Question 1. (5 points) **(180 points remaining)**
What is the product of all the (positive integer) factors of 18? Express your answer as a 4 digit integer.

Solution 1. The positive integer factors of 18 are 1, 2, 3, 6, 9, 18.

We can multiply these together directly. Alternatively we might notice that

$$1 \times 18 = 2 \times 9 = 3 \times 6 = 18$$

so that

$$1 \times 2 \times 3 \times 6 \times 9 \times 18 = 18^3 = 5832$$

The answer is 5832.

Question 2. (5 points) **(175 points remaining)**
A car travels from A to B at the rate of 20 km/h and then returns from B to A at the rate of 80 km/h, without stopping. What is the average speed for the round trip, in kilometres per hour?

Solution 2. Let the distance between A and B be d.

Time taken to travel A to $B = d/20$.
Time taken to return B to $A = d/80$.
So total time $= d/20 + d/80$.

Total distance travelled $= 2d$.

So the average speed is

$$\begin{aligned}\frac{\text{distance}}{\text{time}} &= \frac{2d}{d/20 + d/80} = \frac{2}{1/20 + 1/80} \\ &= \frac{2}{5/80} = \frac{2 \times 80}{5} = 32\end{aligned}$$

The answer is 32 km/h.

Question 3. (5 points) **(170 points remaining)**
In MUMSland coins are shaped like regular polygons. All coins have the same side length but they vary in the number of sides they have. There are some coins that are shaped like squares, and others that are shaped like hexagons. There is another type of coin, so that the square, hexagon and the third type of coin fit together snugly at a vertex, with no hole between coins (see the diagram below).

How many sides does the third type of coin have?

Solution 3. We use the fact that if a regular polygon has n sides, then each interior angle is $\frac{n-2}{n} \times 180°$.

The interior angles in a square are 90°.
The interior angles in a hexagon are 120°.

Since the three coins fit nicely together at a point, their interior angles add up to 360°.

So the third type of coin must have interior angles of 150°.

Let the third type of coin have n sides. Then

$$\frac{n-2}{n} \times 180^\circ = 150^\circ$$

$$\frac{n-2}{n} = \frac{150}{180} = \frac{10}{12}$$

So $n = 12$ and the third coin has 12 sides.

Question 4. (5 points) **(165 points remaining)**

In MUMSland money comes in the currency of proofs. Proofs are divided up into smaller denominations called propositions and lemmas. 1 proof is worth 10 propositions. 1 proposition is worth 10 lemmas. There are coins in circulation worth 1 lemma, 2 lemmas, 5 lemmas, 1 proposition, 2 propositions, 5 propositions, 1 proof and 2 proofs.

Lori is about to travel on a hot air balloon to her economics tutorial but she needs to pay the fare, and must pay exactly. She knows the fare is more than 1 proof and less than 3 proofs. At least how many coins must she carry to be sure of carrying the exact correct fare?

Solution 4. For this question, it is much easier to obtain the answer than to prove it rigorously!

Note that the currency is identical to a dollars-cents currency, with 1c, 2c, 5c, 10c, 20c, 50c, \$1, \$2 coins. We will use the language of dollars and cents for this solution as it is simpler.

Consider how many coins we need to obtain the amounts of \$1.88 and \$2.99.

Note that, if we have a collection of coins worth \$1.88, then there must be a subcollection of coins worth exactly 8c. (This is not entirely obvious – the same argument would not apply for \$1.11 = \$1 + 5c + 2c + 2c + 2c). Of the remaining \$1.80 in coins, similarly there must be a subcollection of coins worth exactly \$1. Thus we can divide the collection into three parts worth \$1, 80c and 8c respectively.

We can do the same for a \$2.99 collection into parts worth \$2, 90c and 9c.

For the 8c and 9c subcollections, taken together, at least 4 coins are required, since at least 3 coins are required for 8c, at least 3 coins are required for 9c, and the sets of 3 coins are not identical, using at least 4 distinct coins in total. These 4 or more coins must all be 5c or less.

Similarly for the 80c and 90c subcollections, at least 4 coins are required. If these subcollections include 5c or smaller denominations, then at least 5 coins are required. In either case al the coins are 50c or smaller.

And obviously, for the \$1 and \$2 subcollections, at least 2 coins are required. If these subcollections include 50c or smaller denominations, then at least 3 coins are required.

