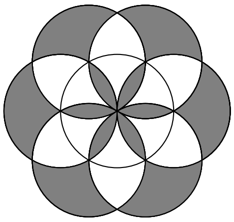
University Maths Olympics 2019

Questions & Answers

1. (20 points) Seven circles, each with radius 1, overlap to form the composite figure shown below. Find the area of the shaded region.



Answer: 2π

2. (20 points) You have a large jar filled with coloured marbles. It contains 1001 blue marbles, 1000 red marbles, and 1000 green marbles.

   You randomly draw two marbles out of the jar, then put one or two marbles in (decreasing the total number of marbles in the jar by one or keeping the number constant) using the following rule:

   - If the two marbles are blue and green, put a red marble into the jar.
   - If the two marbles are green and red, put a red marble into the jar.
   - If the two marbles are both red, put two blue marbles into the jar.
   - If the two marbles are anything else, put a green marble into the jar.

   Repeat this process until one marble remains. What colour is it?

Answer: Red

3. (20 points) Let x, y and z all exceed 1 and let w be a positive number such that $\log_x w = 24$, $\log_y w = 40$ and $\log_{xyz} w = 12$. Find $\log_z w$.

Answer: 60

4. (20 points) ***Change Walker Now***
   What integer value does the following sum equal?

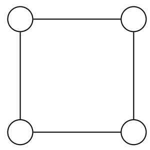
$$\frac{1}{2} + \left(\frac{1}{3} + \frac{2}{3}\right) + \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) + \left(\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5}\right) + \ldots + \left(\frac{1}{100} + \ldots + \frac{99}{100}\right)$$

Answer: 2475

5. (25 points) A graph is a collection of vertices (points) joined by edges (line segments). A pair of vertices are adjacent if they are connected by an edge.

   Colouring a graph refers to assigning colours to its vertices. A graph is said to be properly coloured if every pair of adjacent vertices receive different colours.

   Consider the below graph with 4 vertices and 4 edges. This graph is denoted as C_4. If you are permitted to use 6 different colours, how many proper colourings does C_4 have?



Answer: 630

6. (25 points) Find the sum of all solutions to the equation

$$(x^2 + 5x + 5)^{x^2-10x+21} = 1$$

Answer: 2

7. (25 points) Alice invited seven of her friends to a party. At the party, several pairs of people shook hands, although no one shook hands with themselves or shook hands with the same person more than once.

   After the party, Alice asked each of her seven friends how many people they shook hands with during the party, and was surprised when they responded with seven distinct positive integers.

   Given that her friends were truthful, how many hands did Alice shake?

Answer: 4

8. (25 points) ***Change Walker Now***
   A *root of unity* is any complex number that yields 1 when raised to some positive integer power n.

   If $w \neq 1$ is an n-th root of unity, then find the value of

$$1 + w + w^2 + w^3 + \ldots + w^{n-1}$$

Answer: 0

9. (30 points) Find the number of real solutions (x, y, z) of

$$\begin{cases} (x+y)^3 = z \\ (y+z)^3 = x \\ (z+x)^3 = y \end{cases}$$

Answer: 3

10. (30 points) Suppose a square is cut by two parallel lines at perpendicular distance $6cm$ apart such that

    - one line has an endpoint at the lower left corner of the square and the other has an endpoint at the upper right corner of the square, and
    - the square is divided into three regions, (a parallelogram and two triangles), of equal area.

    What is the area of the square in cm^2?

Answer: 468

11. (30 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a bijective function, and

$$f'(x) = \frac{x^{100}+1}{x^{100}+2}$$

If $f(1) = \alpha$, what is $(f^{-1})'(\alpha)$?

Answer: $\frac{3}{2}$

12. (30 points) ***Change Walker Now***
A bag contains 100 marbles, some of which are red, the rest of which are blue. There are more red marbles than blue marbles. If you were to draw two marbles (without replacement) from the bag, you'd be just as likely to get different-coloured marbles as you would be to get marbles that were the same colour. How many blue marbles are there in the bag?

Answer: 45

13. (35 points) Let $ABCD$ be a unit square and $\lambda_i = 1$ or $\lambda_i = -1$ for $i \in \{1, 2, 3, 4, 5, 6\}$. Find the *maximum* of $\left\|\lambda_1\overrightarrow{AB} + \lambda_2\overrightarrow{BC} + \lambda_3\overrightarrow{CD} + \lambda_4\overrightarrow{DA} + \lambda_5\overrightarrow{AC} + \lambda_6\overrightarrow{BD}\right\|$. Where $\|\cdot\|$ represents the standard Euclidean norm.

Answer: $2\sqrt{5}$ or $\sqrt{20}$

14. (35 points) A conical reservoir with its vertex pointing downward has a radius of 10 metres and a depth of 20 metres. Suddenly, a leak springs and water begins to empty the cone at its vertex and into an empty cylindrical basin with radius 6 metres and height 40 metres. At the moment the depth of the reservoir reaches 16 metres and is decreasing by 2 metres per minute, how fast is the height of water in the basin changing? Give your answer in m/min.

Answer: $\frac{32}{9}$

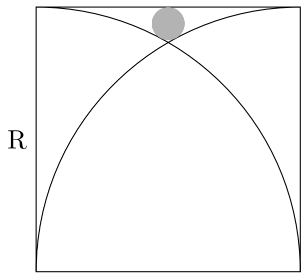
15. (35 points) A point is chosen randomly (by uniform distribution of area) on the inside of a unit circle. Find the expected value of its distance from the center of the circle.

Answer: $\frac{2}{3}$

16. (35 points) ***Change Walker Now***
Let $f(x) = x^4 + x^3 + x^2 + x + 1$. Find the remainder when $f(x^5)$ is divided by $f(x)$.

Answer: 5

17. (40 points) Depicted below is a square of side length R, with circular arcs of radius R on two adjacent corners. A circle (shaded grey) is drawn so it is tangent to the arcs and edge of the square. Find the radius of the circle in terms of R.



Answer: $\frac{R}{16}$

18. (40 points) Consider 3 random chords drawn in a circle. (Each chord is established using the random endpoints method, i.e. the endpoints of any chord are two points chosen randomly, uniformly and independently on the circumference of the circle.)

Find the probability that there is *exactly one* point of intersection (inside the circle) among these 3 random chords.

Answer: $\frac{2}{5}$ or 0.4

19. (40 points) A sequence $\{a_n\}$ is defined by $a_1 = k$ and $a_{n+1} = a_n - \frac{1}{(n+1)!}$ for $n \geq 2$ such that k is a constant real number.

Also, $\sum_{m=1}^{\infty} a_m = K$, where $K \in \mathbb{R}$.

Find the sum of k and K.

Answer: $e - 1$

20. (40 points) Find the positive integer n with exactly 12 divisors $1 = d_1 < d_2 < \cdots < d_{12} = n$ such that the divisor with index $d_4 - 1$ (that is $d_{d_4 - 1}$) is $(d_1 + d_2 + d_4)d_8$.

Answer: 1989

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Answer Table

Question	Points	Answer
1	20	2π
2	20	Red
3	20	60
4	20	2475
5	25	630
6	25	2
7	25	4
8	25	0
9	30	3
10	30	468
11	30	$\frac{3}{2}$
12	30	45
13	35	$2\sqrt{5}$ or $\sqrt{20}$
14	35	$\frac{32}{9}$
15	35	$\frac{2}{3}$
16	35	5
17	40	$\frac{R}{16}$
18	40	$\frac{2}{5}$ or 0.4
19	40	$e-1$
20	40	1989