
Paradox

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THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY



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An infinite number of mathematicians walk into a bar. The first one orders a beer. The second orders half a beer. The third, a quarter of a beer. The bartender says: "You're all idiots", and pours two beers.

Words from the Editor

Welcome to 2013's first issue of Paradox, the magazine produced by the Melbourne University Mathematics and Statistics Society (MUMS). Over the past few years, this publication has morphed from a problem-solving extravaganza to a forum for bad maths jokes, then a recovery period coinciding with a pre-occupation about a now world-famous past Vice-President, and then a forum for bad maths jokes again.

But there's more to Paradox than just bad maths jokes. Have you ever wondered what the secrets to happiness are... for numbers? Do you want to avoid making a common mistake when estimating your retirement funds? Or do you want to have a crack at some puzzles that will make you never look at Sudokus the same way again? Well then, never fear, Paradox is here!

Old heroes return once more to Paradox, now accompanied by flashbacks from the past. Rather than being there to make up for a lack of submissions (I'm overflowing with them at the moment!), flashbacks are there to reintroduce you to the content of Paradoxes old alongside those of Paradoxes new.

The curious should note that Paradox has existed since 1981 but the online archive only has issues since 1996. The super-curious will find that there was in fact another MUMS publication once upon a time in a galaxy far far away... discovered by yours truly only when it was dropped before his eyes in the MUMS Room when the Maths Library was shutting down. More in this saga will unfold... eventually!

As usual, many thanks go to all of our contributors, without whom Paradox could not exist. Having broken the all-time record for the number of issues last year, Paradox is radically aiming to beat that record yet again, but it will only be possible with your help! Please keep contributing articles, puzzles, news, reviews, jokes, problems, cartoons, or anything else that has something to do with maths, stats, or logic to paradox.editor@gmail.com. With my somewhat non-mathematical background, perhaps you could get away with submitting something even far removed from maths, but no promises there, even from this Radical Editor!

Yours radically,
Kristijan Jovanoski
Paradox Editor

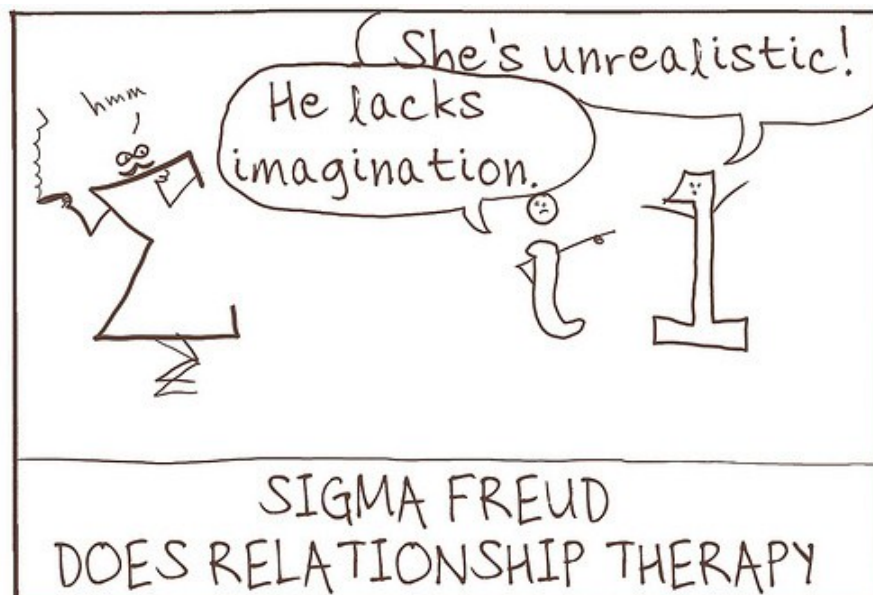
Words from the President

As we charge our calvalry through the vanguard of a new year, we give praise to the years of Mathematics past and look to the future battles of Mathematics future.

Soon, the fight for the wallspace of Richard Berry will be won, with the evil blankness banished to the foul realms below and replaced with the shining paragon of virtue: glorious posters for our most eminent society. Off in the distance, the war cries of our Friday Seminar series can be heard echoing over the throng of petulant first-years eager to immerse themselves in the blood-pumping world that we provide.

In-depth I hear you read to yourself? A new in-depth seminar series will be making its way to our rustic shores in the second portion of this fine semester. In addition, the annual Puzzle Hunt will reveal itself in a frenzy of solving that the singers will recount for years to come. Much will become of this year. Follow us and see!

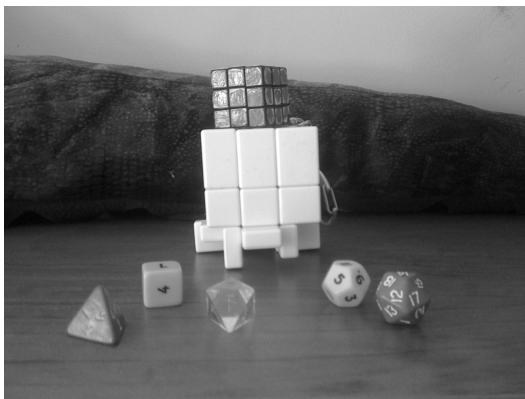
Onwards to glory,
President Imperator
Giles David "Limbrusher" Adams
MUMS President



The Adventures of Rubik's Turtle

— Dougal Davis and Jinghan Xia

Episode 5: The Vault of Mysteries



Previously, Rubik's Turtle and Rubik's Sidekick retrieved the Platonic Gems of Polyhedra from the villainous STAPLERS OF DOOM!



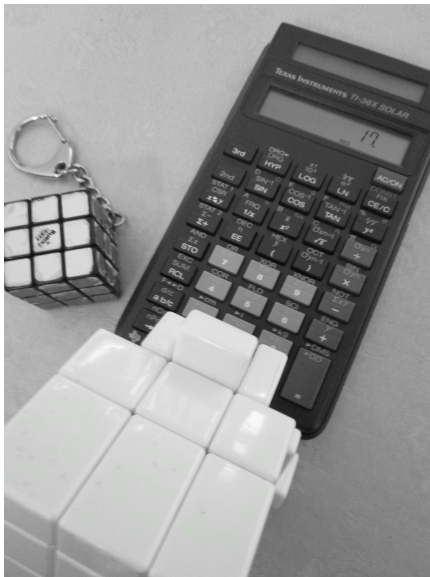
Their mysterious leader, Mother-of-all-Rubiks, gives them their next mission: to unlock the mystical Vault of Mysteries.



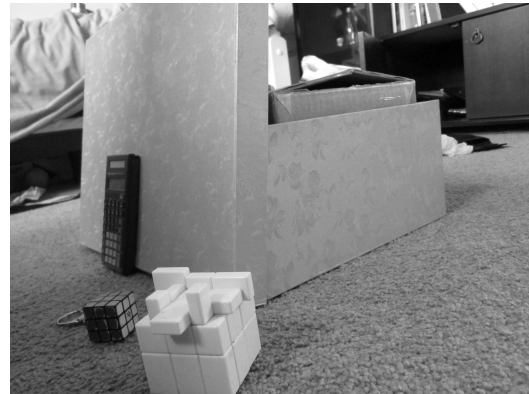
Our heroes arrive at the Vault of Mysteries to find it guarded by a secretive sentinel.



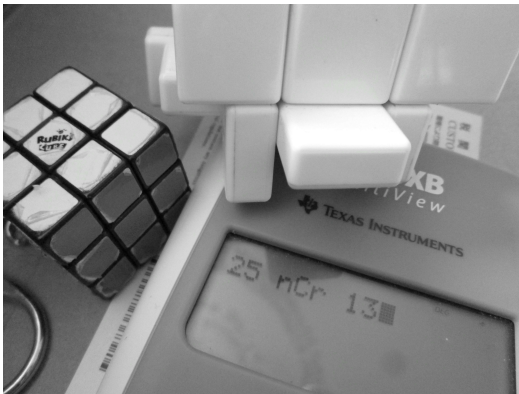
"Halt! I am TI-36X SOLAR. My brethren and I guard this vault. To pass you must enter the answers to our riddles."



