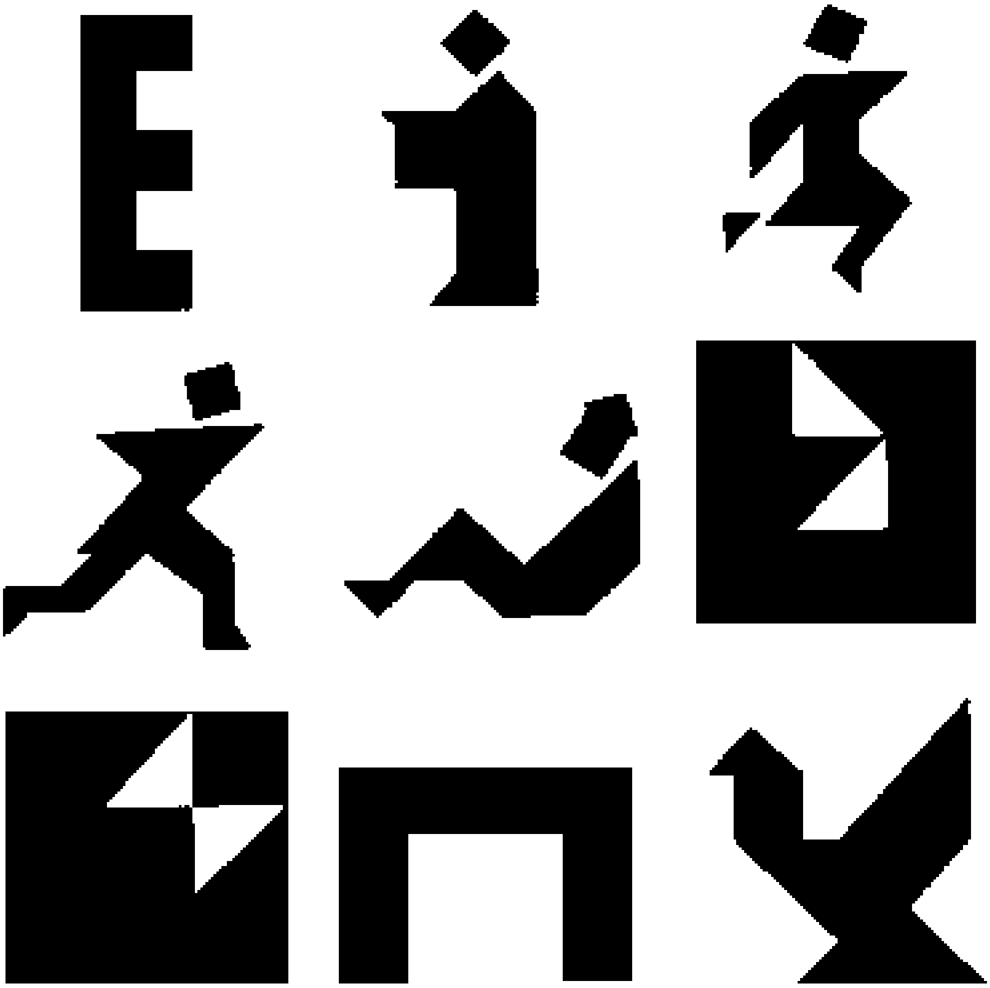
Paradox

Issue 1, 2002

THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY



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Paradox

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Words from the Editor...

Welcome to the first issue of Paradox for 2002. For those unaware, Paradox is a publication produced by the Melbourne University Mathematics and Statistics Society, which is more affectionately known as MUMS.

Since the early days pre-dating staples, Paradox has provided an entertaining and informative read for those of mathematical bent. This year, Paradox promises to continue the trend of being a feast for the mathematical palate. In particular, this issue is stacked with goodies such as an interview with current staff member, Kerry Landman. Remember to take the "Are you a maths geek?" quiz, appearing in the centrefold and brimming with hilarity. We also have articles about geometry, probability and the popular Tangram puzzle for your reading pleasure. And, as always, there are some problems for you to try your hand at, with cash prizes up for grabs.

