org/wiki/Simpson%27s_paradox.

	Males	Females
Law	18/40 = 45%	22/40 = **55%**
Science	10/35 = 28.6%	46/143 = **32.2%**
Arts	120/194 = 61.9%	44/66 = **66.7%**
Overall	148/269 = **55%**	112/249 = 45%

Each faculty is indeed favouring females over males, yet when the data is aggregated this trend is reversed. This is an example of a 'Simpson reversal': the trend in aggregated data can actually be the reverse of trends seen when the data is split into two or more subgroups.

At first glance, a Simpson reversal may seem like an unusual quirk, but in fact it can and does appear in many different real-world situations. Consider the following:

1. It is possible for Mitchell Johnson to finish the first innings of a test match with a better bowling average than Brett Lee, and then to repeat this feat in the second innings of the same match, but to still end up finishing the match with a lower bowling average than Brett Lee.[5]

2. It is possible for the overall unemployment rate to be lower today than in 1980, but for the unemployment rate among those with university education, those without university education but with a high school certificate, and those without high school certificates, to be higher today than in 1980.[6]

3. It is possible for Jetstar to have a better flight-delay record than Tiger in every airport in which they operate together, but for the aggregate data to show that, overall, Tiger has the better flight-delay record.

This Simpson reversal is troubling, even for professional statisticians, because it demonstrates just how difficult it is to summarise statistical data in a way that is not potentially misleading.

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[5]The interested reader might like to search for an occurence of such a phenomenon, and let Paradox know about it. In the baseball context, a famous example comes from the batting averages of Derek Jeter and David Justice: in both 1995 and 1996, Justice had the better average, but when these years are considered together, Jeter had the better average.

[6]This is precisely the case at present in the United States, where much attention has been placed on whether the ongoing recession today is worse than the the recession of 1980: Tuna C. When Combined Data Reveals the Flaw of Averages. *Wall Street Journal*; [updated 2009 Dec 2; cited 2011 Jul 21]. Available from: http://online.wsj.com/article/SB125970744553071829.html.

Take the University of Melbourne scenario, and imagine that a sex discrimination case was brought against the university and eventually proceeded to trial. Which data set should the judge take into account when determining the case: the data split by faculty, the aggregated data, or a combination? What if the judge deferred their opinion to a statistician, and wanted a one-word answer to the question: 'Does the data tend to suggest that the university has a selection bias in favour of males?'

Similarly:

1. In the cricket scenario, who was the better bowler over the course of the test match: Mitchell Johnson or Brett Lee?

2. In the unemployment scenario, is it disingenuous for politicians to claim that the unemployment rate is lower today than in 1980, despite the fact that across all education levels in society the chances of being out of work are actually higher now than in 1980?

3. In the airline scenario, which is the better airline for travellers wishing to avoid delays: Jetstar or Tiger?

Such questions go right to the heart of statistical inference, and any uncertainty about their resolution cannot be tolerated. That is why an appreciation of Simpson reversals are so critical to understanding data presented in terms of proportions or percentages.

The 'paradox' explained

If drawing information based on a Simpson reversal can be troubling, the same is not true of the maths that underpins it, for on a superficial level the reversal is extremely easy to understand. The crux lies in the fact that when proportions are aggregated they are not simply summed, but instead are weighted according to the quantity of data points underlying the proportion.

The fact that the Deans reported only the percentage acceptance figures masked the true story. In fact, the raw acceptance/total applications data suggested that the two sexes did not apply to the faculties in a uniform manner. For instance, a significantly higher number of females applied to the Science Faculty than did their male counterparts. The Science Faculty also

happened to be the faculty with the highest overall rejection rate. So females displayed a preference for a highly competitive faculty, and hence their overall rejection rate was comparatively raised.[7]

Similarly:

1. Brett Lee must have taken more wickets than Mitchell Johnson in the innings where the bowling figures were better for both bowlers, thus reducing his overall bowling average.

2. Compared to 1980, more people now have an university education, and the unemployment rate is lowest among this group. Thus, despite the unemployment rate increasing in every category, the increased weight of the 'university education' sector produces an aggregate downward trend.

3. Tiger Airlines must be flying more planes out of airports where less delays occur. This pushes its overall flight-delay figures downwards.

This explanation of the Simpson revesal makes immediate sense when translated into a context where discrepancies in weighting occur naturally. Consider Australia's tax system, which charges a higher tax rate for those with higher incomes. Given the effects of nautral inflation, it is intuitive that a government could, over a 10-year period, decrease the tax rate in each of the income ranges while reaping a higher overall tax rate from the population.[8] The key is the change in relative weightings: as incomes rise, more people are pushed into higher income regions. While this might colloquially be called 'bracket creep', it can also be understood as a manifestation of a Simpson reversal.