*Rubik's Sidekick: "Uh... try 20.
That's my birthday!"
Rubik's Turtle: "Nah, that's
silly... I'll do my birthday
instead!"*



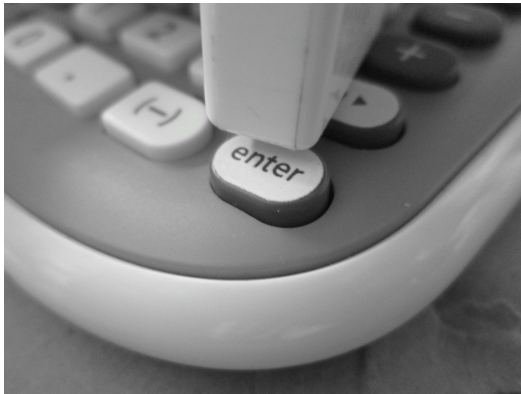
*By an incredible stroke of luck, Rubik's
Turtle's birthday just happens to be 10001
in binary, solving the riddle.
TI-36X SOLAR: "Correct. You may pass
to the next door."*



*"Halt! I am TI-30XB MultiView.
Enter the code!"
Unfortunately, Rubik's Sidekick was
only taught the notation $\binom{25}{13}$ and
Rubik's Turtle spent his maths classes
drawing pictures with his graphics
calculator.*



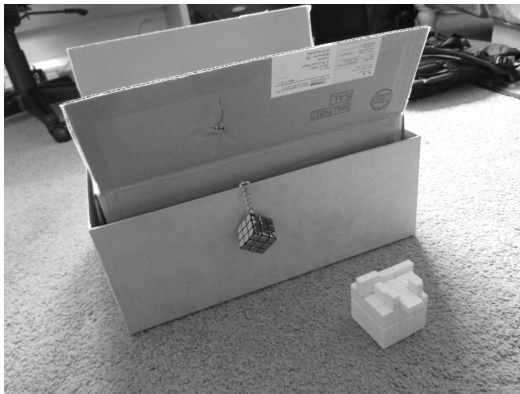
*Rubik's Sidekick: "Hey, it's my turn
to use my birthday now!"
Rubik's Turtle: "But 53 is my
favourite number!"*



*While they are fighting over what to type,
Rubik's Turtle steps on the enter key.*



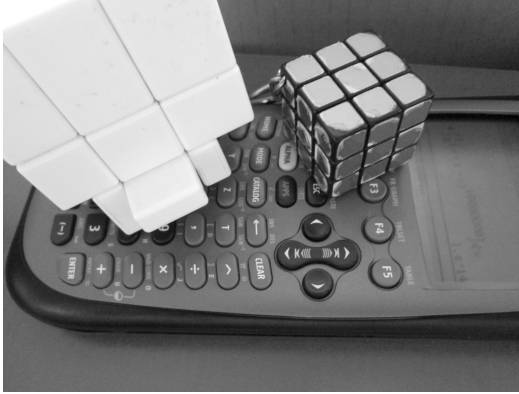
*The probability that Rubik's
Turtle and Rubik's Sidekick
accidentally entered the correct
answer is approximately
 $1.2524 \times 10^{-13} \dots$*



*TI-30XB Multiview: "Correct. You
may pass." The vault swings open to
reveal the next door...*



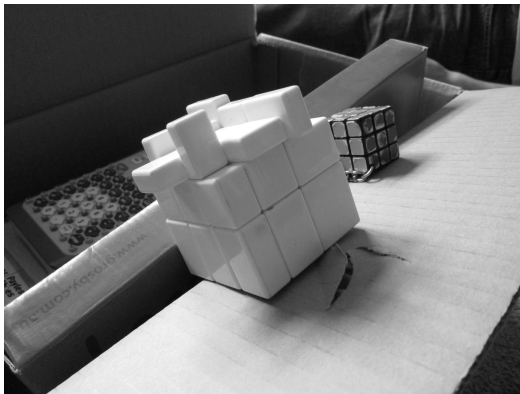
*"Halt! I am TI-89 Titanium.
Warning: May not be fully
simplified. **BUSY.**"
Rubik's Turtle: "Huh?"*



Rubik's Turtle: "Oh no! None of the buttons are responding!"



Rubik's Sidekick: "Uh... Hey! Maybe you should take the batteries out and put them back in..."



*As our heroes put the last battery back into TI-89 Titanium, the calculator says:
"Well done. You have solved the riddle.
You may pass."
Rubik's Turtle: "Huh?"*



*"Halt! I am TI-nspire CAS.
Would you like to calculate,
graph, open a new document,
open a saved document, open a
recent document, view the
current document, view
settings and..."*



Rubik's Sidekick: "Oh no, why are there so many options?"

Rubik's Turtle: "I don't know. But hey, look what I can do!"



TI-nspire CAS: "...draw a table, make a statistical plot, write a... Hey, that's the solution to the riddle I hadn't gotten around to telling you yet! I guess you can pass..."

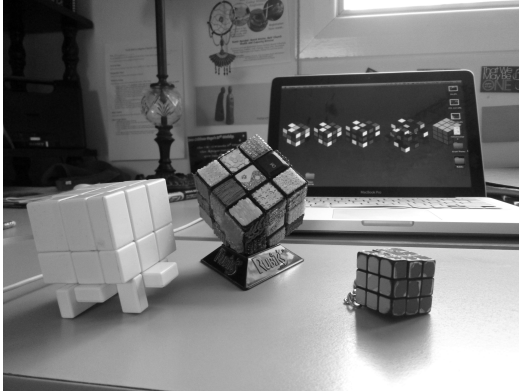


Rubik's Sidekick: "Hey look! There are no more calculators! This must be the inside of the Vault of Mysteries! Hooray!"

Rubik's Turtle: "What? There's just this scrap of paper! What a waste of time!"



Despite their disappointment, our heroes are still environmentally conscious, and throw the scrap of paper into the nearest waste receptacle.

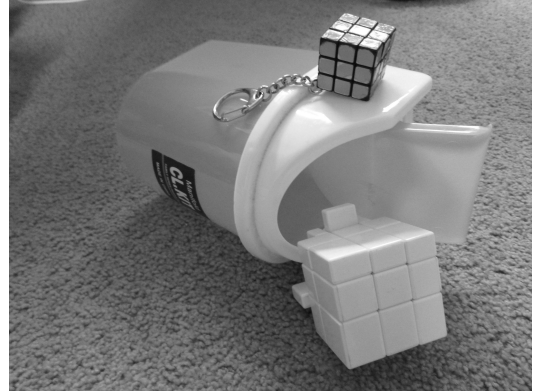


Mother-of-all-Rubiks: "Oh, back so soon! Do you have the directions?"

Rubik's Turtle: "The wha...?"

Mother-of-all-Rubiks: "The directions to the cave! They should have been written on a piece of paper in the Vault of Mysteries."

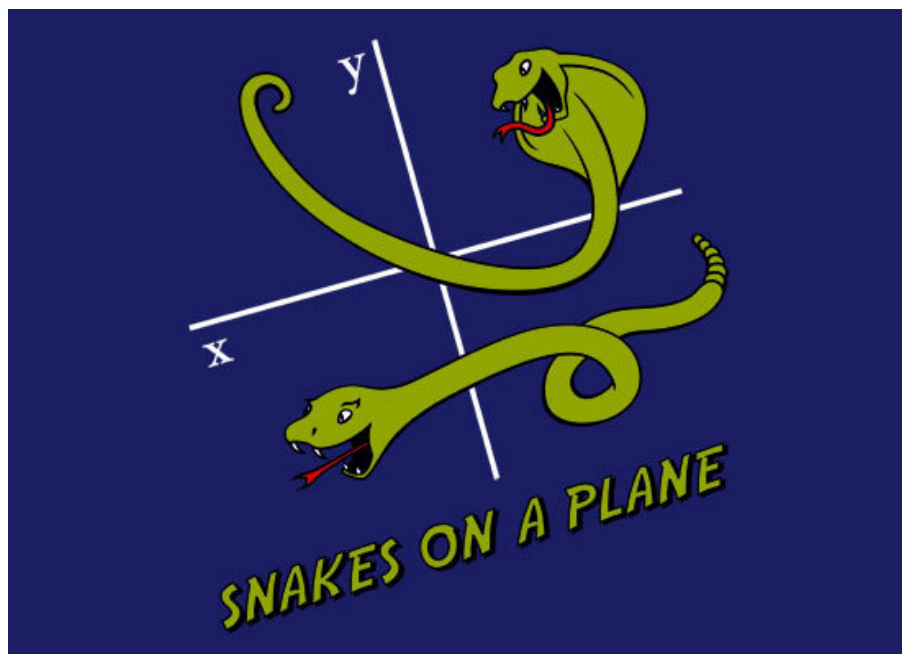
Rubik's Turtle: "Uh..."



Rubik's Turtle: "It's gone!"

Rubik's Sidekick: "Oh no! Whatever will we do?"

Have the efforts of our heroes in the Vault of Mysteries been for nothing? Will they track down whatever villain (or garbage collector) has stolen the "directions"? And why are the direction to this "cave" so important, anyway? Find out in Episode 6 of *The Adventures of Rubik's Turtle*!



Interview with a MUMS Alumnus

Background

Simon Pampena aka “The Mathemaniac”, 37, is a leading maths communicator and stand-up mathematician. After graduating with a Bachelor of Science from the University of Adelaide and being awarded First Class Honours for his postgraduate diploma in Pure Mathematics from the University of Melbourne in 2000, Simon worked for the Foster’s Group as a Statistical Analyst (Sales) for seven years. Outside his day job, Simon began his foray into the comedy scene, becoming a Raw Comedy finalist in 2004.