We are always interested to hear from our readers, so if you have any comments or contributions, please email us (`paradox@ms.unimelb.edu.au`).

— Norman Do, *Paradox* Editor

... and some from the President

In addition to Paradox, MUMS organises many events throughout the year, including barbecues, trivia nights, seminars on interesting mathematical topics (don't worry, there's no prior knowledge required), and the much loved Maths Olympics.

I would encourage you all to come along and get involved in MUMS events, no matter what stage of a degree you're at. We advertise our events around the Maths building, on our notice board (opposite the drinking fountain), on our web site `http://www.ms.unimelb.edu.au/~mums/`, and via an email list. (You can subscribe by sending an email with the subject line "subscribe mums-events" to `mums@ms.unimelb.edu.au`.)

Furthermore, if you're interested in helping organise or run MUMS activities, please send an email to the above address. We're always interested in having more helpers!

I hope to see you at some of our events during second semester this year!

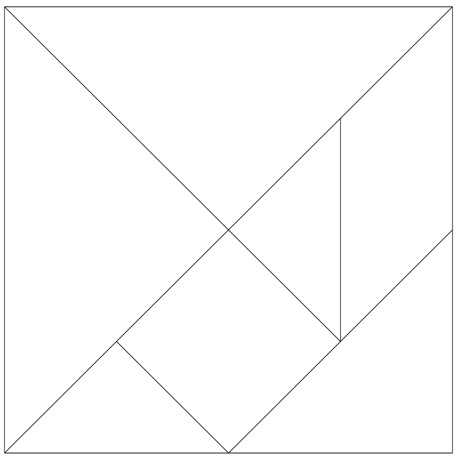
— Luke Mawbey, President of MUMS

The Do-It-Yourself Tangram Kit

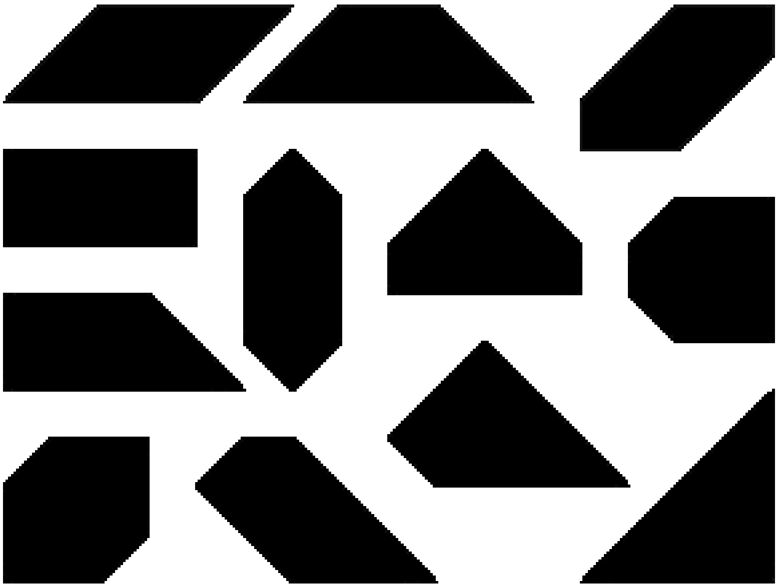
The Tangram is a puzzle thought to have originated in China over 1,000 years ago and has been popular ever since. This is due to its simplicity in nature and yet it provides a vast number of possibilities and problems for the Tangram enthusiast.

It is made up of seven geometric shapes known as tans - one square, two small triangles, one medium triangle, two large triangles and one parallelogram - all cut from a large square as shown below. The most popular Tangram puzzles involve recreating a shape seen only in silhouette by rearranging all of the pieces without allowing any overlap. For example, some of these Tangram shapes appear on the front cover of this edition of Paradox.