To better understand the reversal, it is useful to write the statements of the reversal in algebraic terms (for simplicity, we'll assume that in our university example there were only two faculties). The statements are:

$$\frac{a}{b} < \frac{A}{B} \qquad \frac{c}{d} < \frac{C}{D}$$

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[7]In this context, a Simpson reversal could have occurred either if females showed a slight preference for a faculty that had significantly higher rates of rejection, or had a strong preference for a faculty that had a slightly higher rate of rejection.

[8]With the exception, of course, of the rate in the tax-free income range, which can only be kept constant.

and

$$\frac{a+c}{b+d} > \frac{A+C}{B+D}$$

where a and b are the acceptance/total applications data for males for one faculty, c and d for another; and the upper case letters the same data for females.

To see how such an arrangement is possible, we can visualise these proportions as vectors (a,b), (A,B), (c,d) and (C,D) with (a,b) being steeper than (A,B), (c,d) being steeper than (C,D), but $(A+B,C+D) = (A,B) + (C,D)$ being steeper than $(a+b,c+d) = (a,b) + (c,d)$:
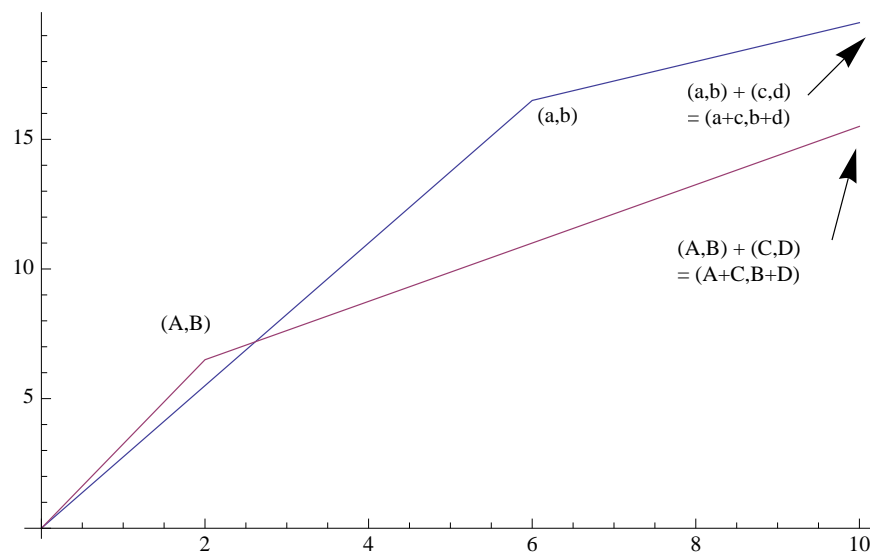

Figure 1: A Simpson reversal depicted in vector form.

Interpreting the data as probabilities can also assist in explaining the reversal. If we let the two faculties be $F1$ and $F2$, A indicate acceptance, and M and M' indicate male and female, then the reversal can be stated as:

$$\mathcal{P}(A|M \cap F1) < \mathcal{P}(A|M' \cap F1),[9]$$

$$\mathcal{P}(A|M \cap F2) < \mathcal{P}(A|M' \cap F2),[10]$$

$$\mathcal{P}(A|M) > \mathcal{P}(A|M').[11]$$

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[9]Interpreted as the probability of acceptance into Faculty 1 given that you are a male is less than the probability of acceptance into Faculty 1 given that you are a female.

[10]Likewise for Faculty 2.

[11]Interpreted as the probability of acceptance into both faculties given that you are a male is greater than the probability of acceptance into both faculties given that you are a female.

Then, using the total law of probability, we can expand:

$$\mathcal{P}(A|M) = \mathcal{P}(A|M \cap F1)\mathcal{P}(M|F1) + \mathcal{P}(A|M \cap F2)\mathcal{P}(M|F2),$$

$$\mathcal{P}(A|M') = \mathcal{P}(A|M' \cap F1)\mathcal{P}(M'|F1) + \mathcal{P}(A|M' \cap F2)\mathcal{P}(M'|F2).$$

From this, it can readily be seen that the Simpson reversal depends on the relative weighting of $\mathcal{P}(M|F1)$, $\mathcal{P}(M|F2)$, $\mathcal{P}(M'|F1)$ and $\mathcal{P}(M'|F2)$, which precisely encodes the preference that males and females show for each faculty.

Decision making: how to avoid becoming Simpson's donkey

So far, we have seen that a Simpson reversal occurs in many real-world situations, and that, although counterintuitive, there is no mathematical reason why such a reversal should not occur. So is that the end of the story?

Far from it.