Since then, he has written, produced, and performed nine maths comedy shows for the Melbourne International Comedy Festival, Darwin Arts Festival, Adelaide Fringe, and Edinburgh International Science Festival. He is also a regular on ABC1’s *Catalyst*, Channel 10’s *The Project*, Discovery’s *You Have Been Warned*, and the *You Can Do Maths* TV campaign; awarded five national touring grants from the Federal Government for Australian Science Week from 2008 to 2012; and is the current National Numeracy Ambassador appointed since 2009. He will be performing his new show *The Peer Revue* at this year’s Adelaide Fringe Festival.

What sparked your lifelong interest in maths?

I was interested in science fiction from a very young age. *Star Wars* was to blame! But my interest in science fiction was soon replaced with a fascination for science fact. I learnt a derivation of $E = mc^2$ from a book in my school library and became obsessed with it. It showed how binomial expansion on the Lorentz factor in the relativistic mass equation gives way to a conservation of energy equation. The inference is that a mass at rest is equivalent to energy. I was blown away. The most famous equation in history was produced by maths! It was from that point on that I really got into maths.

What got you interested in stand-up comedy? Were you the class clown?

I have always loved comedy. I would listen to my parent’s comedy records over and over again when I was a kid. There was no YouTube back then! I suppose I would have been considered one of the funny guys at school but I definitely wasn’t the funniest guy. There were a few people who were comedic geniuses at my school who also happened to be close friends. I was always hanging out with the funniest guys. Perhaps I was studying them!

I came into my own once I took to the stage for the first time. While others are funny in conversation, I found that I could be funny on stage. Something strange happens to me in front of an audience where I occupy some sort of super-ego and I get a lot funnier. It was a big surprise to me the first time it happened. Once I made the connection between what I had experienced on stage and what I had listened to on all those comedy records, I knew I could be a stand-up. I had personal realisation that I understood how stand-up comedy worked... which doomed me to never being able to give it up!

When you studied maths at university, did you know that you would combine your two interests to become a maths comedian?

I always had a sense that I was a good science/maths communicator but the idea of combining maths and comedy only occurred to me once I had performed stand-up for a year. I realized that to be a truly great comic you need to find your own voice and go into new territory. At the same time, all of the artist people I was hanging out with absolutely hated maths. I couldn't understand their hostility to the subject I loved. I then took it upon myself to communicate how awesome maths was through stand-up. I had no idea how I was going to achieve that but I had confidence that I was going to solve the problem and make it work somehow!

Was there a particular moment in your life that inspired you to become a stand-up mathematician?

I didn't think I would become a professional stand-up mathematician. It was only through my tenacity that this career has happened. It took me a long time to work out what I was doing artistically and luckily, by the time I had worked myself out, the government was looking around for people to help the crisis in maths education. It was good timing really. But my intentions were always to communicate maths to lay people. I was never interested in just performing to 'nerds'.

Why do you think it is important for people to be interested in maths, and for Australia to perform better at maths globally?

Being numerate is as important as being literate. While no one is going to argue about the merits of reading and writing, the fundamentals of mathematics are ignored. And I see the biggest benefit of being numerate is the ability to solve problems. The better Australia is at problem solving, the better our future will be.

You've described yourself as "sensitive, artistic" and "creative, social". Is this unusual for a mathematician, who is usually stereotyped as being introverted, nerdy, and anti-social?

I think there are a lot of people out there who have the capacity to be mathematicians who are also creative and social. Unfortunately, they choose not to go into the profession due to 'uncoolness' and ignorance. It's a great loss to maths that we don't attract more of these people because the introverted and nerdy need good communicators in their midst.

Why do you think mathematicians are seen as "uncool"? Do you think social media, the rise of hipsters, and shows like *Letters and Numbers* are slowly reforming this stereotype?

I think mathematicians are seen as uncool because they are usually not great communicators. We need to get better at explaining how cool our area of intellectual exploration is. Maths is as complicated and beautiful as music but far less accessible. Beyond the stereotype, we need to get better at communicating mathematical ideas. By the way, speaking of *Letters and Numbers*, I was integral in finding Lily Serna for the show! SBS called me and I called Lashi Bandara from ANU and he gave me Lily's number which I then passed onto SBS. And no you can't have Lily's number :)

What does a maths communicator usually do? What kinds of people are suited to becoming maths communicators? Do you have to be outgoing, talkative, and love the spotlight? Do you need to travel a lot?

A good maths communicator is someone who understands some maths and who can explain it to someone who doesn't have a clue what they know. That sounds like a truism but that's where good communication starts. You need to be able to see things from your audience's perspective. It helps to do this live as you can see any confusion that might be creeping in their faces. You need to explain what *you know* with the *knowledge they have*. You can use analogies, diagrams, stories...even stand-up comedy! The objective is to take an audience through a series of discoveries in a journey of mathematical discovery. Being outgoing, talkative or extroverted is important only because a nervous speaker is distracting: it gets in the way of good communication. But if you really love maths and love sharing your knowledge then it's surprising how easy it is to overcome nerves on stage...I recommend you try it!

A (Rough) Guide to Undergraduate Maths

What sort of mathematician are you?

At Melbourne University, mathematics is broadly split into four fields: pure mathematics, applied mathematics, statistics and operations research. The question is, what's the difference between these fields? If you're a first year, fresh out of high school (to whom this article is mostly targeted), there's a good chance that the only two fields of mathematics that you're familiar with are *Specialist Mathematics* and *Mathematical Methods*. Which to be quite frank are the most useless titles for subjects ever. You might as well have called them *Maths* and *Harder Maths*. There's nothing particularly *special* or *methodical* about either. High school has hardly prepared you for the diversity and broad utility of university mathematics. So here I'll list just a few basic pointers that may help you with deciding just what sort of mathematician you are.

Pure Mathematics

Are you the sort of person who finds the elegance of mathematics attractive? Do you enjoy proving seemingly useless but nevertheless interesting results? If you answered yes to these questions, then pure maths may be the way forward for you.

In the pure maths specialization, more than any other, you'll discover why maths is regarded as an art as well as a science (though, quite possibly, only mathematicians put it like this). Pure mathematics is about studying the underlying concepts that make all maths work. And besides that, it's just really cool stuff.

The variety of material in the pure maths specialization makes it particularly interesting. You'll find out that solving polynomials isn't just as simple as using the quadratic formula. In fact, you'll even see why there is no quintic formula. You'll discover that a punctured torus (a donut surface with a hole in it) is essentially the same as two circles joined at a point. Just don't tell any bakers that one, it may blow their minds and result in some strange looking donuts later on. These are just tiny fragments of what you'll learn studying pure maths, but just as a small warning, this specialization is not for the faint of heart.

Applied Mathematics

Do you really like formulae? Would you like to see how maths can be used in the real world? And most importantly, do you really *really* like calculus? If the answers to these questions are yes, then you should be looking into applied maths.

Applied mathematics is about trying to model complicated systems and then poking around with the inputs to see how things change under certain conditions. Applied mathematics has applications in just about every field you can think of. In the applied maths specialization, you learn techniques and skills that will enable you to solve certain types of equations which commonly crop up in the real world, such as modeling river flows or how human cells reproduce. Just be prepared for a lot of calculus.

Probability, Statistics, and Stochastic Processes

You see statistics all the time. Figures, percentages and ratios are thrown up all the time in the modern world. But do you ever wonder how meaningful these numbers are? When you play a card game, do you ever wonder, “well that was unlikely, but exactly *how* unlikely was it?” We all know that smoking is bad for you, but how exactly do you *prove* this? If these are things that you’ve wondered about, then you should be looking into probability, statistics and stochastic processes.

In probability you’ll learn how to calculate the probability of certain events happening, and study various distributions occurring naturally in the real world. An important use of probability is its application to statistics and stochastic processes. In statistics you’ll learn how to properly analyse a data set. By creating statistical models you’ll be able to test the effects of certain variables on others.

Stochastic processes is about modeling random processes that occur in the world. For example, you can model the number of people who walk into a shopping centre. You can even attempt to make money by modeling financial markets, though personally I wouldn’t recommend this off just your undergraduate subjects.

Discrete Mathematics and Operations Research

So we all spend plenty of time bagging out the government for being slow, inefficient, wasteful, or more often than not all of the above. But how would you make it better? Do you spend time thinking about how you could make processes faster, more efficient and just better in general? If these questions appeal to you, then you should be looking into discrete mathematics and operations research.

This specialization is all about decision making. And decision making is hard. Just think about a can of baked beans, and the path it travels from the farm where the original beans are grown, to your dinner plate. There's at least a dozen different processes that have to happen before it reaches you. Now the question is, what's the best way to do this? You'd want to reduce time, but also costs, and then on top of that increase quality. All of a sudden your choices aren't all so clear cut. Operations research deals with these sorts of issues in a scientific manner to help with decision making. And with society becoming more complex, and processes becoming more numerous, there's no doubt that this field is important.

Now what?