So trace out the Tangram figure below, cut out the seven tans, and have a go at recreating either the thirteen convex figures shown on the next page, or the Tangram puzzles shown on the front cover.



The Tangram caught the eye of one of world's most famous puzzle experts, Sam Loyd, whose book, The Eighth Book of Tan, helped to popularise the Tangram in the Western world. The puzzle also caught the eye of mathematicians Fu Traing Wang and Chuan-Chih Hsiung who published an article in American Mathematical Monthly, proving that there are only thirteen convex Tangrams possible. A convex figure is one where the line segment joining any two points of the figure remains entirely within it. (That is, it has no holes and all corners stick outwards.) The proof shows that there are only thirteen convex figures which can be made from the seven tans without overlap. Apart from the obvious square Tangram, the remaining twelve convex Tangrams are shown below.



For something a little different, consider the following puzzle which is based, more or less, on actual happenstance:

Karl made two Tangram sets as gifts - one to be sent to his sister and the other to his brother. The instructions were simply to assemble all the pieces into a square. Karl's sister brought hers back and declared (correctly) that the solution was impossible. Examining her set, they discovered that Karl had made a mistake in packing and had accidentally put two pieces into the wrong box, so one person got a set of five pieces and the other got nine. Embarrassed, Karl suggested that they phone their brother and explain the mistake. But his sister reflected for a moment and then said, "No that won't be necessary - he can make a square with his set." Can you tell who got the two extra pieces and what shape or shapes they were?

— Joseph Healy

References: The Tangrams Book, Randy Crawford, 1998, Carlton Books Limited, `http://www.johnrausch.com/PuzzlingWorld/chap01b.htm`, American Mathematical Monthly, Volume 49, Issue 9 (November, 1942), 596-599 (A Theorem on the Tangram)

Probability Approximations

A basic question in probability is: conduct n trials with $Pr(\text{success}) = p_1, \ldots, p_n$ respectively. What is the distribution of the number of successes, X?

This simple exercise has a very diverse range of applications. In finance, the trials could represent the fate of a stock on successive days with a success occurring if the price crosses some barrier. The trials could also be hourly rainfalls, or numbers of calls at a telephone exchange. Another important application is to DNA sequential study. A DNA sequence is formed from a 4-letter nucleotide alphabet {A (adenine), C (cytosine), G (guanine), T (thymine)}, with bases A and T forming a complementary pair, C and G forming the other pair. A palindrome of length, say $2s$, is a symmetric structure defined as sequence of s bases followed immediately by its inverted complementary sequence. For example, GCGCATGCGC constitutes a length 10 palindrome. Palindromes are of interest because they point to markers for replication in genes. For a sampled DNA sequence, at each base, the trial could be the occurrence of a starting point of a palindrome of at least, say, 10 bases (a success) and X would be the number of palindromes of at least 10 bases observed in the sequence.

It is well-known that if the trials are conducted independently and under identical conditions so that $p_1 = \cdots = p_n = p$, then X follows the binomial distribution $B(n,p)$. However, it is rather difficult to ensure the identical conditions and independence, so it is very natural to ask for approximations of the distribution of X.

For simplicity, we assume the trials are independent and $p_1, \ldots, p_n$ are small but unnecessarily equal. As the binomial distribution $B(n,p)$ is a particular case of X, we look at the binomial distribution first. It is a very old result that the Poisson distribution $Pn(np)$ closely approximates $B(n,p)$ if p is small and np is moderate. In fact, an elementary argument leads to

$$Pr(Y = k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{e^{-np}(np)^k}{k!}[1 + O(np^2)].$$

