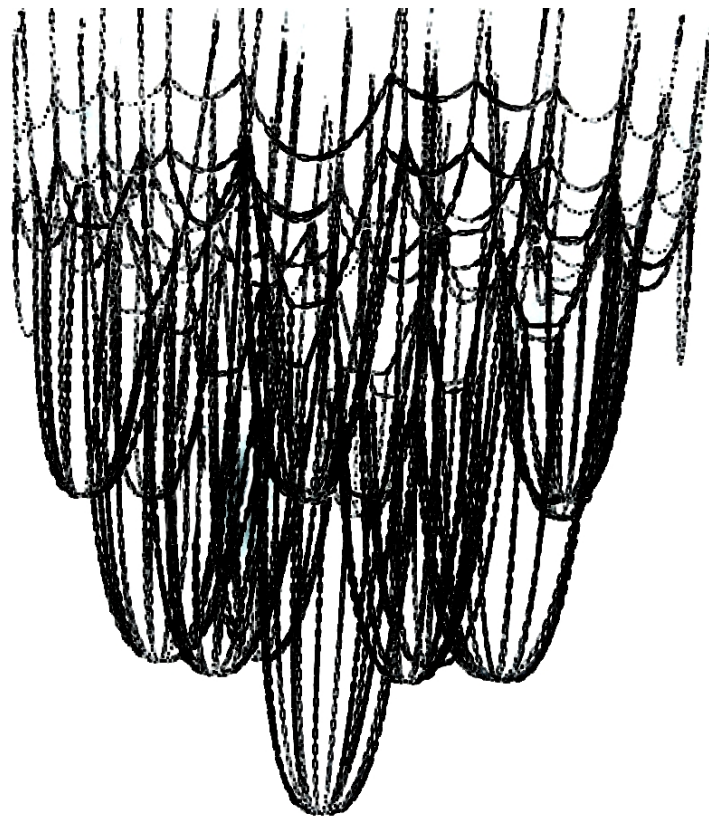

Paradox

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Words from the Editor

Welcome to the second edition of Paradox, the magazine produced by the maths and stats students society (MUMS). This edition is the first of the post-James Wan era, Paradox's longest-serving editor ever having finally retired his orange highlighters and tEx manual at the end of last semester.

In 1996 Paradox consisted of two editions, both 10 pages long. In 2001 Paradox had progressed to three editions averaging 14 pages. Since 2006, the year James took the helm, Paradox has been produced three times a year, and is on average over 30 pages long. This is as much a mark of his professionalism, as it is evidence of an unmatched ability to unearth (terrible) maths jokes, anecdotes and problems.

If imitation is the sincerest form of flattery, we begin this edition with a tribute; a selection of the best (or worst) of Paradox jokes from the last three years.

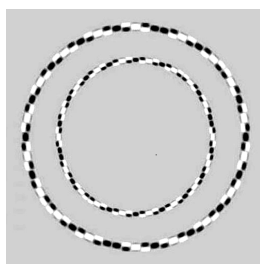
Also in this edition you will discover why hanging chains can make you a famous architect, why you should never work hard again, and why the value of ten dollars is not obvious. You might also realise how much *you* value ten dollars by solving one of Paradox's cash prize problems.

Paradox is the magazine of the maths students society, and hence of all maths students. It is only as good as the people that contribute to it. If you hear a good joke, a mind-bending puzzle or even if you simply overhear your maths lecturer saying something witty or stupid, Paradox wants to hear from you (either drop into the MUMS room, or use the Paradox email on the inside front cover). Article submissions, in any form, are gladly accepted.

And, for goodness sake, go see Terry Tao's lecture!! See back cover for details.

— Stephen Muirhead

An optical illusion: These two figures are perfect circles.



Words from the President

Apparently it is the task of the President of MUMS to write the President's words in Paradox. Who would have figured? At least for this edition, I do have some momentous news. After long, hard, difficult research in the dark depths of the Maths and Stats library, it has been discovered by James Wan, the esteemed former editor of Paradox, that MUMS was formed in 1929. This means that MUMS is officially 80 years old this year! Hip, Hip, Hooray! We will be accepting any cakes that are offered to us in celebration. Unfortunately, due to the serious global economic crisis (best excuse ever), MUMS will not be holding a epic birthday party of win, and you can blame this on all the finance graduates (not the statistics students, they're cool).

In other less momentous news, MUMS has revamped our website, and made it far less orange/eyehurty and now its a much more peaceful blue. I encourage you all to go check it out, keep up to date with MUMS events, and find all the broken links for us. Also, we've rearranged the furniture in the MUMS room. Feel free to come in and attempt to prove to us that it is still isomorphic to the original formation. Glittering prizes await the first to do so!

With the semester just having started, MUMS will be soon rolling out all our big ticket events, most notably the Math Olympics, the epic test of endurance, skill, agility and mathematical cunning. Don't forget to partake in our (mostly) weekly seminars about what I consider to be the much more fun and interesting side of mathematics. Additionally, don't be afraid to poke your heads into the MUMS room. We won't bite! In fact, if you ever need any help with your maths, there's a good chance you can find it in the MUMS room. And in case you're worried about not knowing anyone in the MUMS room, for easy reference, the sleeping immobile figure on the couch is yours truly. Oh, and keep on the lookout for the next edition of Paradox, which should be coming out later in the semester. Gotta build up those libraries of paradoxen!

— Han Liang Gan

Cover Picture: Spanish architech Antoni Gaudí is famous for his heavy use of hanging chains (catenary curves) in his work. This model was created using a series of inter-connected hanging chains. A mirror situated beneath the model inverts the viewpoint, revealing the intended design of the building. For more details see the article *Catenary Curves*.

A Tribute to James Wan

Q: What do you call a young eigensheep? A: A lamb, duh!

Q: Why didn't Newton discover Group Theory? A: Because he wasn't Abel.

"What's your favourite thing about mathematics?" "Knot Theory!" "Yeah, me neither!"

Q: How can you tell when a mathematician is extroverted? A: When he is talking to you he stares at your shoes instead of his.

Did you hear about the statistician that was thrown in jail? He now has zero degrees of freedom.

Lecturer: Today we'll be studying Abelian groups. Student: What?! I hardly know two!

Zenophobia is the irrational fear of convergent sequences.

Mathematican puns are the first sine of madness.

∞

Three statisticians go hunting. When they see a rabbit, the first one shoots and misses on the left. The second one shoots and misses on the right. The third one shouts: "We hit it!"

An engineer thinks his equations are an approximation of reality, a physicist thinks reality is an approximation of his equations. A mathematician doesn't care!

When the logician's son refused to eat his vegetables for dinner, his father threatened him: "If you don't eat your vegetables, you won't get any ice-cream." Frightened by this prospect, the son quickly finishes his vegetables. The father, bemused that his idle threat had worked, sends his son to bed without any ice-cream.