So maybe I've given you some idea about what the different specializations are. The question that remains is what subjects to do. The best advice I can give is to pick up a copy of the course advice booklet produced by the Maths Department. It's bright orange and you can pick it up from the front office in the Richard Berry building. If you're a first year, the choices are actually remarkably simple.

First year subjects

In first year, maths students, regardless of your intended specialization, will take a first-year mathematics and statistics package. The first-year package is a good little set of subjects that will give you an introductory glimpse into all the fields of maths. You are then able to narrow your focus on any of the specializations. What you pick depends on how you did in high school. Most packages are two subjects, one in first semester, one in second, and there's the option of also taking the breadth subject Critical Thinking with Data.

For students who did Specialist Maths, it depends on what your raw study score was:

- ≥ 38 – take Accelerated Mathematics 1 in Semester 1 and Accelerated Mathematics 2 in Semester 2.
- ≥ 27 – take Calculus 2 in Semester 1 and Linear Algebra in Semester 2.
- < 27 or not Specialist Maths – take Calculus 1 before taking Calculus 2 and Linear Algebra.

After first year

Once you get into second or third year you'll need to specialize into one of the four fields I've talked about above. Because I'm lazy, I'm not going to go into all the subjects but instead refer you to the department's course advice booklet.

Disclaimers, advice and conclusion

As I'm sure my law student friends will tell me, I need to put in a disclaimer thingy here. The material above is all purely my opinion, I cannot stress enough that if you are looking for course advice there are people, very friendly and nice people even, in the Maths and Stats Learning Centre (MSLC) who are far more qualified than yours truly to help you out with subject selection. Understandably, some people find it easier to talk to peers, so while we will always recommend that you speak with the MSLC, also feel free to drop into the MUMS room in the maths building and have a chat with some of us. We're also very friendly and nice people!

The field of mathematics and statistics is an enormous field with an incredible variety of content. Hopefully I've given some people an insight into how awesome maths is at uni, and offered some helpful advice. I've certainly had no regrets doing maths at uni, and I encourage you all to do as much maths as possible, though admittedly, I may be slightly biased.

— Han Liang Gan

What Makes Numbers Happy?

Take any natural number and add the squares of its digits together. Do this again for the resulting sum and proceed in this way until you find yourself looping within a finite sequence of numbers. This process defines a sequence of numbers whose elements are the sums of the squares of the immediately previous element, and the end-behavior of such a sequence defines what are strangely called *happy numbers*. For the sake of a definition, let's call such a sequence that starts with any natural number the *happy sequence* of that number. We can therefore define a happy number as one whose happy sequence ends with the number one.

To better illustrate this, let's look at two examples. First, an example of a happy number is 13, as $1^2 + 3^2 = 10$, and $1^2 + 0^2 = 1$, and 1^2 equals itself again. The happy sequence associated with the number 13 would be $\{13, 10, 1, 1\}$. A number that is *unhappy* is the number 11, as its happy sequence is $\{11, 2, 4, 16, 37, 58, 89, 145, 42, 20, 4\}$ and it can be seen that this sequence ends with the infinitely reoccurring loop $\{4, 16, 37, 58, 89, 145, 42, 20\}$. The fact that this closed loop exists is extremely interesting! By extension, the end nature of all happy sequences is interesting, and we will find that a peculiar conclusion can be made about them.

We should note that any number appearing in the happy sequence of another number shares the happiness of that initial number. So every number appearing in 7's happy sequence $\{7, 49, 97, 130, 10, 1, 1\}$ is also happy because seven is itself happy. Another little property of happy numbers that is quite interesting is that neither rearranging a number's digits nor inserting zeros at any point has any effect on the happy sequence that that number generates except for the first element. For example, 331 produces the same happy sequence $\{331, 19, 82, 68, 100, 1, 1\}$ as 1033 $\{103, 19, 82, 68, 100, 1, 1\}$ because they share exactly the same non-zero digits.

These aren't the only interesting quirks about happy numbers. In this article, we'll explore two more quirks: the first concerns how happy sequences end, while the second concerns how they are distributed.

How Happy Sequences End

A quick observation of the happy sequences of the first ten natural numbers, and even the first twenty, shows that they all end in just two ways. The first occurs when the starting number is happy and so will by definition end with

the finite sequence $\{1\}$, while the second occurs when the number is not happy and instead ends with the finite sequence $\{4, 16, 37, 58, 145, 42, 20\}$. And indeed, this observation extends to all natural numbers. No matter how large, every single natural number will eventually end in one of the two ways described above. This peculiar conclusion about the end nature of all happy sequences is one of the most interesting points about happy numbers because it implies that no number other than 1 is the sum of the squares of its own digits. Thankfully, the proof of such a conjecture is fairly straightforward and hopefully easy to understand!

We can see that for any starting number n of length m , the maximum possible next element in the happy sequence would occur when n is the number with nines occupying all of its place values. This implies that the maximum possible next element would be $9^2 * m$. For example, if we consider a three-digit number (when $m = 3$), then the number 999 will give the highest next value of $9^2 + 9^2 + 9^2$ or $9^2 * 3 = 243$. For a four-digit number (when $m = 4$), the number 9999 will give the highest next value of $9^2 + 9^2 + 9^2 + 9^2$ or $9^2 * 4 = 324$. This means that any other three or four-digit number less than those described above will produce a number less than 243 or 324 respectively under the happy sequence process.

A critical point is that after a certain number, the next element in that numbers happy sequence will always have one less digit than the starting number n . This is important because it will ultimately simplify our analysis of happy number sequences and tell us that if a certain number has a large enough length, then the length of the subsequent elements of its happy sequence will be strictly less than the initial starting value and indeed one digit less. Because $9^2 * m$ represents the maximum possible value of the next element in the happy sequence of a number of length m , and $10^{(m-1)}$ represents the first number of length m , $9^2 * m < 10^{(m-1)}$ refers to the length m of a number required such that the maximum possible next element of its happy sequence has strictly fewer digits than its starting value. Since $m \geq 4$ fits this inequality, any number with four or more digits will have fewer digits under the happy sequence process. Thus, the sums of the squares of the digits of any number greater than 999 will necessarily be at least one digit shorter under the happy sequence process.

Next, let's try to determine if all natural numbers eventually get to a two digit number under the happy sequence process. To do this, we need to consider what occurs to numbers within the interval $[100, 999]$. We have already determined that any positive natural number greater than 999 will eventually find

its way into this interval under the happy sequence process and if the number is already less than 100, then it is already a two-digit number. Within the $[100, 999]$ interval, the number that has the greatest next value in its happy sequence is 999, with a value of $9^2 + 9^2 + 9^2 = 234$. This means that all the numbers less than 999 necessarily have the sum of the squares of their digits being strictly less than 234, and that 234 in a way represents a “least upper bound” when it comes to all of the possible next values of any number in the interval $[100, 999]$ under the happy sequence process.

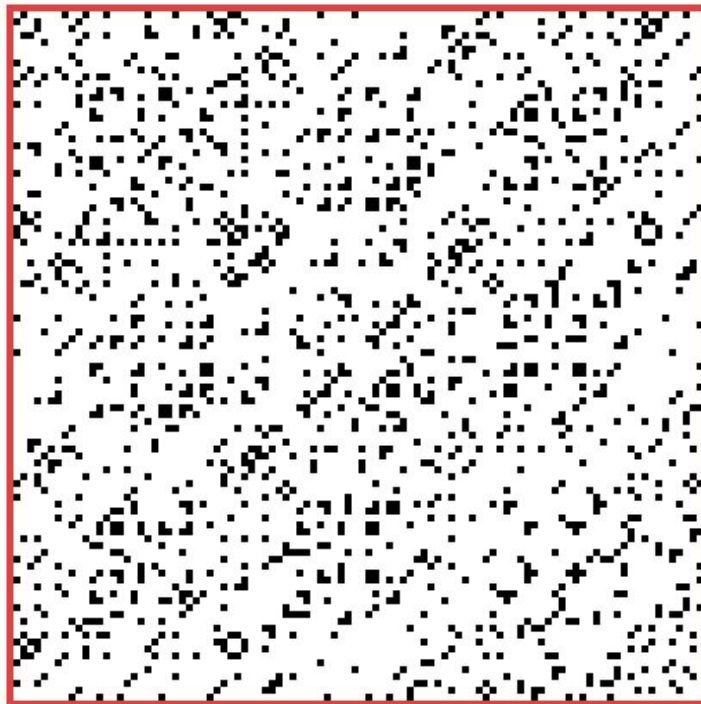
What if we analyse the interval $[100, 234]$ in the same way as the last interval? It's worth doing this since the last interval showed that all natural numbers greater than 99 eventually arrive within this interval. Similarly, within this interval, the number that has the greatest next value in its happy sequence is 199, with a value of $1^2 + 9^2 + 9^2 = 163$. Continuing in the same way within the interval $[100, 163]$, the number 159 has the next greatest value of $1^2 + 5^2 + 9^2 = 107$. Within the next interval of $[100, 107]$, 107 has the next greatest value of $1^2 + 7^2 = 50$. Hence, all natural numbers greater than 99 eventually make their way into the range $[1, 99]$ under the happy number process.