So, these subcollections in total require 4+4+2 coins. It is easy to verify that there cannot be an overlap between subcollections (eg 90c and 8c subcollections) allowing a smaller number of coins, since then there must be 50c or 5c coins requiring larger subcollections. So at least 10 coins are required.

The following 10 coin collection can easily be verified to allow every possible fare to be achieved:

$$\$2, \$1, 50c, 20c, 20c, 10c, 5c, 2c, 2c, 1c$$

So the answer is 10.

Question 5. *Change Runner Now!* (5 points — 160 points remaining)

Sam's credit card number has 14 digits. But the number is special — the sum of every four consecutive digits is 16. The first digit is 1, the second digit is 6, and the twelfth digit is 3.

What is the seventh digit in Sam's credit card number?

Solution 5. Consider five digits in a row on Sam's credit card a, b, c, d, e.

Then $a + b + c + d = 16$ and $b + c + d + e = 16$.

But then $a + b + c + d = b + c + d + e$ so we must have $a = e$.

This tells us that the digits in Sam's credit card number repeat every 4 digits!

We are given the first digit is 1, the second is 6, and the twelfth is 3.

By the repetition every 4 digits, the fourth digit must be the same as the twelfth, which is 3.

Since the first 4 digits must add up to 16, the third digit must be 6.

The seventh digit is the same as the third digit, which is 6.

Question 6. (5 points) (155 points remaining)

Geordie baked some loaves of bread today and decided to give them away in the following fashion: the first person who came received half the bread he baked plus half a loaf. The second person received half of what was left plus half a loaf, and so on. After the fifth person had left, Geordie realised that all of his bread was gone. How many loaves of bread did Geordie bake?

Solution 6. Suppose a person Fred comes to visit Geordie. Let there be x loaves when Fred arrives, and y when he leaves. Thus $y = \frac{x}{2} - \frac{1}{2}$. Equivalently $x = 2y + 1$. So if there were y loaves when Fred leaves, then there are $2y + 1$ loaves when Fred arrives.

There were 0 loaves remaining when the fifth person left, hence $2 \times 0 + 1 = 1$ when he arrived. Hence there were $2 \times 1 + 1 = 3$ loaves when the fourth person arrived; $2 \times 3 + 1 = 7$ loaves when the third person arrived; $2 \times 7 + 1 = 15$ when the second person arrived; and $2 \times 15 + 1 = 31$ at the start.

Thus Geordie baked 31 loaves of bread.

Question 7. (5 points) (150 points remaining)

What is the smallest non-negative value of the following expression?

$$\pm 1^3 \pm 2^3 \pm 3^3 \pm 4^3 \pm 5^3$$

Solution 7. 5^3 is a lot bigger than $1^3, 2^3, 3^3$ or 4^3. In fact, if we choose the minus sign for 5^3, then we can't get a positive value — the highest we can get is

$$+1^3 + 2^3 + 3^3 + 4^3 - 5^3 = -25.$$

So we must choose the plus sign for 5^3.

Then the smallest value we can get is clearly

$$-1^3 - 2^3 - 3^3 - 4^3 + 5^3 = 25$$

Since this is non-negative, 25 must be the answer.

Question 8. (5 points) (145 points remaining)

A trapezium is divided into four triangles by its diagonals. The areas of the top and bottom triangles (adjacent to the parallel sides) are 25 and 49 square light years respectively. Find the area of the trapezium in square light years.

Solution 8. INSERT PICTURE!!!

Let the areas of the remaining triangles (equal areas by symmetry) both be x square light years. Let perpendicular heights from both parallel sides to the point of intersection of the diagonals be h_1 and h_2 as shown.

Consider triangles with area 25 and $25 + x$. Both have the same base, so the ratio of areas is equal to the ratio of heights. Thus $\frac{25+x}{25} = \frac{h_1+h_2}{h_1}$, and $\frac{x}{25} = \frac{h_2}{h_1}$. A similar argument with triangles of area 49 and $49 + x$ shows $\frac{49}{x} = \frac{h_2}{h_1}$. Thus

$$\frac{x}{25} = \frac{h_2}{h_1} = \frac{49}{x}$$

Solving for x gives $x = 36$

Question 9. (10 points) (135 points remaining)

Joe loved cantaloupes. He put a spherical one in his cubical cantaloupe fridge and it just fit — touching all of the fridge's (very thin) sides and lid.