Biologist think they are Biochemists. Biochemists think they are Physical Chemists. Physical Chemists think they are Physicists. Physicists think they are God. And God thinks he is a Mathematician.

Philosophy is a game with clear objectives and no rules. Mathematics is a

game with clear rules but no objectives.

∞

For his epitaph, Erdős suggested: "I've finally stopped getting dumber."

George Bernard Shaw: "Statistics show that, of those that contract the habit of eating, very few survive."

John von Neumann: "In mathematics you don't understand things, you just get used to them."

Aaron Levenstein: "Statistics are like a bikini. What they reveal is suggestive, but what they conceal is vital."

Quotes from Maths Lecturers

In response to the article from the last edition, several new quotes have been sent in by readers. Here is a selection. Please send in any more that you hear!

Barry Hughes

- "You don't have to be the Brain of Brisbane to solve this PDE."
- "It will look at you and scream: 'Eigenfunction me!' "
- "The probability of getting something right decreases exponentially with its length."

Paul Norbury

- "One day, lecturing Algebra, I was supposed to be proving that there existed infinitely many primes. That was too easy, so instead I proved that there were finitely many. Sometimes proving false things can be useful."

Craig Westerland

- In response to a request for a joke. "A physicist and an engineer are on a deserted island. Seeing a hot air balloon flying overhead the engineer yells out: 'Where are we!?' To which the response comes from above: 'You are below me!' The physicist immediately says: 'Ahh, a mathematician.' When the engineer looks incredulous, asking how he knew, the physicist responds: 'Well, what he said was at once entirely correct, and entirely useless.' "

Egyptian Fractions and Harmonic Series

“God created the integers; all else is the work of Man.” At least, according to Leopold Kronecker, a 19th century German mathematician who believed that irrational numbers didn’t exist. This sounds a little silly, but let’s think about it for a moment. The information capacity of the observable universe is finite, so two quantities which differ in decimal expansion only at digits beyond this information capacity can never be distinguished. As far as the real world is concerned, every supposedly irrational number is exactly equal to a rational one, so why should we care about numbers that aren’t rational?¹

Our lifelong romance with the rationals begins at an early age, and like all relationships, has its ups and downs. First, we’re taught that $\frac{23}{10}$ is *wrong*, because it is *vulgar*. Later, we’re taught that $2\frac{3}{10}$ is *wrong*, because it can be confused with multiplication. Finally, we’re taught that 2.3 is *wrong*, because it suggests an error of ± 0.05 .

So what is *right*? We could resort to hideous compromises like $23 \times \frac{1}{10}$, $2 + \frac{3}{10}$ or 2.3000.... We could invent weird and wacky alternatives like 2.2999... or $\int_0^\infty 23e^{-10x} dx$. Or, to finally get to the topic of this article, we could use the long-winded but strangely elegant $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10}$.

This is called an *Egyptian fraction*, characterised by being a sum of distinct *unit fractions*. The ancient Egyptians indeed wrote their fractions like this, but they also allowed integers, so they would’ve just written $2 + \frac{1}{5} + \frac{1}{10}$.

Of course, to be useful as a notational system for rationals, we first need to know whether we can write every (positive) rational this way. The answer is yes, and in fact, the proof is quite easy:² *Given a positive rational number $\frac{x}{y} < 1$, subtract the largest unit fraction possible, that is, subtract $\frac{1}{z}$ where $\frac{1}{z} \leq \frac{x}{y} < \frac{1}{z-1}$. Then $\frac{x}{y} - \frac{1}{z} = \frac{xz-y}{yz}$, but $y > x(z-1)$ by our choice of z , so $xz - y < xz - x(z-1) = x$. Thus, the numerator is strictly smaller, so we must eventually end up with a numerator of 0. That means the unit fractions we subtracted add up to exactly the original number, and it’s pretty clear that they’re distinct.*

This proves it for positive rationals less than 1, so now take any positive rational q . The harmonic series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$ diverges, so we can pick the largest partial sum

¹We have omitted complex numbers from this discussion. In fact, they’re much more useful than irrationals, the reason being they can’t just be approximated by rationals.

²Paragraphs in italics are proofs; the busy, impatient and/or downright lazy reader should feel free to skip them.

$s = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}$ which is less than or equal to q . Then, s is a sum of unit fractions, and $q - s < \frac{1}{n+1}$, so we can apply the above argument to $q - s$. The unit fractions we obtain all have denominator at least $n + 2$, so they can't be repeats of the ones we've already picked for s , and hence we have a bona fide Egyptian fraction.

Applying this method to our favourite number 2.3 gives $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{60}$. Wait, 60? What gives? Our original expansion only needed denominators up to 10, so how come we need 60 now? Well, we proved that an Egyptian fraction representation exists, but we didn't say anything about our algorithm being a good one. In fact, much nastier examples exist, but at least we know we can always find one, however bad it might happen to be.³

For those who find this hard to believe, here's a nice intuitive analogy: If you have one of each Australian coin, what amounts of change can you make? Anything less than \$4 that doesn't contain the digits 4 or 9 – and the reason 90c doesn't work is because the 20c and 50c coins are too far apart; if we had a 35c coin in addition to the existing ones, we'd be fine. If we think of the unit fractions as an infinite pocketful of distinct coins, with no big gaps in their values, it makes sense that we can make any amount of change we want.

What's next? Well, we can make any rational number, so how about our unphysical friends, the irrationals? Irrationals can't be written as any finite sum of any rationals, but they can certainly be written as an infinite sum of them, for example, $\pi = \frac{3}{1} + \frac{1}{10} + \frac{4}{100} + \dots$. But what if we want the summands to be distinct unit fractions? Can every positive real number be written as a generalised Egyptian fraction?

As it turns out, we can. The proof is a little harder than before, but not too much so: For any positive real number r , let a_1 be the largest unit fraction less than r , a_2 the largest unit fraction less than $r - a_1$ which is not a_1 , and so on, with a_n the largest unit fraction less than $r - a_1 - \dots - a_{n-1}$ which has not already been picked. Then the partial sums $s_n = a_1 + \dots + a_n$ are monotone increasing and bounded above by r , so if you remember your first year maths, this means they must converge!

What do they converge to? Well, the limit s can't be any bigger than r , and if it were any less, then $r - s > 0$, so we can pick an integer $m > \frac{1}{r-s}$. In fact, we can pick m so that $\frac{1}{m}$ isn't equal to any a_n , since if every possible m were taken, then by divergence of the harmonic series, the sum of the a_n would be infinite. There are only finitely many unit fractions greater than $\frac{1}{m}$, but infinitely many distinct a_n , so some $a_n < \frac{1}{m}$. But $\frac{1}{m} < r - s < r - s_{n-1}$, and we picked a_n to be the largest unit fraction

³http://en.wikipedia.org/wiki/Greedy_algorithm_for_Egyptian_fractions.

less than $r - s_{n-1}$ which hasn't already been picked, so this is a contradiction. Hence, the limit must be r .