The next part of the proof is a bit of cop-out since I need to write the words ‘exhaustive’ and ‘search’ together in the same phrase. That said, it still proves what it needs to prove, and I have yet to see any other poof (that I understand) demonstrating the same thing. So far, we have established that any natural number eventually gets within the range $[1, 99]$ under the happy number process. If we were to show that all of the numbers in the range $[1, 99]$ are either happy or unhappy, then that would mean that any numbers which eventually produce a number in this range via the happy sequence process are also either happy or unhappy. It turns out that an exhaustive search of all the numbers in the range $[1, 99]$ shows that every one of them is either happy or unhappy.

In summary, since all of the numbers in the range $[1, 99]$ are either happy or unhappy, and because all natural numbers will eventually get within this range under the happy sequence process, then all natural numbers must also be either happy or unhappy. And by definition, this implies that all natural numbers happy sequences will either end happily with the loop 1 or unhappily with the loop $\{4, 16, 37, 58, 145, 42, 20\}$. It is a curious thing that within our number system such an arbitrary process defined on merely the digits of numbers should lead to such a startling binary inherent in all numbers! It also seems curious that the $\{4, 16, 37, 58, 145, 42, 20\}$ end sequence is the only sequence that is closed within the natural numbers other than the trivial sequence of $\{1\}$.

How Happy Numbers Are Distributed

The second thing that I would briefly like to talk about concerning happy numbers has to do with their distribution. The picture below¹ shows the distribution of happy numbers. I have included it because I really like visual representations of mathematical patterns. They can also give you an insight into quite interesting properties of things that you would not have considered otherwise, as is the case here with happy numbers.



This is a 100×100 grid where the top-left cell represents the number zero and the bottom right number represents the number 9999. The numbers increase from left to right, and all of the cells representing happy numbers in the grid are filled in black while the rest are left white. The thing to notice here is that the image appears to be symmetric about the main left-right diagonal. I would never have thought that happy numbers did this! I never would have thought purely just from the description and definition I gave earlier of them that they would somehow manifest themselves in this way. Nonetheless, we can see in the picture above that some underlying symmetry is inherent in the distribution of happy numbers, and that we should be able to explain it using the definition and properties of happy numbers.

¹From <http://www.shaunspiller.com/happynumbers/>.

The key to explaining the symmetry of happy numbers in the picture lies in two subtle points. The first has been mentioned previously, that is, the property of happy numbers that makes their happiness invariant under the addition of zeros and the rearrangement of their digits. More specifically to the issue of symmetry however, is the fact that the complete reversal of the digits of a happy number or the mirroring of a happy numbers digits does not affect that numbers happiness. For example, because the number 103 is happy and the number 301 is just the reverse of 103, we can conclude without any doubt that 301 is also happy. The second subtle point lies in how the grid is set up. The fact that the grid is a 100×100 grid starting with 0 and not a 115×115 grid starting with 1 contributes to the symmetry.

If we take a look at a magnified version of the top-left corner of the 100×100 grid, then we can begin to see how the symmetry comes about. If we take the 103 cell and look for its reflection along the main diagonal, we find the cell representing 301, the same number as 103 but with its digits reversed. If we take another happy number in this magnified grid such as 203 and look for the cell representing its reverse (302), then we find that it is also in a cell which is just a reflection of 203 about the diagonal. Even with the happy number 1 we can see that its reflection about the diagonal is also just the number 001 with its digits reversed, 100.

0	1	2	3	4
100	101	102	103	104
200	201	202	203	204
300	301	302	303	304
400	401	402	403	404

We can extend this symmetry to all numbers in the grid because we are using a 100×100 grid and we just so happen to be using a base 10 number system. These two things together imply symmetry because there are exactly 100 cells along the first row, just enough so that when the next row is started below the first, the digits start again in the third column of the numbers. Thus, symmetry is produced because the dimensions of the grid align up exactly with the base we are using to represent our numbers. Similar symmetry would have been seen if instead of a 100×100 grid, we used a 10×10 grid or any other grid sized with a power of 10. This result even implies that we would see a similar symmetry using a $6^2 \times 6^2$ or 36×36 grid in combination with base 6 numbers because the dimensions of the grid would align perfectly with the base used.

Now we can say that for every happy number on the grid, there will be exactly one other on the opposite side. This is because the grid we are using reflects numbers with reversed digits about the main diagonal, and happy numbers retain their happiness when their digits are reversed.

It Depends on How They Are Written

Amongst all of these interesting properties of happy numbers, it should also be noted that the process which defines them is extremely specific to the way that the numbers are written as opposed to the intrinsic property of quantity that numbers represent. This means that a number that is happy in base ten may not necessarily be happy in another base. For example, 13 is happy in base ten, but in hexadecimal 13 is D_{16} (D base sixteen). D_{16}^2 squared would be 13_{10}^2 which is 169_{10} or $A9_{16}$. $A9_{16}$ would then go to $A_{16}^2 + 9_{16}^2$, which is $10_{10}^2 + 9_{10}^2$, which is in turn equal to 181_{10} or $B5_{16}$. Continuing in the same way, we can construct a sequence $\{D, A9, B5, 92, 55, 32, D\}$, which implies that D_{16} is unhappy since the sequence does not end with a 1_{16} and would continuously loop around the same sequence of numbers.

D_{16} and 13_{10} both represent the same numerical quantity and have many of the same mathematical properties, the only difference being that I have chosen to represent them differently. D_{16} and 13_{10} are both prime numbers and odd, but curiously they are not both happy. Because both D_{16} and 13_{10} represent the same numerical value, you would expect that a property of both dependent on or based off their intrinsic numerical value would not change simply because its representation has changed. As such, the nature of happy numbers and happy sequences can be said to be less influenced by the nature of numbers themselves and more affected by the way those numbers are written. It turns out that in other bases there are other ways for happy sequences to end other than just the two that we showed always occurred in base 10. The sequence in hexadecimal above is an example of one of them.

Conclusion

Happy numbers are a great example of a topic in mathematics that is simple enough for anybody to try to investigate themselves in the comfort and privacy of their own home. The concepts involved are basic enough so as to not look as scary and as menacing as many other topics in maths. Furthermore, anyone with a basic knowledge of computer programming could easily construct a program to help them investigate happy numbers faster and more efficiently than solely by hand. If you can't be bothered doing that, there are hundreds of examples of code on the web that will effectively do whatever it is that you need from happy numbers. All it takes is a little curiosity and a little spare time.

Flashback: Taxicab Numbers

G. H. Hardy once arrived at Srinivasa's Ramanujan's¹ residence in a cab numbered 1729. He commented that the number seemed to be rather dull and hoped that it was not an unfavourable omen, to which Ramanujan immediately replied:



No, it is a very interesting number; it is the smallest number expressible as the sum of two [positive] cubes in two different ways.

Generalizations gave rise to the notion of the n th 'taxicab number', typically denoted $Ta(n)$ or $Taxicab(n)$. It is defined as the smallest number that can be expressed as the sum of two positive algebraic cubes in n distinct ways. $Ta(1)$ is rather simple:

$$Ta(1) = 2 = 1^3 + 1^3$$

However, $Ta(2)$ is much larger; in fact, it is 1729, the number that has now been immortalised as the Hardy-Ramanujan number:

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

First discovered in 1657 by Bernard Frénicle de Bessy, $Ta(2)$'s famous story featuring Ramanujan brought the concept of taxicab numbers to prominence and in 1954, G. H. Hardy and E. M. Wright proved that such numbers exist for all positive integers n . While their proof is easily converted into a program to generate such numbers, it does not claim whether the numbers generated are actually the *smallest* and hence cannot be used to find the actual value of $Ta(n)$.

¹This article originally appeared in Issue 4 of Paradox in 2011. See Ramanujan's biography in that issue on the Paradox archive online to read more about how Hardy 'discovered' Ramanujan.

Why must the algebraic cubes be positive? Because if negative numbers are allowed, then we can form more (and smaller) instances of numbers that can be expressed as the sums of two cubes in n distinct ways. Such numbers are known as *cabtaxi* numbers, but for the sake of brevity, they shall not be discussed any further.

The most famous taxicab number has made a few appearances in popular culture. 1729 and its interesting property is mentioned by the somewhat-insane mathematician Robert played by Anthony Hopkins in the 2005 film *Proof*, while in the animated television series *Futurama*, the spaceship is designated as Nimbus BP-1729 in Season 2 and the robot character Bender's serial number is revealed to be none other than 1729 in a Christmas card he receives in the episode *Xmas Story*.