However an alien civilization angry at human worship of cantaloupes destroyed all cantaloupe plantations on Earth. But Joe was prepared for an escalation of anti-cantaloupist violence and had purchased a cantaloupe-fridge cryogenic suspension device. To preserve his cantaloupe for future generations, Joe put the fridge inside the spherical suspension chamber and it just fit.

The chamber wasn't very well packed, so Joe decided to remove the fridge, puree all his cantaloupes (all identical) and pour the puree directly into the chamber. How many pureed cantaloupes can Joe fit in the cantaloupe suspension chamber? (The answer need not be a whole number, and should be expressed exactly).

Solution 9.

Let the radius of a canteloupe be r. Then by two applications of Pythagoras' theorem, the distance from the centre of the fridge to its corner is $r\sqrt{3}$. Since the corner of the fridge touches the chamber, the radius of the chamber is also $r\sqrt{3}$.

The volume of a canteloupe is $\frac{4}{3}\pi r^3$. The volume of the chamber is $\frac{4}{3}\pi \left(r\sqrt{3}\right)^3$. Thus the number of pureed canteloupes required is

$$\frac{\frac{4}{3}\pi \left(r\sqrt{3}\right)^3}{\frac{4}{3}\pi r^3} = \frac{r^3 3\sqrt{3}}{r^3} = 3\sqrt{3}.$$

Question 10. *Change Runner Now!* **(10 points — 125 points remaining)**
Claire and her grandson were born on the same day of the year. One year on that day Claire noted that her age had become an integral multiple of her grandson's age and furthermore, this phenomenon would be repeated for the following five birthdays as well. How old was Claire when she made these observations?

Solution 10. Let Claire's age be b and her grandson's age be a. So we have:

$$\begin{array}{rcl} a \mid b & \Leftrightarrow & a \mid b-a \\ a+1 \mid b+1 & \Leftrightarrow & a+1 \mid b-a \\ a+2 \mid b+2 & \Leftrightarrow & a+2 \mid b-a \\ a+3 \mid b+3 & \Leftrightarrow & a+3 \mid b-a \\ a+4 \mid b+4 & \Leftrightarrow & a+4 \mid b-a \\ a+5 \mid b+5 & \Leftrightarrow & a+5 \mid b-a \end{array}$$

Thus the least common multiple of $a, a+1, a+2, a+3, a+4, a+5$ is a factor of $b-a$. Among the six consecutive numbers, there must be multiples of 3, 4, 5, so the least common multiple is a multiple of 60. Thus $b-a$ is a multiple of 60. Clearly $b-a = 60, 120$ are the only realistic answers.

Thus we have $a, a+1, a+2, a+3, a+4, a+5$ all factors of 60, or all factors of 120. But

$$\begin{aligned} \text{Factors of } 60 &= \{1,2,3,4,5,6,10,12,15,20,30,60\} \\ \text{Factors of } 120 &= \{1,2,3,4,5,6,8,10,12,15,20,24,30,40,60,120\} \end{aligned}$$

so the only possible value of a is 1, and Claire was 61 or 121 (both answers are correct).

Question 11. (10 points) **(115 points remaining)**
In Norm's favourite maths textbook, on each page the page number appears once, starting from page 1 (numbers are written in decimal notation). 642 digits are used in Norm's favourite textbook to number pages. How many pages are in Norm's favourite textbook?

Solution 11. On pages 1 to 9 there are 9 digits used.

From pages 10 to 99 there are 90 pages, with 2 digits on each, so 180 digits are used.

This leaves $642 - 180 - 9 = 453$ digits remaining, and above 99 each page uses 3 digits, so this leaves $453 \div 3 = 151$ pages. So the number of pages is $99 + 151 = 250$.

Question 12. (10 points) **(105 points remaining)**
$WXYZ$ is a square of side length 30. V is a point on XY and P is a point inside the square with PV perpendicular to XY. $PW = PZ = PV - 5$. What is PV?

Solution 12. INSERT PICTURE!!!