Pulling unit fractions out of your pocket to make up any positive real number is a neat party trick, but incredibly, even cooler things happen once we add some negative signs. We all know that $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$, but slightly less well known is the fact that $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2$. It makes sense that alternating signs cause it to converge—the partial sums jump back and forth between 0 and 1, the jumps getting smaller until it finally settles down; knowing that it converges, $\log 2$ certainly seems like a plausible limit.

What isn't quite as plausible is that the limit depends on the order in which these signed unit fractions are added, without changing anything else. For example, if instead of alternating odd and even denominators (with their positive and negative signs respectively), we take one odd for every two evens, we get $(\frac{1}{1} - \frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6} - \frac{1}{8}) + \dots = (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{6} - \frac{1}{8}) + \dots = \frac{1}{2} \log 2$, since each term is now exactly half the corresponding term in the original arrangement. We've all grown up thinking that order doesn't matter when adding things together, but when there are infinitely many things and some of them are negative, it does!

How much does it matter? As it turns out, a lot. The *Riemann series theorem*⁴ tells us that given any *conditionally convergent series* – that is, a convergent infinite sum which no longer converges when the signs on the summands are removed – can be rearranged so as to converge to any real number, to diverge to $+\infty$ or $-\infty$, or to have no limit at all.

Going back to our initial example, we could really have some fun and write $2.3 = (\frac{1}{1} + \dots + \frac{1}{49}) - \frac{1}{2} + (\frac{1}{51} + \dots + \frac{1}{351}) - (\frac{1}{4} + \dots + \frac{1}{14}) + \dots$.

— James Zhao

Puzzle 1

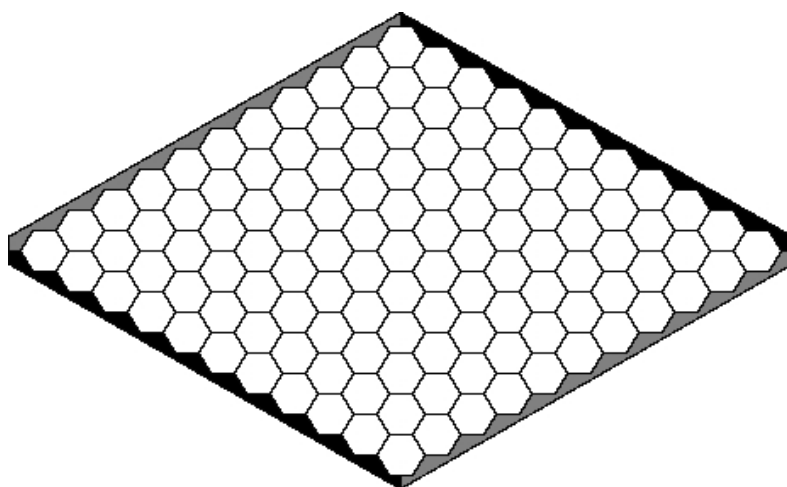
You have 14 balls and three paper cups. How can you place the balls in the cups such that each cup contains an odd number of balls?

⁴Proof omitted: It's not too hard to understand, but is a bit long for this article. It can easily be found online if desired.

Beautiful Concepts

Hex

The game of Hex is played on a hexagonal grid, such as the one below.



Two players take turns to place a token, the blue player trying to connect the blue edges with blue tokens, and the red player trying to connect the red edges with red tokens.

The game was first invented by Piet Hein in 1942, whilst contemplating the four-colour theorem⁵ — a fascinating result not proven until 1976.⁶ In 1947, Nash independently invented the game while at Princeton University. David Gale, an older student, whom Nash told about the game, made the first board and donated it to the common room, where games (particularly chess and go) had always been a popular means of recreation⁷ (much like the MUMS room!). However, when Parker brothers marketed the game in 1952, under the name Hex, Nash accused Gale of selling the game without his permission — an allegation which he always denied.

Perhaps surprisingly, Nash proved that the game can never end in a draw, that is, someone has to win! His proof used the Brouwer fixed point theorem.⁸

⁵http://www.swarthmore.edu/NatSci/math_stat/webspot/Campbell,Garikai/Hex/history.html.

⁶http://en.wikipedia.org/wiki/Four-colour_theorem.

⁷<http://maarup.net/thomas/hex>.

⁸http://en.wikipedia.org/wiki/Brouwer_fixed_point_theorem.

David Gale later proved Nash's result to be equivalent to the theorem, and generalised to n dimensions.⁹

Given Nash's result, it is easy to argue that Hex is a first player win, regardless of the board size; it's an exceedingly common game theoretical argument known as 'strategy stealing'. Firstly, observe that the two-game satisfies the following conditions:

1. It is deterministic (there is no chance involved).
2. Both players know the exact position at all times.
3. There are two results: win/loss.
4. The game can't go on forever.

From the above, it follows that either Player 1 has a winning strategy, or Player 2 has a winning strategy.

Suppose player 2 has a winning strategy. We will derive a contradiction by showing that Player 1 has a winning strategy, namely the following:

- Start with any move.
- Player 2 moves.
 - If you are able to, follow his winning strategy.
 - If you already have all of the necessary tokens on the board to execute his winning strategy, then play any other move.
- Repeat.

With this strategy, Player 1 follows Player 2's winning strategy, but always has an extra token on the board. This contradicts our original assumption, so we in fact deduce that Player 2 does *not* have a winning strategy (and hence Player 1 does).

The good news is that the above argument doesn't tell us *how* to win Hex. In other words, Hex is still a fun game! Traditionally it's played on an 11x11 board, although Nash himself advocated 14 as the optimal board size.

⁹The American Mathematical Monthly, Vol. 86, No. 10. (Dec., 1979), pp. 818-827.

Nash Equilibria

Having found something to do with his buddies in his spare time, Nash then went on to explore something really useful, namely the concept of Nash equilibria. Again, Nash wasn't the inventor, but again, he made the greatest contribution.

Long before Nash, the French economist, philosopher and mathematician Antoine Augustin Cournot adopted the concept in his theory of oligopoly (1838). In Cournot's model, firms would set their quantity of output to maximise their profits *given the output of other firms*. Similar logic would belie other oligopoly theories, which were to follow, such as Bertrand's, in which firms instead set their prices given other firms' prices.

Let's look at a more concrete example; a classic one known as the *Prisoner's Dilemma*. In its classical form:¹⁰

Two suspects are arrested by the police. The police have insufficient evidence for a conviction, and, having separated both prisoners, visit each of them to offer the same deal. If one testifies (defects from the other) for the prosecution against the other and the other remains silent (cooperates with the other), the betrayer goes free and the silent accomplice receives the full 10-year sentence. If both remain silent, both prisoners are sentenced to only six months in jail for a minor charge. If each betrays the other, each receives a five-year sentence. Each prisoner must choose to betray the other or to remain silent. Each one is assured that the other would not know.

We can summarise the above using a payoff matrix:

	Betray	Abide
Betray	(-5, -5)	(0, -10)
Abide	(-10, 0)	(-0.5, -0.5)

Step into the first player's shoes.