What about the other taxicab numbers? Only the first six are known to date, and the ones after $Ta(2)$ were found with the help of computers. John Leech found $Ta(3)$ in 1957:

$$\begin{aligned} 87,539,319 &= 167^3 + 436^3 \\ &= 228^3 + 423^3 \\ &= 255^3 + 414^3 \end{aligned}$$



It was a while until $Ta(4)$ was found to be a whopping 6,963,472,309,248 in 1991 and $Ta(5)$ was found to be 48,988,659,276,962,496 in 1999. Only three years ago, $Ta(6)$ was recently announced to be an incredibly huge 24,153,319,581,254,312,065,344! When will $Ta(7)$ be discovered?

But it can be even more challenging: what if a taxicab number is restricted such that it is not divisible by any cube other than 1^3 ? If we describe it as a *cubefree* taxicab number T and write it as $T = x^3 + y^3$, then the numbers x and y must be relatively prime for all pairs (x, y) .

Among the taxicab numbers $Ta(n)$, only $Ta(1)$ and $Ta(2)$ are cubefree taxicab numbers. But $T(3)$, the smallest cubefree taxicab number with three representations, was discovered by Paul Vojta (unpublished) in 1981 while he was a graduate student and it was not $Ta(3)$:

$$\begin{aligned}T(3) &= 15,170,835,645 \\&= 517^3 + 2468^3 \\&= 709^3 + 2456^3 \\&= 1733^3 + 2152^3\end{aligned}$$

Have any more of these numbers been discovered? In 2003, $T(4)$ was found to be a colossal 1,801,049,058,342,701,083 — taxis may be small but taxicab numbers and their cubefree companions are certainly not!

— Kristijan Jovanoski

Pure mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere — Archimedes did it in a bathtub.

— Richard J. Trudeau, *Dots and Lines*

How to Estimate Your Retirement Funds

“Mate, I reckon I’m ready to retire. I mean, I’m currently 65 and the life expectancy for blokes is around 80. So I have around 15 years left. I’ve got around \$1,500,000 in investments and I reckon \$100,000 per year will afford me a very comfortable retirement, not even accounting for interest!”

This argument involving primary school maths—an application of division—and some basic knowledge of how interest works seems logical enough. However, as we shall discover, nothing is as it seems...

The fundamental flaw comes from the fact that 80 represents the life expectancy *at birth*. When someone is 65 years old, we should actually be looking at their life expectancy *at 65* instead! If you manage to reach such an age, it turns out that your life expectancy is actually greater than 15 years, so you should expect to live beyond 80 years old.

Let’s first check out some statistical evidence from the Australian Bureau of Statistics (ABS) and then let’s get into some maths using elementary probability. Finally, we’ll use good-old intuition to see why this all makes sense.

The Australian Life Tables

The ABS publishes the Australian Life Tables at regular intervals. A life table essentially provides information about the mortality of the average individual. An extract of their most recent publication is displayed in Table 1.¹

Observe that the life expectancy at birth is approximately 80 years old (79.7). However, the life expectancy at age 65 is not $79.7 - 65 = 14.7$. It is 19.1 instead, showing that the 65-year-old earlier underestimated his life expectancy by a significant number of years! He is instead expected to die at $65 + 19.1 = 84.1$ years of age.

This result can be stated more generally: a person who has not yet died will on average, have a total lifetime longer than their life expectancy at birth. The longer you live, the higher your age at death in comparison to when you were born. Comforting news for those who value their longevity.

¹<http://www.abs.gov.au/AUSSTATS/abs@.nsf/DetailsPage/3302.0.55.0012009-2011?OpenDocument>.

Age x	Life Expectancy at age x
0	79.7
1	79.1
2	78.2
3	77.2
\vdots	\vdots
64	19.9
65	19.1
66	18.3
67	17.5
68	16.7

Table 1: 2009–2011 Australian Life Tables for males.

Using Probability

Here we get into the maths and prove that the total life expectancy of a person who has not yet died is greater than their life expectancy at birth. First, we need to construct a model for an individual's future lifetime.²

Let the random variable T_x denote the future lifetime of an individual now aged $x \geq 0$. We make a simplifying assumption that T_x is a non-negative absolutely continuous random variable.³ This assumption is reasonable since we do not expect someone's mortality to experience an instantaneous jump at any point in their life, provided that they have not had some life-threatening procedure such as a major medical operation.

Our next key assumption harks back to the work of Halley of Halley's Comet fame⁴ who made the following insightful observation:

$$\mathbb{P}(T_x > t) = \mathbb{P}(T_0 > x + t | T_0 > x).$$

²This construction follows from *Actuarial Mathematics for Life Contingent Risks* by David C. M. Dickson, Mary R. Hardy, and Howard R. Waters.

³For example, we could let T_x be an exponential random variable with parameter μ . Although this leads to results that are easy to use, we do not have an accurate model to describe human mortality in this case.

⁴See Halley, E. (1693). An estimate of the degrees of the mortality of mankind, drawn from curious tables of the births and funerals at the city of Breslaw; with an attempt to ascertain the price of annuities upon lives, *Philosophical Transactions* **196**, 596-610.

This essentially says that the probability that an individual aged x survives t years is equal to the probability that a newborn survives up to the age $x + t$, given that they have already survived up to the age x .

Finally, we can make a proposition explaining that $\mathbb{E}(T_x) + x$, the total life expectancy of a person now aged x , increases with age. As an immediate corollary, we have $\mathbb{E}(T_{65}) + 65 > \mathbb{E}(T_0) + 0$, thereby justifying the result from the life tables: $84.1 = 19.1 + 65 > 79.1 + 0 = 79.1$. That is, a person aged 65 years will on average have a total lifetime longer than a newborn.

prop For $x > 0$,

$$\frac{d}{dx} [\mathbb{E}(T_x) + x] > 0.$$

prop

Proof. Let's use some elementary results taught in first and second-year university mathematics to calculate the expectation using the tail of the distribution function, conditional probability, the quotient rule, and the fundamental theorem of Calculus.

$$\begin{aligned}
 \frac{d}{dx} [\mathbb{E}(T_x) + x] &= \frac{d}{dx} \left[\int_0^\infty \mathbb{P}(T_x > t) dt + x \right] \\
 &= \frac{d}{dx} \left[\int_0^\infty \mathbb{P}(T_0 > x + t | T_0 > x) dt + x \right] \\
 &= \frac{d}{dx} \left[\int_0^\infty \frac{\mathbb{P}(T_0 > x + t)}{\mathbb{P}(T_0 > x)} dt + x \right] \\
 &= \frac{d}{dx} \left[\frac{\int_x^\infty \mathbb{P}(T_0 > t) dt}{\mathbb{P}(T_0 > x)} \right] + 1 \\
 &= \frac{\mathbb{P}(T_0 > x)[- \mathbb{P}(T_0 > x)] - \left[\int_x^\infty \mathbb{P}(T_0 > t) dt \right] \frac{d}{dx} \mathbb{P}(T_0 > x)}{[\mathbb{P}(T_0 > x)]^2} + 1 \\
 &= \left[- \frac{\frac{d}{dx} \mathbb{P}(T_0 > x)}{\mathbb{P}(T_0 > x)} \right] \left[\frac{\int_x^\infty \mathbb{P}(T_0 > t) dt}{\mathbb{P}(T_0 > x)} \right] \\
 &= \left[- \frac{\frac{d}{dx} \mathbb{P}(T_0 > x)}{\mathbb{P}(T_0 > x)} \right] \mathbb{E}(T_x) > 0
 \end{aligned}$$

This follows from observing that $T_x > 0$ implies $\mathbb{E}(T_x) > 0$, $\mathbb{P}(T_0 > x) > 0$ and the distribution function is non-decreasing so $-\frac{d}{dx}\mathbb{P}(T_0 > x) = \frac{d}{dx}\mathbb{P}(T_0 \leq x) > 0$. \square

Intuitive Arguments

How can we use intuition to see that a person aged 65 years will on average have a total lifetime longer than a newborn? Let's explore two ways to go about this: the first by contradiction and the second essentially non-rigorous.

Suppose that it is not the case that the life expectancy at 65 is greater than 15 years, that is, a person who is aged 65 should not expect to live beyond 80 years old. This means $\mathbb{E}(T_{65}) \leq 15$. Now consider an 80-year-old. Following this argument, this person will have a life expectancy of $80 - 80 = 0$. Hence, $\mathbb{E}(T_{80}) = 0$. But since T_{80} is a non-negative random variable, we have a result in probability theory that states $T_{80} = 0$ almost surely.⁵ This result says that as soon as someone turns 80, we should almost surely expect to see them drop dead instantaneously. This rather alarming conclusion is positively absurd and it follows that our hypothesis is false.

Alternatively, let's consider a newborn named Mr X. He can die at any age in the range $(0, \infty)$. Now consider a person aged $x > 0$ named Mr Y and let's make $x = 65$ for simplicity. He can die at any age in the range $(65, \infty)$. Notice that he cannot die in the range $(0, 65]$ because he is already 65 years old and so no longer has the risk of dying in that range. On the other hand, Mr X does, as he is a newborn. Since Mr X has a risk of dying young (i.e. in the range $(0, 65]$) and Mr Y does not, Mr X should have a lower total life expectancy than Mr Y.