This in turn gives the estimate of error of Poisson approximation as

$$d(Y, Pn(np)) = \sum_{k=0}^{\infty} |Pr(Y = k) - Pn(np)(k)| = O(np^2),$$

where $Pn(np)(k) = e^{-np}(np)^k/k!$ and $d(Y, Pn(np))$ is called the total variation distance between Y and $Pn(np)$. This gives an upper bound for any absolute difference of corresponding probabilities for the two distributions. More careful analysis would give $d(Y, Pn(np)) \leq 2np^2$. For example, if $n = 200$, $p = 0.1$, then the bound is 4. However, direct calculation, using a computer package like Minitab or Excel, gives $d(Y, Pn(20)) = 0.0509504$. This suggests the Poisson distribution is a natural choice of approximation to the distribution of X, but the bound here is obviously too pessimistic. In order to justify the Poisson approximation, more must be done.

There has been a lot of research in this field and the most successful approach was introduced by Stein in 1971 in the context of the central limit theorem and adapted to Poisson approximation by Chen in 1975. Hence for the Poisson distribution, the approach is called the Stein-Chen method. Its idea is based on the fact that Z follows $Pn(\lambda)$ if and only if $E[\lambda g(Z+1) - Zg(Z)] = 0$ for all bounded functions g on the non-negative integers. Intuitively, if $E[\lambda g(X+1) - Xg(X)]$ is nearly zero for all such bounded g, then X must have a distribution very close to $Pn(\lambda)$. Some elementary calculation involving conditional expectation gives

$$d(X, Pn(\lambda)) \leq \sup_{\text{suitable } g} \|\Delta g\| \sum_{i=1}^{n} p_i^2,$$

where $\lambda = p_1 + \cdots + p_n$. The estimations of $\sup_{\text{suitable } g} \|\Delta g\|$ and its counterparts in other approximations turn out to be the most difficult part in this field. Some sophisticated analysis yields $\sup_{\text{suitable } g} \|\Delta g\| \leq \frac{2(1-e^{-\lambda})}{\lambda}$. For $n = 200$, $p = 0.1$, the bound gives $d(Y, Pn(20)) \leq 0.2$, which is much closer to the true value 0.0509504.

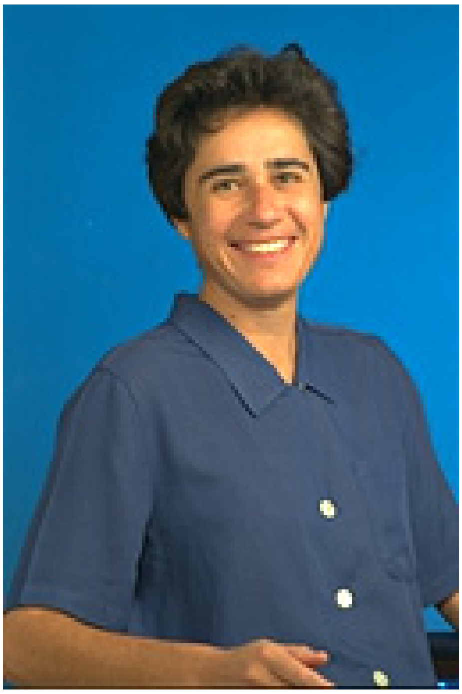
The bound of $\sup_{\text{suitable } g} \|\Delta g\|$ can also be estimated using Markov chains, thus enabling probability rather than more general analysis as the basis of the approach. In fact, our research shows this approach is much better than the analytical approach in the sense that it could be easily adapted to estimate the counterparts in other approximations. We found a large class of approximating distributions as convenient as the Poisson distribution but with orders of higher precision of approximation.

To explain how the new distribution works, let us assume the first 70 trials have $p = 0.1$, followed by 9 trials with $p = 1/3$ and 2 trials with $p = 2^{-0.5}$, then $d(X, Pn(11, 4142)) = 0.132$, and the total variation distance between X and the new distribution with two parameters is 0.00080!