Let A be the foot of the perpendicular from P to WX as shown, so $AXVP$ is a rectangle. Since $PW = PZ$, P lies halfway between WX and ZY, so $AP = XV = VY = 15$. Let $PW = x$ so $PV = x + 5 = AX$ and $WA = 30 - AX = 30 - x - 5 = 25 - x$. Then by

Pythagoras we have

$$\begin{aligned} WA^2 + AP^2 &= PW^2 \\ (25-x)^2 + 15^2 &= x^2 \\ 850 = 50x \end{aligned}$$

Therefore $x = 17$ and $PV = x + 5 = 22$.

Question 13. (10 points) (95 points remaining)

$PQRS$ is a rectangle in which $PQ = 2PS$. T and U are the midpoints of PS and PQ respectively. QT and US intersect at V. What is the area of $QRSV$ divided by the area of PQT? Express you answer as a fraction in simplest terms.

Solution 13. INSERT PICTURE!!!

We note that as T and U are midpoints of PS and PQ, QT and SU are medians of triangle PQS and V is the centroid of this triangle. Drawing in the remaining side and median of PQS, and noting that $|PQS| = \frac{1}{2}|PQRS|$, we can use the well-known fact that the medians of a triangle divide it into 6 triangles of equal area.

Thus

$$|PQT| = \frac{1}{2}|PQS| = \frac{1}{4}|PQRS|$$

and

$$|QRSV| = |QRS| + |QSV| = \frac{1}{2}|PQRS| + \frac{1}{3}|PQS| = \frac{1}{2}|PQRS| + \frac{1}{3}\frac{1}{2}|PQRS| = \frac{2}{3}|PQRS|$$

so

$$\frac{|QRSV|}{|PQT|} = \frac{\frac{2}{3}|PQRS|}{\frac{1}{4}|PQRS|} = \frac{8}{3}.$$

Question 14. (10 points) (85 points remaining)

Jolene has a 13 pound measuring weight that broke into three pieces as the result of a freak tsunami. When the pieces were subsequently weighed, it was found that the weight of each piece was a whole number of pounds and that using a two-pan balance, the three pieces could be used to weigh every integral weight between 1 and 13 pounds. What were the weights of the pieces in pounds, in increasing order?

Solution 14. Note that, for example, Jolene can weigh a 5 pound weight either by putting 5 pounds on one pan of the balance, or by putting weights on both pans differing by 5 pounds (eg 9 pounds on one pan and 4 on the other).

The only way to weigh a 12 pound weight is to put 12 pounds on one pan and nothing on the other – any other combination would total more than 13 pounds. As the weights of the pieces are integral numbers of pounds, two of the pieces add up to 12 pounds, and the remaining piece is a 1 pound weight.

To weigh a 10 pound weight, there are only two possibilities – 1 pound on one side and 11 pounds on the other; or a 10 pound weight on one pan only. In the first possibility, as there are only three pieces, they must be 1,1,11. But in this case there is no way to weigh 3 pounds.

Thus there are pieces totalling 10 pounds – either one piece, giving the pieces as 1,2,10 pounds; or two pieces, giving 1,3,9 pounds. But in the first possibility there is no way to measure a 4 pound weight. Thus the pieces must weigh 1,3,9 pounds, and it is easy to check that using these pieces any integral number of pounds can be weighed.

Question 15. *Change Runner Now!* **(10 points — 75 points remaining)**

Let a, b and c denote the angles of elevation of a tower measured at horizontal distances of 100, 200 and 300 metres from the tower, respectively. If $a+b+c$ is a right angle, find the height of the tower in metres.

Solution 15. Let the height of the tower in metres be h. So $\tan a = \frac{h}{100}$, $\tan b = \frac{h}{200}$, $\tan c = \frac{h}{300}$. Since $a+b+c = 90°$, $\tan c \tan(a+b) = 1$. Expanding using the addition formula for tan gives

$$\begin{aligned}
\frac{\tan c(\tan a + \tan b)}{1-\tan a \tan b} &= 1 \\
\tan c(\tan a + \tan b) &= 1 - \tan a \tan b \\
\frac{h}{300}\left(\frac{h}{100}+\frac{h}{200}\right) &= 1 - \frac{h}{100}\frac{h}{200} \\
\frac{3h^2}{60000} &= 1 - \frac{h^2}{20000} \\
h^2 &= 10000
\end{aligned}$$

so $h = 100$ and the tower is 100m high.