Consequences

Now that we've established that a 65-year-old should expect to live beyond 80 years, what does this mean for the bloke thinking about his retirement plan earlier? By underestimating his life expectancy, he has overstated his yearly allowance and will discover that his retirement may not be as luxurious as he expected when he runs out of funds too early! That's why it's important to apply the correct maths when making important life decisions.

— Timothy Lee

⁵This is proposition 4.11 from the standard textbook used in the third-year probability course, *Probability* by Alan F. Karr. It states: if $X \geq 0$ and $\mathbb{E}(X) = 0$, then $X = 0$ almost surely.

SUDOKION: Spatial-Logic Puzzles

In Issue 4 of Paradox in 2012, readers were introduced to SUDOKION, a new range of spatial-logic puzzles.

Grid Orientation and Perception of Challenge

A matter of interest to the author not raised in the 2012 article is how the orientation of a SUDOKION grid affects the player's perception of his or her ability to complete the puzzle and the degree of challenge the puzzle poses.

Owing to its regular shape, a typical Sudoku grid has no orientation. The clues may be arranged in a way that gives the puzzle rather than the grid an incidental orientation but such an effect can usually only be achieved by offering the player more clues than would ordinarily be necessary, thus making the puzzle easier and less exciting to solve.

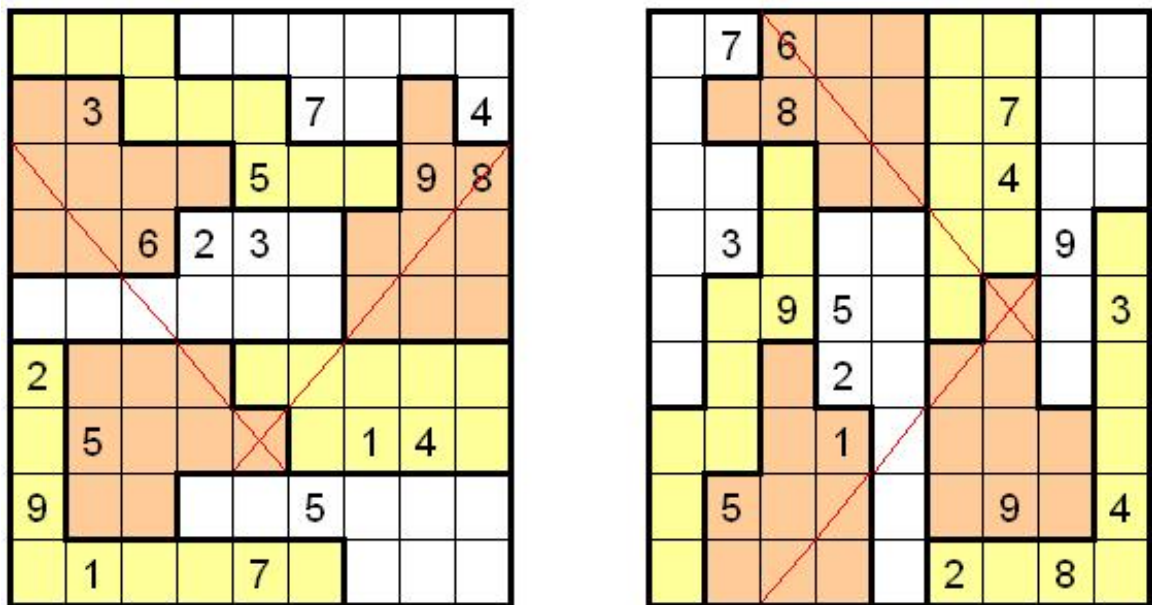


Figure 1: Every row, column cluster, and the red 'V' line must contain the numbers 1 to 9.

SUDOKION grids, on the other hand, often possess an easily-identifiable orientation. Both **Logikions** above are the same puzzle; the second is turned at 90 degrees and the clues are renumbered (so that the reader has two puzzles to complete and an opportunity to test the observation). The puzzle on the left is oriented east-west, i.e. there is a bias towards horizontal elongation of the clusters, while the other puzzle has a north-south orientation.

Although we have tested only a small cohort we have found that players often rate the same puzzle more challenging when it is presented with an east-west orientation rather than a north-south orientation. In this case, the first Logikion is more likely to be rated more challenging than the second. Perhaps, as lists of figures are usually displayed in columns rather than rows, our brains have a preference for scanning puzzles north-south rather than east west.

V and X Pandemonions

One of the delights of making spatial-logic puzzles such as SUDOKION is the opportunity to experiment - to see how far the concept can be taken. The two puzzles below are in the mid-range of complexity in the SUDOKION range.

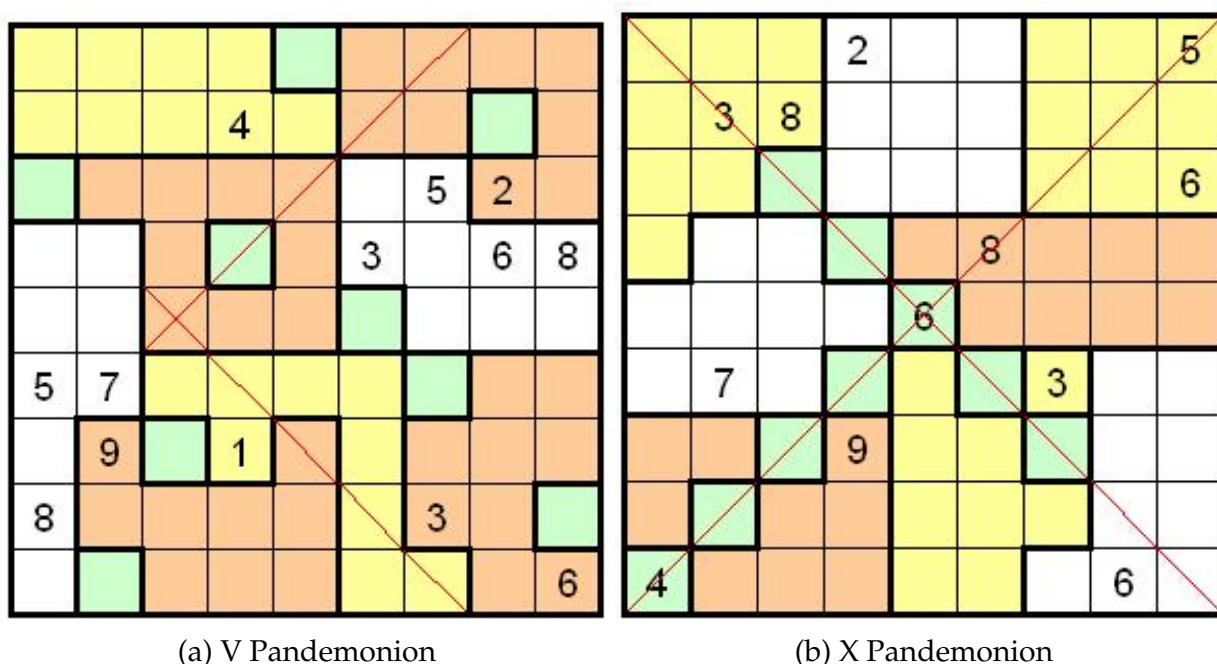


Figure 2: Every row, column and cluster, including the fragmented green cluster, and the red 'V' line (or both red diagonal lines) must contain the numbers 1 to 9.

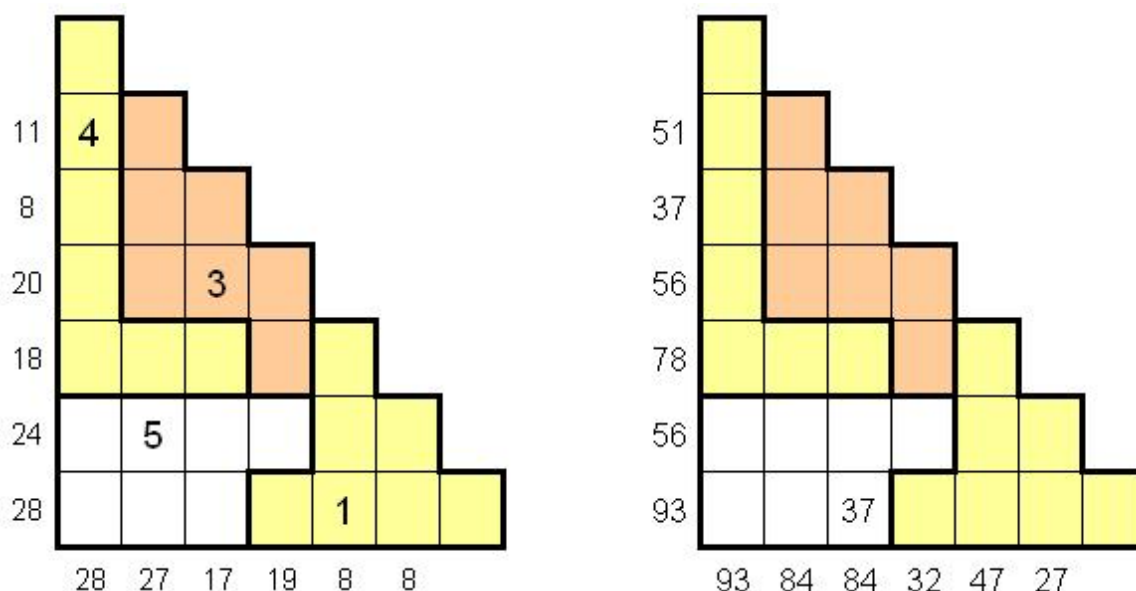
The X-type puzzles are a strange group. Despite all my attempts to do otherwise I have only been able to create a 81-cell (9×9) X Pandemonion with all the fragmented green cells on one or other of the diagonals. I don't have sufficient knowledge of mathematics to be able to prove abstractly that such a puzzle must always locate the fragmented cells on the diagonals but practice shows almost definitely that this is the case. Why?