— Professor Tim Brown and Dr Aihua Xia

Kerry Landman – Industrial Problem Solver

In the field of mathematical endeavour, which might seem impractical to the layperson, Kerry Landman has made her name by marketing mathematics to industry. As an Associate Professor in the Department of Mathematics and Statistics at the University of Melbourne, she balances her time between teaching undergraduate subjects, supervising postgraduate students and doing what she loves – solving problems using maths.



Kerry grew up in Melbourne and throughout her schooling, had strong interests in maths and physics. She completed a science degree with honours at Melbourne University, majoring in mathematics and statistics. At this stage, she decided that rather than look for a job, she would invest her time in maths and subsequently began a PhD. "Uni was different back then," Kerry explains. "There weren't computers like there are now. We had just one room with punch card computers... and I had to hand-write my thesis!"

Kerry's PhD thesis involved the newly burgeoning area of chaos theory – this straddled the fence between pure and applied maths. After completing her PhD, Kerry accepted a position at Massachusetts Institute of Technology (MIT) where she decided to concentrate solely on applied maths, using her knowledge to solve problems about physical phenomena. At MIT, she investigated the reasons behind red blood cells changing their shape. This was the beginning of Kerry's ongoing interest in applying mathematical methods to biomedical problems.

After a couple of years at MIT, Kerry was still unsure as to whether she wanted to pursue a life in academia. So she decided to accept a job at the Environmental Protection Agency. Kerry then spent three years in Dallas at Southern Methodist University before returning to her home town to work as a consultant for CSIROmath. In the mid 1980's, Kerry accepted a position at Melbourne University where she has remained ever since, plying her trade as a mathematical problem solver.

But no matter where Kerry worked, she always used mathematics to find solutions to new and interesting problems. These were often posed by people from a diverse range of areas as demonstrated by the following examples of Kerry's past work:

Indoor pollution Kerry's job was to investigate how radon gas managed to seep from the ground into people's houses, causing indoor pollution.

Windscreen wipers A large number of road accidents are caused by restricted visibility due to faulty windscreen wipers. Thus, Kerry was approached and asked to model and simulate the performance of various windscreen wiper designs. This mathematical approach had significant advantages over manufacturing the different designs and physically testing them.

Colloids (particles suspended in a fluid) Understanding these is very important in industries such as ceramics and minerals processing. Kerry's research into colloids began about fifteen years ago, but her results have only recently been used for industrial application.

Cooking cereal grains Amazingly enough, a large amount of mathematics goes into the cooking of cereal grains, which is required to make such breakfast nourishments as Vita Brits. Kerry spent some time looking at how such an industrial process can be improved. She jokingly says that her dream is "to one day see the words 'thermomathematically enhanced' used as advertising on the front of a cereal box!"

More recently, Kerry has been Director of the Mathematics in Industry Study Group (MISG) whose aim is to find solutions to problems posed by various companies. This allowed people to see the power and versatility of mathematics in business and industry. Currently, Kerry's pet project has been in conjunction with the Royal Children's Hospital, investigating the growth of the nervous system in developing embryos. This research aims to develop understanding of why some babies are born with nervous system deficiencies in their gastrointestinal tract.

It must be said that Kerry doesn't fit in with the stereotype of the eccentric, absent-minded mathematician as often portrayed in films. Firstly, she is a woman working in a field where traditionally there has been a significant gender imbalance towards male mathematicians. When asked about this, Kerry commented that "the situation at Melbourne Uni is probably better than at other places" and that "schools don't really encourage girls to study science". Furthermore, mathematics is not all that Kerry likes to do; in her spare time, she indulges in various pastimes such as walking, swimming and listening to music. She also enjoys being with her family, being out in the bush and travelling.

Kerry continues to do what she loves – solving problems using maths. She remarks that, "Maths sheds light on what's happening... Mathematicians know how to ask the right questions and can find similarities between seemingly unrelated problems."

Every triangle is equilateral!

The following is a well-known argument which "proves" the amazing fact that every triangle is equilateral! Of course, the argument is fallacious, since it is easy enough to construct a triangle which is not equilateral. See if you can spot the error...