Question 16. (10 points) **(65 points remaining)**

A mathematician had nine children when he died — quadruplets aged 17, quadruplets aged 18, and one nineteen year old – all born on the same day of the year. His assets, totalling \$100,000, according to his will, were to be divided and put into super-speculative interest bearing trusts. The trust fund compounded the amounts for each child annually until they turned 21, upon which they could access the amount.

As each child turned 21 they were overjoyed to discover that each individual inheritance had compounded to the same amount of \$400,000. The interest rate had remained constant over the entire period.

If you had put \$1 into the fund a year ago, how many dollars would you have gained in the last year? Express your answer exactly.

Solution 16. Let the amount originally given to each of the 17-year-olds be A, the amount given to the 18-year-olds be B and the amount given to the 19-year-old be C. Let I be the factor by which amounts are multiplied each year – so if there was \$1 in the fund one year, the next year it would have increased to \$$I$.

Now $4A + 4B + C = 100000$ and $I^4A = I^3B = I^2C = 400000$. Substituting $C = I^2A$, $B = IA$ into the first equation gives

$$A(4 + 4I + I^2) = 100000 \Rightarrow A = \frac{100000}{(I+2)^2}.$$

Now we solve $I^4A = 400000$ to obtain I:

$$\begin{aligned} \frac{100000I^4}{(I+2)^2} &= 400000 \\ I^4 &= 4(I+2)^2 \\ I^2 - 2I - 4 &= 0 \\ (I-1)^2 - 5 &= 0 \end{aligned}$$

so $I = 1 \pm \sqrt{5}$. But $I = 1 - \sqrt{5} < 0$ is unrealistic so $I = 1 + \sqrt{5}$. If you had put \$1 in the fund a year ago you would now have $\$(1 + \sqrt{5})$ and would have gained $\$\sqrt{5}$.

Question 17. (15 points) **(50 points remaining)**

How many pairs of natural numbers (u, v) are there such that $1 \le u \le v$, with the property that the least common multiple of u and v is 2000?

Solution 17. We will first drop the consideration that $u \le v$, and just find pairs of natural numbers (u, v) with least common multiple 2000.

Let $u = 2^a5^b$, $v = 2^c5^d$. So the least common multiple is $2^{\max(a,c)}5^{\max(b,d)} = 2000 = 2^45^3$. We can restate the problem now as: find all sets of four non-negative integers $\{a, b, c, d\}$ with $\max(a, c) = 4$ and $\max(b, d) = 3$.

Consider a and c. If $a = c$ then there is only one possibility: $a = c = 4$. If $a \ne c$ then one of them must be 4 – there are 2 choices as to which one. The other one can then be $0, 1, 2$ or 3 – there are 4 choices. It is clear that we have then counted all possible $\{a, c\}$ exactly once, so there are $1 + 2 \times 4 = 9$ possibilities.

Now consider b and d with a similar argument. If $b = d$ then $b = d = 3$. If $b \ne d$ then one of them is 3, and the other is one of $0, 1, 2$. This gives $1 + 2 \times 3 = 7$ possibilities.

Thus there are $9 \times 7 = 63$ sets of four non-negative integers $\{a, b, c, d\}$ with $\max(a, c) = 4$ and $\max(b, d) = 3$. So there are 63 pairs of natural numbers (u, v) with least common multiple 2000.

If now $u \le v$, then we have counted every pair twice, except the one where $u = v$, in which case $u = v = 2000$. So there are 31 pairs we have counted twice and 1 pair we have counted once – a total of 32 pairs.

Question 18. (15 points) **(35 points remaining)**

The positive fractions can be written out in the following pattern, occupying an infinite number of positions:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \frac{4}{2}, \frac{3}{3}, \frac{2}{4}, \frac{1}{5}, \ldots$$

For instance the number $\frac{1}{3}$ occupies position number 6 and the number $\frac{2}{6}$ $\left(=\frac{1}{3}\right)$ occupies position number 42.

Consider the first five positions occupied by $\frac{1}{2}$, or by fractions equal to $\frac{1}{2}$. What is the sum of the numbers of these five positions?

Solution 18. (Note there was a minor typo in the original statement of the question! However this didn't affect the problem or its solution.)