Similarly, despite many attempts I have never been able to make a 64-cell (8×8) X Pandemonion in any configuration. I have also tried on many occasions to make X Logikions (similar to the first two puzzles in this article)—without success—and have concluded that they are mathematically impossible and that, in certain matters, nature favours complexity.

V Pandemonions are far more forgiving. I can place as many or as few fragmented cells on the line. V Pandemonions work well in 49-cell (7×7) and 64-cell grids and are more challenging on average than their X equivalents.

Triangoleums

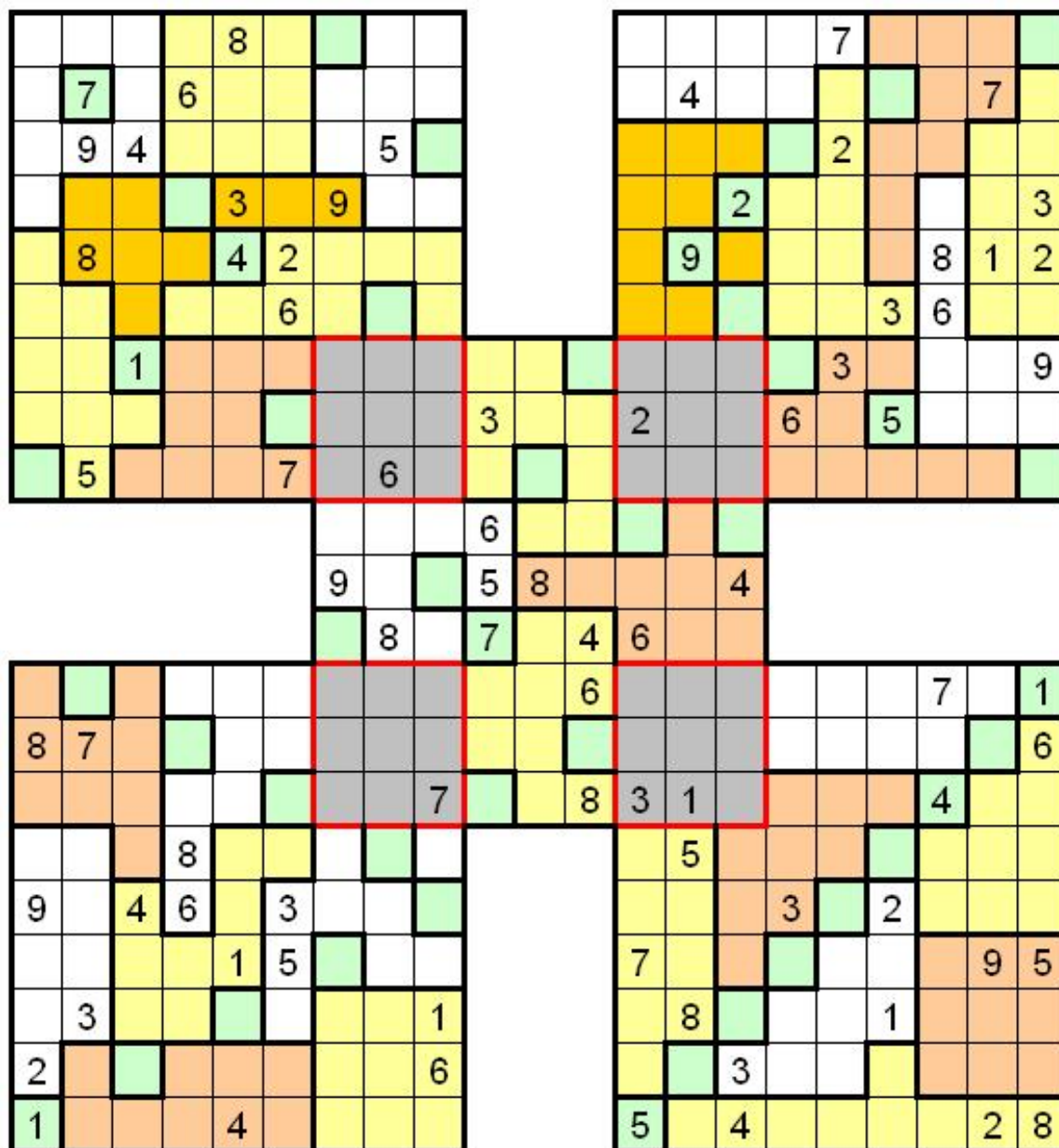
Most of the puzzles I make are true spatial-logic puzzles; values within the puzzle may be represented by symbols other than numbers, e.g. letters, emoticons, pictures, etc. There are also arithmetic spatial-logic puzzles such as the two Triangoleums below. To solve these, you need at least to use at least some arithmetic. The first Triangoleum is relatively easy, while the second definitely requires more thinking.



Every cluster as well as the 'A', 'B' and 'C' sides of the Triangoleum must contain the numbers 1 to 7. No number may appear more than once in any row or column. The numbers at the left and bottom show the sum of their respective rows and columns. The numbers of the second Triangoleum constitute a Fibonacci series, but not the 0, 1, 1, 2, 3, 5, etc. series. Part of the solution involves the discovery of the Fibonacci series relevant to the puzzle (all numbers are positive integers).

Super Pandemonion

The Super Pandemonion is the sort of puzzle you can get your teeth into over a wet long weekend. The challenge level of the featured puzzle is mid-range.



The Super Pandemonion has five sectors. Every row, column and cluster (including the fragmented green cluster) must contain the numbers 1 to 9. The grey clusters surrounded by the heavy red line are common to the centre and its adjoining sectors.

About

All SUDOKION are hand-made by the author, Stephen Jones, co-founder of Muddled Puzzles with his wife Michele Day.

Solutions:

[http:](http://www.muddledpuzzles.com/paradox_solutions_2013_1.pdf)

[//www.muddledpuzzles.com/paradox_solutions_2013_1.pdf](http://www.muddledpuzzles.com/paradox_solutions_2013_1.pdf).

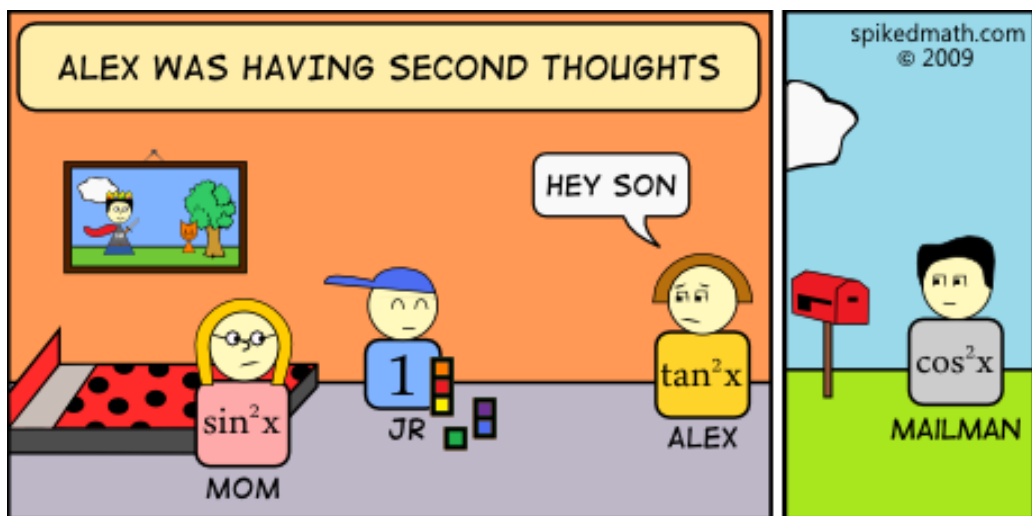
More free puzzles:

http://muddledpuzzles.com/quantum_edition.pdf.

Email:

2178309@muddledpuzzles.com

— Stephen Jones



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