Case 1: Player 2 betrays me

- If I abide, then I cop 10 years in prison.
- If I betray him, then I only cop five years.

¹⁰http://en.wikipedia.org/wiki/Prisoner%27s_dilemma.

Case 2: Player 2 abides

- If I abide, then I get 6 months.
- If I betray him, then I go free!

In both cases, Player 1 is better off betraying Player 2, so he will do so. We say that betrayal *strictly dominates* abiding, for Player 1. The same argument holds for Player 2. So both players will testify (with 'best play'), and will reach the outcome $(-5, -5)$. If the players could cooperate, however, they could both remain silent and achieve a more favourable outcome for both of them, $(-0.5, -0.5)$, and the police would get nothing! It's no wonder police choose to question them separately!

When ten to the enemy's one, surround him; When five times his strength, attack him; If double his strength, divide him. . .

— Sun Tzu, The Art of War

In general, if every possible outcome for strategy A was no worse than it would be with strategy B, and at least one possible outcome was better (for action A), then we say that strategy A *dominates* strategy B. If every outcome for strategy A is better than the corresponding strategy B one, then we say A *strictly dominates* strategy B. Clearly any strictly dominant strategy is also dominant. If every player follows a dominant strategy, then we're certainly in Nash equilibrium.

An example of a modern day prisoner's dilemma is advertising. Suppose that Qantas and Virgin have a duopoly over the Australian airline industry. Should they advertise?

For simplicity, suppose they only have two options: to advertise, or to not advertise at all. Advertising draws customers, but costs money as well. Let's make up semi-reasonable relative profit figures:

	Advertise	Don't Advertise
Advertise	(4, 4)	(6, 3)
Don't Advertise	(3, 6)	(5, 5)

With no advertising, they might take 50 per cent of the market each. When Qantas advertises but Virgin doesn't, Qantas may attract 70 per cent of the market, but incur some cost, resulting in a (6, 3) outcome. Here advertising is the dominant strategy, so we'd expect the (4, 4) outcome. Again, however, there is a better outcome: if the firms cooperate, they could both not advertise and both achieve a better outcome.

In practice the firms could of course covertly agree to both not advertise (unlike in the classical prisoner's dilemma, where the players are isolated), so let's look at yet another economic example. Dominant strategies do not have to exist for a Nash equilibrium to exist; Nash's concept is more general, as we shall see.

Suppose two French people are driving towards one another on a street in a wide, un-laned street in Park Orchards (which is in Victoria, Australia, for those of you who don't know). They both know you're meant to drive on the left here, but they both recognise their fellow countryman immediately, and think that the other may well drive on the right. We get the following payoff matrix:

	Drive on the Left	Drive on the Right
Drive on the Left	(100, 100)	(0, 0)
Drive on the Right	(0, 0)	(100, 100)

For a given driver, neither strategy dominates the other. However, there are two Nash equilibria: $[L, L]$ and $[R, R]$.

Now we can define Nash equilibrium more generally. A set of strategies is a Nash Equilibrium if no player can do better by changing their strategy, given that other players' strategies remain the same. The final phrase is akin to the Latin phrase *ceteris paribus* – meaning “with other things the same” – which is ubiquitous in economic theory.

In general there doesn't have to be a Nash equilibrium at all, and – as we've seen – there can be more than one! In many situations, however, there is precisely one. In this case, the Nash equilibrium strategy set will be adopted under certain conditions:¹¹

1. The players all will do their utmost to maximize their expected payoff as

¹¹http://en.wikipedia.org/wiki/Nash_equilibrium.

described by the game.

2. The players are flawless in execution.
3. The players have sufficient intelligence to deduce the solution.
4. The players know the planned equilibrium strategy of all of the other players.
5. The players believe that a deviation in their own strategy will not cause deviations by any other players.
6. There is common knowledge that all players meet these conditions, including this one. So, not only must each player know the other players meet the conditions, but also they must know that they all know that they meet them, and know that they know that they know that they meet them, and so on.

— Sam Chow

Why you should never work hard again...

Picture the following situation: you are working for the day in a café, a temporary job filling in for a friend who is ill. Given the provisional nature of the work, the all important issue of pay has not yet been discussed.

At the start of the day, you see an excellent opportunity to slack-off while pretending to clean tables out of sight. You reason that your boss has already predetermined your pay, hence there is nothing to be gained by working hard, and you pass the rest of the day in the shadowy recesses of the café. Your boss, meanwhile, has also reasoned that, since you will only find out your pay at the end of the day, you will put the same effort into the work no matter how generous he is, and so proceeds to strike off a zero from the standard daily pay-rate. At the end of the day you part equally unsatisfied, the café owner having received shoddy work, and you having received a pittance for your efforts. Rational self-interest has conspired to undermine the optimum outcome: hard work and generous pay.

1 An iterated Prisoner's Dilemma

Those who have studied some game theory, or have read the previous article *Beautiful Concepts*, may have noticed that this is an application of the Prisoner's Dilemma. Here the workplace dynamic between employee and boss has been simplified into the following payoff chart:

	Pay Well	Pay Badly
Work Well	(Good,Good)	(Very Bad, Very Good)
Work Badly	(Very Good, Very Bad)	(Bad,Bad)

And in classic Prisoner Dilemma fashion, there is an outcome (Work Well, Pay Well) which is far better for both parties than the Nash Equilibrium (Work Badly, Pay Badly).

Yet if we believe what we read above, people in real-life work surprisingly hard, and bosses in real-life pay surprisingly well. Is this merely a lack of rational self-interest? Or is something wrong with our model? The answer is, of course, that real jobs do not usually last for only one day. Indeed, if the café owner wished you to return the next day, you may well wish to impress him by working that little bit harder, in the hope of superior pay the following day.

Thus real work environments are better modelled by *iterated* Prisoner's Dilemmas, with the assumptions that a new *decision* (Level of Pay, Quality of Work) is made at the beginning of each time-scale (day, week, month, year etc), that this decision is made independently by both parties, and that once made, the decision must be stuck to. So, if you turn up for work in the café the following day, the boss must decide at the *start of the second day*, what he will pay you for that day's work, and you must *simultaneously and independently* decide whether or not to work hard on that day. If we imagine working at the café an extended, indefinite, period of time, we might think of this as an infinite series of daily decisions.

2 An infinite workplace

Armed with this model, we can attempt to explain why real-life workplaces are *not* exclusively inhabited with slackers. Let's call the decision pair (Boss,

Employee) at time x , $d_x = (b_x, e_x)$, and let's define a *b-strategy* and an *e-strategy* to be rule-books that uniquely determine decisions b_x and e_x respectively, based on the decision history $\{d_1, \dots, d_{x-1}\}$. An example of such a *b-strategy* might be pay well for the first day, then pay well whenever the employee worked hard the day before, otherwise pay badly. These strategies can clearly become very complex and convoluted. Whereas in the classic Prisoner's Dilemma there is a single rational *b-strategy* and *e-strategy*, in the iterated analogy determining rational strategies becomes much more complex. Indeed, depending on the strategy chosen by the opponent, a strategy that may be successful in one instance may seriously backfire in another. For example, an overly 'nice' strategy, where the boss usually pays well, is open to exploitation by an unscrupulous employee. No strategy will produce an optimum outcome in all cases, therefore this is no truly rational strategy that should always be followed. In other words there is no Nash Equilibrium!