Examining the sequence, we see that there is 1 term with $\{\text{numerator}\}+\{\text{denominator}\} = 2$, then 2 terms with $\{\text{numerator}\}+\{\text{denominator}\} = 3$, then 3 terms with $\{\text{numerator}\}+\{\text{denominator}\} = 4$, and so on. Within each of these sections, we see that the fractions are arranged in order of increasing denominator, from 1 upwards.

So, to find, for example $\frac{5}{10}$, this is in the section with $\{\text{numerator}\}+\{\text{denominator}\} = 15$. So there are previous sections with $\{\text{numerator}\}+\{\text{denominator}\} = 2, 3, \ldots, 14$ – and the total length of these previous sections is $1 + 2 + \cdots + 13 = \frac{13\times 14}{2}$. Within its section, $\frac{5}{10}$ is the 10^{th} term, so it is the $\frac{13\times 14}{2} + 10$'th term of the overall sequence.

By the same argument, the fraction $\frac{a}{b}$ occupies position number $\frac{(a+b-2)(a+b-1)}{2} + b$ in the overall sequence.

Now the first five fractions in the sequence equal to $\frac{1}{2}$ are clearly $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}$. So:

$$\begin{array}{lll} \frac{1}{2} \text{ occupies position} & \frac{1\times 2}{2} + 2 & = 3 \\ \frac{2}{4} \text{ occupies position} & \frac{4\times 5}{2} + 4 & = 14 \\ \frac{3}{6} \text{ occupies position} & \frac{7\times 8}{2} + 6 & = 34 \\ \frac{4}{8} \text{ occupies position} & \frac{10\times 11}{2} + 8 & = 63 \\ \frac{5}{10} \text{ occupies position} & \frac{13\times 14}{2} + 10 & = 101 \end{array}$$

The sum of these positions is $3 + 14 + 34 + 63 + 101 = 215$.

Question 19. (15 points) (20 points remaining)

Find all positive integers n such that $2^{200} + 2^4 + 2^n - 2^{103}$ is a perfect square.

Solution 19. To find an answer, we might note that $2^{200} = (2^{100})^2$, $2^4 = (2^2)^2$, and that

$$\left(2^{100} + 2^2\right)^2 = 2^{200} + 2^4 + 2 \times 2^{102} = 2^{200} + 2^4 + 2^{103}$$

is a perfect square, which is quite close to the expression given. In fact, if we note that $2^{104} - 2^{103} = 2^{103}$, then letting $n = 104$ gives us exactly this perfect square!

While this would have been sufficient for the purposes of the competition, we'll show that this is the only answer. First we note that we can factorise $2^{200} + 2^4 - 2^{103} = (2^{100} - 2^2)^2$. This gives

$$\left(2^{100} - 2^2\right)^2 + 2^n = x^2$$

for some positive integer x. Now we can rewrite and factorise as follows:

$$\begin{aligned} 2^n &= x^2 - \left(2^{100} - 2^2\right)^2 \\ &= \left(x - 2^{100} + 2^2\right)\left(x + 2^{100} - 2^2\right) \end{aligned}$$

Since both the expressions in brackets are integers, they must be powers of 2. Then we have

$$x - 2^{100} + 2^2 = 2^a \quad (1)$$
$$x + 2^{100} - 2^2 = 2^b \quad (2)$$

where $a + b = n$.

Subtracting (1) from (2) now gives

$$\begin{aligned} 2^b - 2^a &= 2 \times 2^{100} - 2 \times 2^2 \\ 2^a\left(2^{b-a} - 1\right) &= 2^{101} - 2^3 \\ 2^a\left(2^{b-a} - 1\right) &= 2^3\left(2^{98} - 1\right) \end{aligned}$$

Since 2^a and 2^3 are powers of 2, while $2^{b-a} - 1$ and $2^{98} - 1$ are odd, we must have $a = 3$ and $b - a = 98$, so $b = 101$. This gives $n = a + b = 104$ as the only possible answer.

Question 20. (20 points) FINAL QUESTION!

Let $x = (4+\sqrt{17})^{714}$. It is known that the decimal representation of x begins 3480003819317.... There are infinitely many digits after the decimal point and N digits before the decimal point. What are the $(N-1)^{\text{th}}$ and N^{th} digits of x after the decimal point, in order?