Consider a new scenario where a Simpson reversal can occur: data collected about a new drug suggests that, when compared with an old drug, it demonstrates increased effectiveness among the female population, similarly demonstrates increased effectiveness among the male population, but demonstrates reduced effectiveness among the population at large.

Imagine that you were a doctor confronted with this data. Should you start using the new drug? Could the correct answer possibly be: 'Only when you know the gender of the patient?'

Intuitively, it seems that the critical consideration is the aggregated data, and hence you should conclude that the new drug is less effective. But where is this intuition coming from? And is it accurate?

Consider, now, the scenario from the other side. Imagine you are a drug company executive faced with aggregate data suggesting that your new drug is worse than its predecessor. Being smart and unscrupulous, you realise that a data set can be broken down into an arbitrary number of subgroups (patients with blue eyes versus patients with non-blue eyes; patients whose names begin with A–K versus patients whose names begin with L–Z etc., some of which you would expect to demonstrate a Simpson reversal. Say, after much

searching, you find the breakdown you need: the new drug is more effective among patients younger than 20, and also more effective among patients older than 20.

You present this information to the hospital. Clearly, the hospital would be foolish to accept your findings. This example demonstrates just how the presentation of data can have a strong potential to mislead.

Imagine now that the disease is lung cancer, and the data shows that the drug is more effective in patients who complain of headaches, and also more effective in patients who do not complain of headaches, but less effective overall. Should the conclusion be different this time? Are we ever justified in considering separate data sets rather than the aggregate data?

The key to this question lies in the causal links between the factors at play. Causation is a controversial topic among statisicians, chiefly due to the impossibility of empirically verifying causal links. Instead, statisticians substitute the notion of *correlation*, the empirical observation that two factors tend to appear together. Yet correlation and causation must not be confused.

Two events may be heavily correlated, yet there may be no direct causal link between them. Instead, they might both be caused by a third event. Take, for instance, the correlation between ice cream consumption and drowning. Both tend to occur in the summer when more people visit the beach or pool, but they are not in themselves causally connected. As such, two events might occur together by mere coincidence, or the causal relationship might be more complicated.

Back to the drugs. Suppose the data is split into two groups, G and G'. Further, let T indicate that a patient is treated effectively, and let N and N' indicate the *act* of taking the new or old drug. In causal calculus this is normally denoted $do(\text{N})$ and $do(\text{N}')$ respectively. Finally, suppose with confidence that the choice of drug *has no causal connection with G*. Thus:

$$\mathcal{P}(G|do(N)) = \mathcal{P}(G|do(N')) = \mathcal{P}(G)$$

$$\mathcal{P}(G'|do(N)) = \mathcal{P}(G'|do(N')) = \mathcal{P}(G')$$

Assume this applies to gender as well as eye colour, name, and age.

Then suppose that the aggregate data demonstrates that the new drug is more

effective than the old drug. Hence:

$$\mathcal{P}(T|do(N)) > \mathcal{P}(T|do(N'))$$

$$\Rightarrow \quad \mathcal{P}(T|do(N) \cap G)\mathcal{P}(G|do(N)) + \mathcal{P}(T|do(N) \cap G')\mathcal{P}(G'|do(N))$$
$$> \mathcal{P}(T|do(N') \cap G)\mathcal{P}(G|do(N')) + \mathcal{P}(T|do(N') \cap G')\mathcal{P}(G'|do(N'))$$

$$\Rightarrow \quad \mathcal{P}(T|do(N) \cap G)\mathcal{P}(G) + \mathcal{P}(T|do(N) \cap G')\mathcal{P}(G')$$
$$> \mathcal{P}(T|do(N') \cap G)\mathcal{P}(G) + \mathcal{P}(T|do(N') \cap G')\mathcal{P}(G')$$

$$\Rightarrow \quad \mathcal{P}(G) \left(\mathcal{P}(T|do(N) \cap G) - \mathcal{P}(T|do(N') \cap G) \right)$$
$$> \mathcal{P}(G') \left(\mathcal{P}(T|do(N') \cap G') - \mathcal{P}(T|do(N) \cap G') \right).$$

which contradicts

$$\mathcal{P}(T|do(N) \cap G) < \mathcal{P}(T|do(N') \cap G)$$

$$\mathcal{P}(T|do(N) \cap G') < \mathcal{P}(T|do(N') \cap G').$$

Hence, where G is not causally connected with $do(N)$, a Simpson reversal is *not* possible. Therefore, in a situation where G is believed not to be causally connected with $do(N)$, but where a Simpson reversal is nevertheless present, the data must either be rigged or the assumption about the causal connection between G and N must be false.

In fact, according to cognitive scientist Judea Pearl, a Simpson reversal is only possible in three causal situations (See Figure 2):[12]

Only if the causal diagram is the one on the left should the separate data be considered instead of the aggregate data. Otherwise, we are conditioning on a factor that resides on the same casual pathway we seek to evaluate.