Yet there are strategies that will at least produce a mutual outcome for both parties better than the one predicted above. This is where both parties adopt a *punishment* type strategy, and announce their intentions to the opponent. Here, the boss would announce to the employee his *b-strategy*: I will pay you well until the day you do not work hard, and for ever after I will punish you with low pay. Similarly the employee would announce to the boss his *e-strategy*: I will work hard until the day you do not pay me well, and for ever after I will punish you with shoddy work. With such announced strategies, rational self-interest does not allow either party to waiver from their own strategy, for any attempt to backstab the opponent to extract a better outcome will result in infinite punishment, incurring losses outweighing any possible gains. With a strategy such as this, rational-self interest will provide the illusion of benevolent co-operation.

In short, in a working relationship running for an infinite amount of time, rational self-interest and the threat of infinite punishment can actually result in decent work and decent pay.

3 Yet none of us work forever...

If approximating our employment model as an infinite series of iterated decisions didn't seem too much of a stretch, in reality this assumption is fundamentally flawed. For, the curious thing about the iterated Prisoner's Dilemma

is that as soon as the length of the iteration becomes known and fixed in advance, no matter how long the time-scale, the rational strategy once again becomes clear; work poorly, pay less. No rational co-operation is possible.

To see this, consider your last day working in the job. On this day, like in the original problem, there is no incentive for you to work hard, and the boss has no incentive to pay you more. No matter any previous history of co-operation, given the imminent end of the relationship rational self-interest takes over to undermine the optimum outcome for both parties. Thus the last day's decision must rationally be (Work Badly, Pay Badly). Now consider the second last day. Given that the decision on the last day is now logically fixed in Nash equilibrium, neither party will be making their decision hoping to influence the final day's decision. Thus this situation is analogous to the decision on the last day, that is, the single Prisoner's Dilemma. Hence the rational decision on the second last day must again be (Work Badly, Pay Badly). We may continue this reasoning as far back as desired. Indeed, a simple inductive argument will show that the rational strategy is to pay less and work less throughout the *whole* working period, *no matter how long*.

Thus, in any work environment where the period of employment is finite and known, no matter how long, there is no possibility of rational co-operation.

4 So, should I be working hard?

In summary, rational cooperation can only exist in a working environment where the period of employment is either infinite (a clear impossibility), or completely unpredictable. In all other situations the only rational decision pair is (Bad Pay, Bad Work), no matter how long the work relationship is iterated.

Thus, the following people should immediately cease hard work:

1. Those who know exactly when they will quit their job (holiday jobs).
2. Those who are unlikely to get fired (bureaucrats, tenured professors, anyone working in France).
3. Those approaching retirement age (perhaps, then, there is a rational reason for age discrimination in the workplace!)

— Stephen Muirhead

Don't trust your instincts!

Here's an easy question. What comes next in this sequence: $\{1, 2, 4, 8, 16, \dots\}$? What about in this sequence: $\{1, 1, 2, 3, 5, \dots\}$? Still easy?

If you were rather clever, you might have guessed 32 and 8, recognising respectively the first five powers of two and the first five Fibonacci numbers. Yet if you were even more clever you would have declined to answer. Why? Because these sequences are not well defined! It depends on which specific sequence I am thinking of.

To see this, consider the following problems:

1. If n points on a circle are connected to each other with straight lines, how many regions have I divided up the circle into (assuming no three lines intersect)?
2. If I have a stack of n coins, how many different 2D towers of coins can I build (assuming each coin on a higher level must touch two on the level below it)?

If you decided to tackle these problems by analysing small values of n you would get the following results; for $n = \{1, 2, 3, 4, 5\}$ the answers are $\{1, 2, 4, 8, 16\}$, and $\{1, 1, 2, 3, 5\}$ respectively. Great, these are famous sequences! Surely, you might reason, the solutions are 2^{n-1} and the n^{th} Fibonacci number.

Yet delve a little deeper and you will find that for $n = 6$ things go arry. Six points divide the circle into 31 regions, not 32, and there are nine towers of six coins, not eight.¹²

So were the original answers 31 and 9? Not a chance! Consider:

1. How many positive divisors does $n!$ have?
2. How many ways are there to partition the integer $n - 1$ into one or more positive integers, where two partions are considered identical if they differ only by the order of the partitioning elements?

Whose initial solutions are, for $n = \{1, 2, 3, 4, 5, 6\}$, $\{1, 2, 4, 8, 16, 30\}$ and $\{1, 1, 2, 3, 5, 7\}$ respectively.¹³ Don't trust your instincts!¹⁴

¹²Try to come up with explicit formulae for a general n .

¹³Where I have followed the convention that there is one partition of 0

¹⁴For more about the weird and wonderful world of integer sequences visit www.research.att.com/~njas/sequences.

Catenary Curves

In case you were wondering, the picture on the front cover is series of chains hanging under their own weight. If that gets you wondering, read on. So what shape does a chain (or string or wire) make when it hangs under its own weight? From when I was a little boy – and yes I was thinking about such things back then – I always imagined that the shape was that of a parabola – OK maybe I only began to think that once I learnt what a parabola was.

Anyway I was in good company, as none other than Galileo Galilei famously claimed this very thing.¹⁵ Of course it seemed to make sense, after all, the parabola was already a significant curve in nature: the trajectory of projectile is parabolic, the parabola was a simple conic section and had a focal point and, perhaps above all, a hanging chain looks like a parabola.

The problem with all these observations it that they are not based on the physics of the situation. So while the statement does not have to be wrong, it would not be surprising to learn that it is. (Perhaps the Catholic Church should have arrested Galileo for this instead.) In fact it was only about 30 years after Galileo's death, near the end of the 15th century. that it was shown by such greats as Gottfried Leibniz, Christiaan Huygens and Johann Bernoulli that the curve was that formed by the cosh function.¹⁶

We shall proceed by parametrising the curve using arc length: $\vec{r}(s) = (x(s), y(s))$.

To get the ball rolling, we shall consider the forces on a section of the chain that starts at s and is δs long. We shall refer to such a section as a *link*.