First we note that for any integer $m \geq 0$,

$$\left(4 + \sqrt{17}\right)^m + \left(4 - \sqrt{17}\right)^m.$$

is an integer. This is because $(4+\sqrt{17})^0 + (4-\sqrt{17})^0 = 2$ and $(4+\sqrt{17})^1 + (4-\sqrt{17})^1 = 8$ are integers, and

$$\begin{aligned} \left(4+\sqrt{17}\right)^m + \left(4-\sqrt{17}\right)^m &= \left[\left(4+\sqrt{17}\right)^{m-1} + \left(4-\sqrt{17}\right)^{m-1}\right]\left[\left(4+\sqrt{17}\right) + \left(4-\sqrt{17}\right)\right] \\ &\quad - \left(4-\sqrt{17}\right)\left(4+\sqrt{17}\right)^{m-1} - \left(4+\sqrt{17}\right)\left(4-\sqrt{17}\right)^{m-1} \\ &= 8\left[\left(4+\sqrt{17}\right)^{m-1} + \left(4-\sqrt{17}\right)^{m-1}\right] + \left(\sqrt{17}-4\right)\left(\sqrt{17}+4\right)\left(4+\sqrt{17}\right)^{m-2} \\ &\quad + \left(\sqrt{17}+4\right)\left(\sqrt{17}-4\right)\left(4-\sqrt{17}\right)^{m-2} \\ &= 8\left[\left(4+\sqrt{17}\right)^{m-1} + \left(4-\sqrt{17}\right)^{m-1}\right] + \left[\left(4+\sqrt{17}\right)^{m-2} + \left(4-\sqrt{17}\right)^{m-2}\right] \end{aligned}$$

so by induction, $\left(4+\sqrt{17}\right)^m + \left(4-\sqrt{17}\right)^m$ is an integer for all integers $m \geq 0$. (Alternatively, for those familiar with recurrence relations, this expression is the solution of the recurrence $x_n = 8x_{n-1} + x_{n-2}$ with $x_0 = 2, x_1 = 8$).

Thus:

$$x = \left(4+\sqrt{17}\right)^{714} = \{\text{integer}\} - \left(4-\sqrt{17}\right)^{714}$$

The second thing to note is that

$$\left(4+\sqrt{17}\right)^{714}\left(4-\sqrt{17}\right)^{714} = \left[\left(4+\sqrt{17}\right)\left(4-\sqrt{17}\right)\right]^{714} = (-1)^{714} = 1$$

so that

$$\left(4-\sqrt{17}\right)^{714} = \frac{1}{\left(4+\sqrt{17}\right)^{714}} = \frac{1}{x}.$$

Combining all this gives us that $x = \{\text{integer}\} - \frac{1}{x}$, and $\frac{1}{x}$ is very small indeed, clearly less than 1.

So to figure out the $(N-1)^{\text{th}}$ and N^{th} digits of x after the decimal point, we can find the digits of $\frac{1}{x}$. We can estimate $\frac{1}{x}$, using the information that x has N digits beginning with $348\cdots$. So:

$$\begin{array}{rcccl} & 3.4\times 10^{N-1} & < x < & 3.5\times 10^{N-1} & \\ & \dfrac{1}{3.5}\times 10^{1-N} & < \frac{1}{x} < & \dfrac{1}{3.4}\times 10^{1-N} & \\ 2\times 10^{-N} < & \dfrac{100}{35}\times 10^{-N} & < \frac{1}{x} < & \dfrac{100}{34}\times 10^{-N} & < 3\times 10^{-N} \end{array}$$

This gives us

$$\frac{1}{x} \quad = \quad \overbrace{0.00\cdots 0}^{N-1 \text{ zeroes}} \quad 2\cdots$$

and now we can do a long subtraction of sorts to obtain the $(N-1)$'th and N'th digits of x, since $x = \text{integer} - \frac{1}{x}$:

$$\begin{array}{cccccccc} & \text{integer} & \rightarrow & & \cdots. & \overbrace{0000000\cdots 0}^{N-1 \text{ digits}} & 00\cdots \\ - & \frac{1}{x} & \rightarrow & - & 0. & 0000000\cdots 0 & 2\cdots \\ = & x & \rightarrow & = & \cdots. & 9999999\cdots 9 & 7\cdots \end{array}$$

So the $(N-1)^{\text{th}}$ and N^{th} digits of x after the decimal point, in order, are 9 and 7.