Proof: Consider an arbitrary triangle $\triangle ABC$. Construct the line which bisects $\angle A$ and construct the line which is the perpendicular bisector of the side BC. There are four possibilities which can arise:

Case 1: The two lines are parallel.

Case 2: The two lines meet at a point D which is inside the triangle.

Case 3: The two lines meet at a point D which is on the triangle.

Case 4: The two lines meet at a point D which is outside the triangle.

Case 1: Since the bisector of $\angle A$ is parallel to the perpendicular bisector of BC, it is perpendicular to the side BC. Suppose the bisector of $\angle A$ meets the side BC at the point X.

Observe that:

1. $\angle BAX = \angle CAX$ since AX bisects $\angle A$;
2. $\angle BXA = \angle CXA = 90°$ since AX is perpendicular to BC;
3. AX is a common side length to $\triangle AXB$ and $\triangle AXC$.

Therefore, $\triangle AXB$ is congruent to $\triangle AXC$ (by ASA). In particular, we have $AB = AC$.

Cases 2, 3 and 4: In each of these cases, the point D is the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of the side BC. From D, drop perpendiculars to the lines AB, AC and BC to create the points E, F and X, respectively. (The lines AB and AC may need to be extended, as shown in the diagram for Case 4.)

Observe that in cases 2 and 4:

- $\angle DAE = \angle DAF$ since X bisects BC;
- $\angle DEA = \angle DFA = 90°$ by construction;
- AD is a common side length to $\triangle ADE$ and $\triangle ADF$.

Therefore, $\triangle ADE$ is congruent to $\triangle ADF$ (by ASA). In particular, we have $DE = DF$ and $AE = AF$.

Observe that:

- $BX = CX$ since DX is the perpendicular bisector of BC;
- $\angle BXD = \angle CXD = 90°$ by construction;
- DX is a common side length to $\triangle DXB$ and $\triangle DXC$.

Therefore, $\triangle DXB$ is congruent to $\triangle DXC$ (by SAS). In particular, we have $DB = DC$ for all cases.

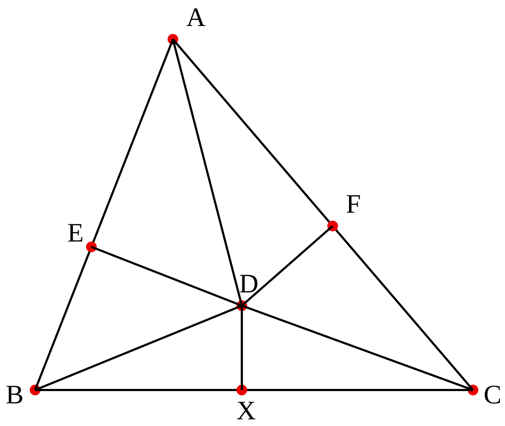
Observe that:

- $\angle DEB = \angle DFB = 90°$ by construction;
- $DB = DC$ as deduced above;
- $DE = DF$ as deduced above.

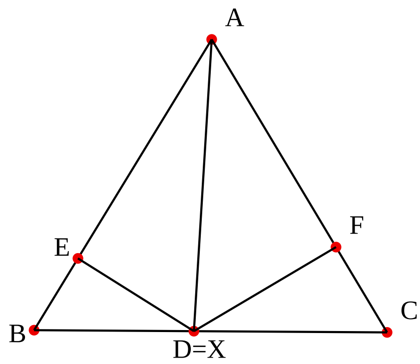
Therefore, $\triangle DEB$ is congruent to $\triangle DFC$ (by RHS). In particular, we have $EB = FC$.