The forces acting on the link will be those of tension and gravity. Gravity will act constantly downwards and will have magnitude mg but the mass of the link depends on its length, so $m = \mu \cdot \delta s$. There will be two tension forces acting in opposite directions, one on the left end of the link and the other right. Because we do not know what magnitude of the tension force is, we will describe it using a vector function $\vec{T}(s)$. The function will take the value $\vec{T}(s)$ at the left end and $\vec{T}(s + \delta s)$ at the right end. As these forces approximately

¹⁵Jürgen Renn, Galileo in Context, 2001

¹⁶The cosh or hyperbolic cosine function can be explicitly written in this form, $\cosh(x) = \frac{e^x + e^{-x}}{2}$. This fact, together with that for its sister function, $\sinh(x) = \frac{e^x - e^{-x}}{2}$ can be used to prove many of the properties of the functions that are necessary but not quoted in this article.

balance each other out, their sum is approximately zero.

$$-\vec{T}(s) + \vec{T}(s + \delta s) + \mu \cdot \delta s \cdot \vec{g} \approx \vec{0}$$

$$\frac{\vec{T}(s + \delta s) - \vec{T}(s)}{\delta s} + \mu \vec{g} \approx \vec{0}$$

The first term on the left should seem familiar to anyone who has ever done any calculus, and we shall manipulate it in the usual fashion.

$$\lim_{\delta s \rightarrow 0} \frac{\vec{T}(s + \delta s) - \vec{T}(s)}{\delta s} + \mu \vec{g} = \vec{0}$$

$$\frac{d\vec{T}}{ds} + \mu \vec{g} = \vec{0}$$

$$\vec{T}(s) = -\mu \vec{g} s + \vec{k}$$

Now $\vec{T}(s) = T(s) \left(\frac{dx}{ds}, \frac{dy}{ds} \right)^{17}$ and $\vec{g} = g(0, -1)$ where $g = 9.8 \text{ m.s}^{-2}$, so we may resolve into components:

$$T(s) \frac{dy}{ds} = \mu g s + k_2 \quad (1)$$

$$T(s) \frac{dx}{ds} = k_1 \quad (2)$$

The constant in equation (1) is inconvenient and may be eliminated by letting $t = s + \frac{k_2}{\mu g}$.

Conveniently, $dt = ds$, and thus derivatives involving s may be substituted with those involving t and we may parameterise by t instead of s . Thus (1) becomes:

$$T(s) \frac{dy}{dt} = \mu g t \quad (3)$$

And (2) becomes:

$$T(s) \frac{dx}{dt} = k_1 \quad (4)$$

¹⁷Note: $T(s)$ is **not** necessarily the magnitude of the tension force as $\left(\frac{dx}{ds}, \frac{dy}{ds} \right)$ may not be a unit vector.

Dividing (3) by (4) gives:

$$\frac{dy}{dx} = \frac{t}{a}$$

Where $a = \frac{k_1}{\mu g}$

To proceed from here, a little basic fact about the arc length parameterisation needs restatement.

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

Recalling that $dt = ds$

$$dt = dx \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$= \frac{a}{\sqrt{a^2 + t^2}}$$

$\frac{dy}{dt}$ may be obtained similarly or by using $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ giving:

$$\frac{dy}{dt} = \frac{t}{\sqrt{a^2 + t^2}}$$

Thus we have

$$x(t) = a \sinh^{-1} \left(\frac{t}{a} \right) + b \quad (5)$$

$$y(t) = \sqrt{a^2 + t^2} + c \quad (6)$$

Now we are done as we have derived the parametric equation:

$$\vec{r}(t) = \left(\sinh^{-1} \left(\frac{t}{a} \right) + b, \sqrt{a^2 + t^2} + c \right)$$

We can use this to plot out the shape of the catenary, provided we know the endpoints of the chain and its length. However, this does not seem very satisfying considering our goal of *classifying* the shape of the catenary and it is not immediately obvious how this will produce a cosh curve.

To show this we must rearrange (5) to express t explicitly and substitute into (6):

$$t = a \sinh \left(\frac{x - b}{a} \right)$$

$$y = \sqrt{a^2 + a^2 \sinh^2 \left(\frac{x-b}{a} \right)} + c$$

Using the identity $\cosh^2(A) - \sinh^2(A) = 1$ we get:

$$y = a \cosh \left(\frac{x-b}{a} \right) + c$$

Thus the shape of the catenary curve is that of the cosh graph.

Returning to the picture on the front cover, it is intended to represent the work of the Spanish architect Antoni Gaudí whose architecture is famous for incorporating both parabolic and catenary arches. One of his drafting techniques was to hang weights from chains and invert the resulting curve to model the arches. Indeed, if we change a few signs here and there we can quite easily show that the catenary arch (an upside down cosh graph) is the ideal arch for supporting its own weight. However, if the arch is to support more than just its own weight and from places other than its turning point, then the ideal shape departs from a catenary. Indeed in a special case where the load is uniform along the x axis, the ideal curve is exactly a parabola. This is more or less what occurs in a suspension bridge. So, given all the suspension bridges I have seen in my lifetime, I guess I may be excused for thinking a hanging chain took the shape of a parabola, but the same cannot be said for Galileo.

—Narthana Epa

Solution to Puzzle 1

Put 11 balls in the first cup, two in the second and one in the third. Then place the third cup inside the second cup (they are paper cups, so they should be able to be squeezed a bit until they fit!).

Puzzle 2

It takes 10 minutes to cross a bridge running at top speed. Every nine minutes a guard comes out of a little office to inspect the bridge. If he sees you crossing he sends you back. How do you cross?

Some basics of utility theory

Why do we need utility theory?

I like to think of utility theory as a way of valuing money. You're probably wondering what I'm talking about. A dollar is worth exactly a dollar, right? It already has a value. Well think about it this way: if you're an ex-investment banker turned street bum, with a net worth of approximately \$50, what would you be willing to do for another \$50? On the other hand, if you were Bill Gates, what then would you be willing to do for another \$50? Obviously the two different positions result in a different valuation of money. This is where utility theory steps in. What utility theory does is, instead of just measuring wealth (W), it measures a function of wealth, typically denoted $U(W)$.

Basic properties of utility functions

- We would like to think that in general, people prefer to have more money than less. Hence, the utility function U should be an increasing function.
- We would also like to think that the more money you have, the less you would value the next dollar. Therefore, most people have a concave utility function.

Examples of utility functions

- The log utility function, $U(x) = \log(x)$
- The exponential utility function, $U(x) = 1 - e^{-cx}$

Uses of utility theory

Utility theory is useful in many fields. It can be used to price investments and make decisions. However, seeing as this is an article for Paradox, what better task could there possibly be than using it to solve and create paradoxen?

The St Petersburg Paradox

The St Petersburg Paradox is a classic paradox in probability and decision making theory. It is effectively a random variable that has infinite expected value. What occurs is, you flip a coin and count the number of tails that occur until you get a heads. You are then paid 2^n dollars, where n is the number of tails that occurred.

The expected value is therefore

$$E = \sum_{n=0}^{\infty} 2^n 2^{-n} = \sum_{i=0}^{\infty} 1 = \infty.$$

So the question is, how much would one be willing to pay to take part in this gamble? From a statistical point of view, if you are offered this game for any arbitrary amount of money, you should be willing to take the bet. Empirically, and rather unsurprisingly people are rather unwilling to offer much money at all to take part in this game. The problem is that people are in general, risk adverse, and do not place much value on small probabilities of making very large amounts of money. Which sounds awfully like a concave utility function...