Returning to the drug example, suppose that effectiveness data demonstrated a Simpson's reversal in either the gender, age, or name data. We can assume, with confidence, that the medication does not *causally* influence the gender, age or name of a patient. Thus, the option presented by diagram (b) is eliminated.

Diagram (c) requires the presence of a hidden factor Y that influences both recovery and the gender, age, or name of a patient. This is plausible in the case of gender or perhaps even name (e.g. the factor could be related to ethnicity)

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[12]Judea Pearl, 'Simpson's Paradox: An Anatomy', Chapter 6, *Causality*, 2009.

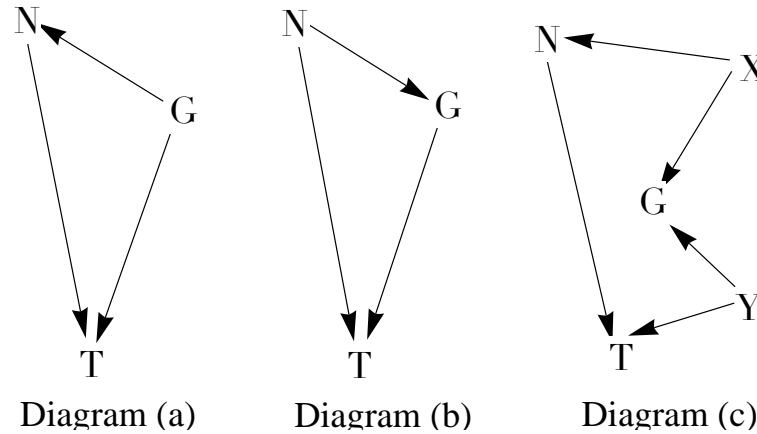Diagram (a)   Diagram (b)   Diagram (c)

Figure 2: The three causal relationships that can lead to a Simpson reversal. In diagram (a), the group influences the type of drug as well as the efficiency of the treatment. In diagram (b), the type of drug influences the group, and the group influences the treatment. In diagram (c), hidden factors X and Y influence all three of the factors.

but this requires further investigation. In the absence of such a factor, the only explanation is diagram (a), which suggests that the factor G influenced the choice of medication. And if this is the case then we should consider the data separately rather than in aggregate.

In contrast, it is logical that headaches might be causally connected to the taking of the new medication. Thus diagram (b) is very plausible, and so we should be more inclined to consult the aggregated data.

Let us apply this analysis to our real-world situations:

1. The innings of a cricket match is not a factor that can be influenced by underlying causes, so diagram (a) applies—the separated data is more appropriate.

2. The education level of society has in all likelihood been caused by factors related to the advance in years from 1980 to 2011, and so diagram (b) applies—the aggregated data is more appropriate.

3. The airline is not a factor that can be influenced by underlying causes, so diagram (a) applies—the separated data is more appropriate.

Finally, we are in a position to resolve the question that we posed above: 'Was

the University of Melbourne demonstrating a gender bias?'

In fact, the answer depends. If females are indeed choosing the 'ultra-competitive' Science Faculty more often than males, then the choice of faculty has a causal connection to the gender of the applicant and we are in a diagram (a) situation. Thus, using the separated data is more appropriate. Conclusion: no gender bias.

On the other hand, if there is a hidden factor, X, that has a causal connection to both the faculty and the gender of the applicant; and also a hidden factor, Y, that has a causal connection to both the faculty and the success of the applicant, then we are in a diagram (c) situation (However, it is not clear what such factors X and Y could be). As a result, using the aggregated data is more appropriate. Conclusion: gender bias.

In sum, Simpson's reversal is an intriguing statistical quirk, a good understanding of which is critical in order to avoid misinterpreting of some forms of data presentation. Interestingly, although the reversal can be easily explained mathematically, it is only through an analysis in terms of causal chains that the power of the reversal to deceive can be truly appreciated.

Moreover, the fact that the reversal is often referred to as a 'paradox' is enlightening in itself, for it suggests that humans are intuitively suspicious of the reversal in certain situations. In fact, as we have seen, where the group factor is not causally related to the other factors, the reversal is *impossible*. Our natural impulse to reject the Simpson reversal might then just be a product of humans' deeply *causal* rather than purely statistical reasoning.[13]

— Stephen Muirhead

> While the individual man is an insoluble puzzle, in the aggregate he becomes a mathematical certainty. You can, for example, never foretell what any one man will be up to, but you can say with precision what an average number will be up to. Individuals vary, but percentages remain constant. So says the statistician.
> — Arthur Conan Doyle

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[13]For more on the link between Simpson reversals and human modes of reasoning, it is worth tracking down Judea Pearl's work *Causality*.