Since we have $AE = AF$ and $EB = FC$:

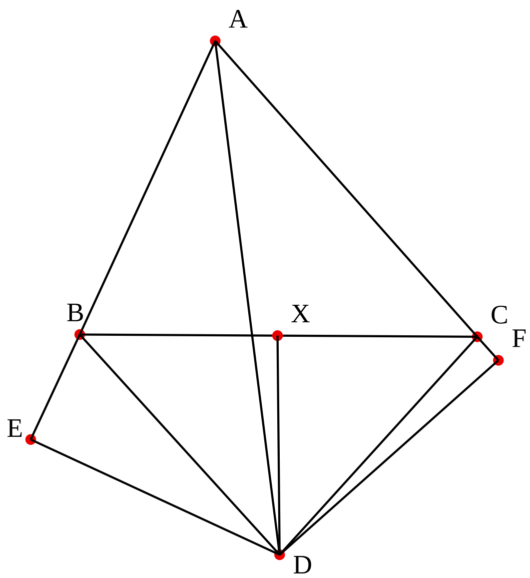
- Case 2: $AE + EB = AF + FC$ so $AB = AC$;



(a) Case 2



(b) Case 3



(c) Case 4

- Case 3: $AE + EB = AF + FC$ so $AB = AC$;
- Case 4: $AE - EB = AF - FC$ so $AB = AC$;

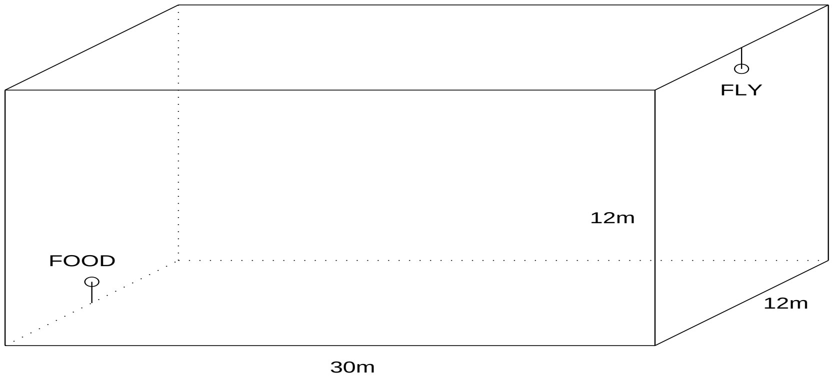
So in all four cases, $AB = AC$. But there was nothing special about the two sides AB and AC. We could have picked any two sides of $\triangle ABC$ and shown that they were equal in the same way. Therefore, we conclude that all three sides of $\triangle ABC$ must be equal. And since we chose $\triangle ABC$ arbitrarily, it follows that every triangle is equilateral!

— Norman Do

Problems

The following are some maths problems for which prize money is offered. The person who submits the best (i.e. clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number. Solutions may be emailed to `paradox@ms.unimelb.edu.au` or you can drop a hard copy of your solution into the MUMS pigeonhole near the Maths and Stats Office in the Richard Berry Building.

1. ($5) Consider a room in the shape of a rectangular prism with a width of 12 metres, a length of 30 metres and a height of 12 metres. A fly stands in the middle of one of the walls, exactly 1 metre from the ceiling (as shown in the diagram below). On the middle of the opposite wall, exactly 1 metre from the floor, is a piece of food which the fly wants to eat. However, the fly is unfortunate enough to have broken one of its wings, and can only reach the piece of food by walking along the walls of the room. What is the shortest distance that the fly can walk to get to the food?



2. (\$5) Is it possible to place the digits from 1 to 9 (each exactly once) into the boxes to make this fraction sum correct?

$$\frac{\square}{\square\,\square} + \frac{\square}{\square\,\square} + \frac{\square}{\square\,\square} = 1$$

3. (\$10) I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to the sum of the squares of the ages of the other three. In one year's time the sum of the squares of the ages of the oldest and the youngest will be equal to the sum of the squares of the other two children. Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

4. (\$10) AD, BE and CF are three chords of a circle which meet each other at angles of $60°$ at the point P. Prove that $PA + PC + PE = PB + PD + PF$.

— Geordie Zhang