The standard solution to the problem is to take a log utility function and compute the expected utility. So the expected utility is:

$$E(U) = \sum_{n=0}^{\infty} \log(2^n) 2^{-n} = \log(2) \sum_{n=0}^{\infty} n 2^{-n}$$

Which is finite and quite easy to calculate. (This is left as an exercise for the reader. Without going into the details, it can be computed using standard infinite sequence tricks.) The actual value of the gamble turns out to be four dollars. So this paradox can be resolved using utility theory. Unfortunately for us this is not always the case...

The Ellsberg Paradox

The Ellsberg Paradox is paradox which is not resolved, but is rather created, by utility theory. It is gambling paradox (the best kind of paradox!), in which empirical observations violate the expected utility theory result.

We are given a box with 300 balls. You are told that 100 of the balls are red, and that of the remaining balls, some of them are black, and some of them are yellow. You are then offered the following pair of choices based on two independent draws from the box with replacement.

In the first choice, based on the first draw, your two options are:

- Gamble A: You win 10 dollars if a red ball is drawn.
- Gamble B: You win 10 dollars if a black ball is drawn.

Your next set of options, based on the second draw, are:

- Gamble C: You win 10 dollars if a red or yellow ball is drawn
- Gamble D: You win 10 dollars if a black or yellow ball is drawn

We let R , B and Y represent the probabilities that this colour is drawn, given what you know about the likely number of Black/Yellow balls picked. Note that, by symmetry, if there is no external factor inducing you to believe that black or yellow balls are preferred, then each of these probabilities is $\frac{1}{3}$. Using a utility function we get the following unsurprising results.

If, empirically, gamble A is preferred to gamble B, then this means

$$R \cdot U(10) + (1 - R)U(0) > B \cdot U(10) + (1 - B)U(0).$$

After a bit of fiddling and assuming an increasing utility function, this will be true if and only if $R > B$. In other words, there *is* some external factor that induces you to believe that there are less black balls in the box than yellow balls, and hence than red balls.

Similarly, if gamble C is preferred to gamble D, that will be true if and only if $R > B$.

What this means is, if there is an underlying assumption that R and B are not equal, then empirically people will make the decision to either prefer gamble A to C *and* gamble B to D, or prefer gamble C to A *and* gamble D to B. The other preference pairings should not be chosen. Otherwise, if you know nothing about the distribution of black and yellow balls, there should be no visible favouring of any preference pairing over any other. However, empirically, people tend to prefer gamble A to B, and gamble D to C. Which violates utility theory.

What does this counterintuitive result show? It show that, in general, people are not rational and do not act according to the predictions of utility theory. Another explanation is that this paradox occurs because people do not like ambiguity. In the first choice of A or B, A is typically chosen as the number of red balls is known precisely and so the conditional probability given the distribution of black and yellow balls is also known. Similarly, D is chosen over C since the sum of black and yellow balls is also known...

Conclusion

So the overriding question is, just how useful is utility theory? We've used it to solve a paradox, and on the otherhand used it to create one (albeit not one that I would consider a genuine paradox). The answer to our question is, useful to a degree. It's useful to make decisions when you have time to analyse the situation properly. However, they key problem is, how do you choose a utility function? Who can honestly sit down and say "Why yes! My valuation of wealth is via a log utility function!" Still, utility theory gave me enough material for a paradox article so maybe it isn't so useless after all...

— Han Liang Gan

Solution to Puzzle 2

After the guard has returned to his office, run across the bridge for eight minutes, then turn around and run back towards the middle of the bridge. When the guard comes out he will see you running towards the original side, and thinking you are trying to cross in that direction, will send you back.

Solutions to Problems from Last Edition

We had a number of correct solutions to the problems from last issue. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

Farshid Jamshidi solved problem 3 and may collect \$3.

Adrian Khoo solved problem 3 and may collect \$3.

Matthew Kotros solved problems 2, 4, 5, 7 and may collect \$16.

Jensen Lai solved all the problems and may collect \$21.

1. Solve $a^2 + b^2 + c^2 = a^2b^2$ in the integers.

Solution: We will use a classic infinite descent argument to show that the only solution is $(0,0,0)$. First we note that $a = b = c = 0$ satisfies the equation. Next, we will assume there exists a non-trivial solution. Let (a_0, b_0, c_0) be the 'smallest' non-trivial solution, in the sense that it minimises $a^2 + b^2 + c^2$. Now, consider the equation modulo 4. Squares are either 0 or 1 modulo 4, and checking all possibilities we find that the only solution is $a^2 = b^2 = c^2 = 0 \pmod{4}$. Thus a_0, b_0 and c_0 are all even, and hence we can construct a 'smaller' solution $(\frac{a_0}{2}, \frac{b_0}{2}, \frac{c_0}{2})$, contradicting the minimality of our original solution.

2. In a triangle ABC , $\angle A = 120^\circ$. Find the length of the angle bisector from A in terms of AB and AC .

Solution: Let D be the foot of the angle bisector. Construct point E on line AC such that $\angle AED = 60^\circ$. As $\angle DAE = \frac{1}{2}\angle BAC = 60^\circ$, we know that ADE is an equilateral triangle. Clearly $\triangle ABC \sim \triangle EDC$, and thus $\frac{EC}{AC} = \frac{DE}{AB}$ hence $\frac{AC-AE}{AC} = \frac{DE}{AB}$. Using our relations $AD = DE = AE$ we get $\frac{AC-AD}{AC} = \frac{AD}{AB}$. Rearranging gives $AD = \frac{AB \cdot AC}{AB+AC}$.

3. If f is a function such that $f(ab) = \frac{1}{2}(f(a) + f(b))$, find $f(1234) - f(4321)$.

Solution: $b = 1$ yields $f(a) = \frac{1}{2}(f(a) + f(1)) \Leftrightarrow f(a) = f(1)$ for all a . Thus $f(1234) - f(4321) = f(1) - f(1) = 0$.

4. Solve the equations $9(x - y)(x^2 + y^2) = 1, 5(x + y)(x^2 - y^2) = 1$ in the reals.

Solution: Since $x \neq y$, we have $9(x^2 + y^2) = \frac{1}{x-y} = 5(x^2 + y^2 + 2xy) \therefore 4x^2 + 4y^2 - 10xy = 0 \Leftrightarrow x^2 + y^2 - \frac{5}{2}xy = 0$. Using the quadratic formula, $x = 2y, \frac{1}{2}y$. $x = 2y$ yields $(\frac{2}{45^{\frac{1}{3}}}, \frac{1}{45^{\frac{1}{3}}})$, whilst $x = \frac{1}{2}y$ yields $(\frac{-1}{45^{\frac{1}{3}}}, \frac{-2}{45^{\frac{1}{3}}})$.

5. You are in the centre of a circular pond 100 metres in radius. There is a zombie on the pond's edge that wants to eat you. The zombie can walk at 4 m/s and can't swim; you can swim at 1 m/s and run at 7 m/s. Can you escape the zombie?

Solution (from Jensen Lai): Firstly I swim 24m towards the zombie. Assuming that the zombie will always walk towards the point on the pond's circumference closest to me, it won't move.

I now begin to swim in a circle of radius 24m concentric with the pond, with the zombie following me around the circumference of the circle. My circle has circumference 48π metres whilst the circumference of the pond is 200π metres which is more than four times longer. As I am travelling at $1/48\pi$ radians per second whilst the zombie is only moving at $1/50\pi$ radians per second, eventually the zombie and I will be on opposite sides of the pond.

I now swim directly away from the centre of the pond towards the edge of the pond covering a distance of 76m in 76s. Meanwhile, the zombie must walk half way around the pond to catch me – a distance of 100π metres taking $25\pi \approx 78.54$ s – which means that I will reach the edge before the zombie can get to me. From this point on I can outrun the zombie and thus escape.

6. Euler proved at least one new theorem every day. To conserve energy, however, he proved no more than 50 theorems in any month. Show that there is a succession of days in a year where Euler proves exactly 125 theorems.

Solution: It's sufficient to prove it for a 365-day year. Consider a_1, \dots, a_{365} , where a_j is the total number of theorems after j days. Also, let $b_j = a_j + 125$ for all j . $a_j \leq 12 \times 50 = 600 \forall j$. Also, $b_j \leq 600 + 125 = 725 \forall j$. Hence $a_1, \dots, a_{365}, b_1, \dots, b_{365}$ are 730 numbers in $\{1, 2, \dots, 725\}$, so the pigeonhole principle tells us that two of them are the same. As Euler proved at least one theorem per day, the a_i are pairwise distinct, and the b_i are also pairwise distinct. Thus $a_j = b_i = a_i + 125$ for some i, j . Then 125 theorems are proven during days $i + 1, \dots, j$.

7. An object initially at $(0, 0)$ moves at all times toward another object initially at $(1, 0)$ and which is moving in the y direction. Both objects have

the same constant speed. Find the path of the first object.

Solution (from Matthew Kotros): Let $\mathbf{f}(t) = (x(t), y(t))$ denote the path of the first object. Now the second object moves along the path $\mathbf{g}(t) = (1, \lambda t)$, where λ is the speed of both objects. Since the first object moves at all times toward the second, the tangent vector must have magnitude λ and be in the direction of $\mathbf{g}(t) - \mathbf{f}(t)$. So we must have

$$\dot{\mathbf{f}}(t) = (\dot{x}, \dot{y}) = \lambda(1 - x, \lambda t - y) / \|(1 - x, \lambda t - y)\|.$$

Thus $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\lambda t - y}{1 - x}$. Now λt is the distance along the path \mathbf{f} that the

first object has travelled in time t , i.e. $\lambda t = \int_0^{x(t)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$.

Hence

$$(1 - x) \frac{dy}{dx} = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx - y,$$

and taking the derivative of both sides w.r.t. x yields

$$(1 - x) \frac{d^2y}{dx^2} - \frac{dy}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} - \frac{dy}{dx}.$$

Letting $w = \frac{dy}{dx}$, we have $(1 - x) \frac{dw}{dx} = \sqrt{1 + w^2}$, which is separable and so $\int \frac{dw}{\sqrt{1 + w^2}} = \int \frac{dx}{1 - x}$. Hence $\sinh^{-1}(w) = -\log |1 - x| + c$.

Now at $t = 0$, we have $\mathbf{f}(0) = (0, 0)$, and so $w = \frac{\lambda \cdot 0 - 0}{1 - 0} = 0$. Thus $c = \sinh^{-1}(0) + \log |1 - 0| = 0$, so that $w = \sinh(-\log |1 - x|)$. We may assume $x < 1$, as the path begins at $(0, 0)$, and so

$$\frac{dy}{dx} = \frac{e^{-\log(1-x)} - e^{\log(1-x)}}{2} = \frac{1}{2(1-x)} - \frac{(1-x)}{2}.$$

Thus $y = -\frac{1}{2} \log(1 - x) + \frac{1}{4}(1 - x)^2 + d$, and since $y = 0$ when $x = 0$, this implies that $d = -1/4$. Therefore $y = -\frac{1}{2} \log(1 - x) + \frac{1}{4}(x^2 - 2x)$.

Paradox Problems

Below are some puzzles and problems for which cash prizes are awarded. Anyone who submits a clear and elegant solution may claim the indicated amount (up to a maximum of four cash prizes per person). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

1. (\$2) In a round robin tournament involving k teams, where every team plays each other exactly once, show that $\sum_k (w_k)^2 = \sum_k (l_k)^2$, where w_k = the number of wins that team k collects and l_k = the number of losses that team k collects.
2. (\$2) Draw n straight lines in a plane such that no three intersect. Show that the resulting regions can be 2-coloured, that is, coloured in one of two colours such that no two bordering regions share the same colour.
3. (\$3) n real numbers are written on the board. Each turn two numbers a and b are erased and replaced with $a + \frac{b}{2}$ and $b - \frac{a}{2}$. Can the set of original numbers every be regained?
4. (\$3) Four points A,B,C and D lie on a circle radius r such that $AB = CD = \sqrt{2}r$, $BC = 6$ and $AD = 8$. Find r .
5. (\$3) On an $n \times n$ chess-board we infect $n - 1$ of the squares. Each minute the infection will spread to a non-empty square if at least two of its four direct neighbours are already infected. Could the infection eventually spread to cover the whole board?
6. (\$4) A triangle ABC has P on AB, Q on BC and R on AC such that $\triangle PQR$ is equilateral. Also, $AP = BQ = CR$. Prove that $\triangle ABC$ is equilateral.
7. (\$5) MUMS-land contains a thousand cities and possesses a dirt-road network such that a person at any city can get to any other city along them. The king of MUMS-land, Han, decides to pave some of the roads. Show that it's possible to pave some of these roads in such a way that every city is connected to an odd number of paved roads.

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Upcoming Events

University Maths Olympics

- The biggest MUMS event of the semester.
- Will take place in the second half of the semester.
- Watch the Richard Berry building for more details closer to the date.

Terry Tao Lecture

- Come see 'The Tao': Australia's own 'Mozart of Maths', the 2006 Field Medalist, and 'probably the best mathematician in the world'¹
- Bring a copy of Paradox to get his signature!
- 6pm Monday 31 August, Copland Theatre.

SUMS Puzzle Hunt

- Warm up for the MUMS Puzzle Hunt!
- The week starting Monday 31 August.
- Google 'SUMS Puzzle Hunt' for details.

¹John Garnett, professor and former chair of mathematics at UCLA.