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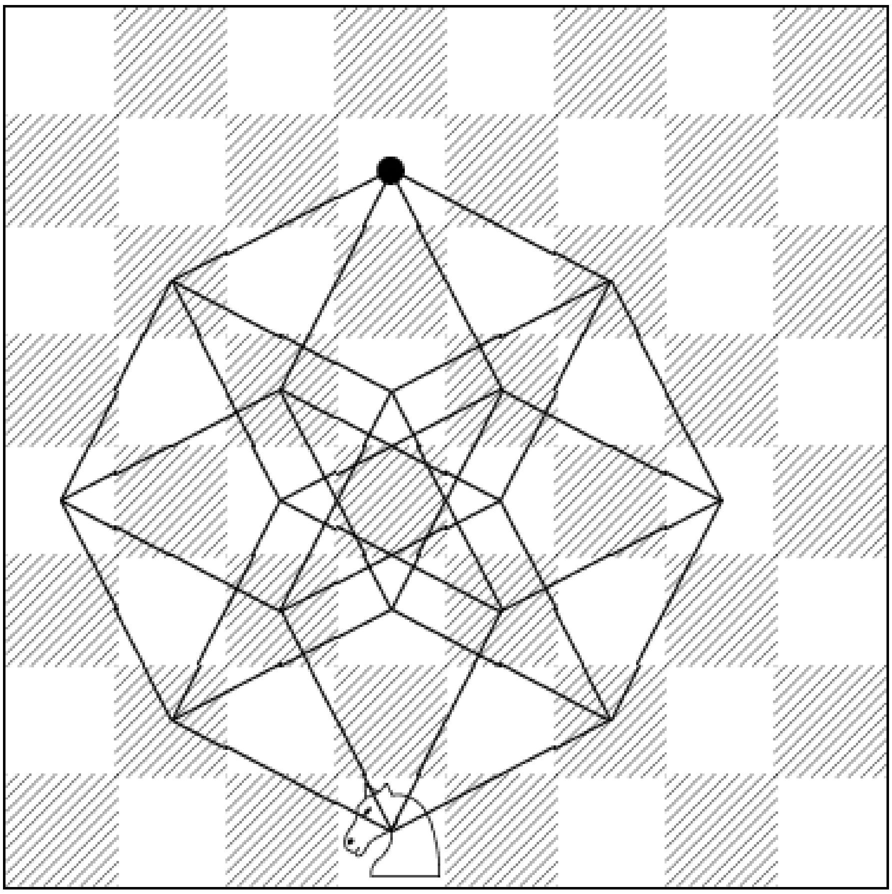
Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society



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In This Edition of Paradox

Regulars

Special Features & Articles

WEB PAGE: www.ms.unimelb.edu.au/~mums/paradox
E-MAIL: paradox.editor@gmail.com
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COVER: The many different ways a knight can move across a chessboard (see page 10 for details).

Maths was fairly easy in Ancient Rome compared to today since x always happened to be equal to 10.

Words from the Editor

Welcome to this year's third and final issue of Paradox, the magazine produced by the Melbourne University Mathematics and Statistics Society (MUMS), although a fourth special edition may follow before the year's end. 2013 has been a great year in general (despite what our illustrious President says on the following page) and we at Paradox wish you all the best with your upcoming exams if you have them!

I did promise that a new Paradox team was going to be put together a month after the last Paradox was published, but you didn't seriously believe me, did you? Nonetheless, I am a man of my word and there is now a vast array of potential contributions you can choose from on the last page of this issue, so turn to the back page of this issue and apply!

It's great fun to be part of the Paradox team, and we have many new and exciting things planned for 2014. We eagerly await a new Paradox hero to fill the shoes of the valiant Rubik's Turtle, as well as regular contributions on topics that haven't featured much in Paradox lately.

In this issue of Paradox, find out what Kazhdan-Lusztig Conjectures are and how fashionability can lead any mathematician astray, how many times a knight can move around a chessboard without visiting any square twice, as well as a blast from the past of bad proofs submitted by first-years a long, long time ago. The hardest Sudokion spatial-logic puzzles also await you during this SWOTVAC and onward, so enjoy the 'productive' and fun (mostly fun) procrastination to be found within Paradox.

To contribute to the next issue, just contact me at `paradox.editor@gmail.com` and I wish you all a relaxing and memorable summer ahead!

— Kristijan Jovanoski

An infinite number of mathematicians walk into a bar. The first one orders a beer. The second orders half a beer. The third orders a third of a beer. The bartender bellows, "Get the hell out of here, are you trying to ruin me?"

Words from the President

It's been a sad semester for our beloved university. It started out fine, with the joy of every student coming back to see more wonderful MUMS seminars and other assorted events, as if all had forgotten the horror of the Roman invasion just a few months earlier. But, when we were just about to take our 'mid' semester break, the unthinkable happened: a team from Monash took out the coveted University Maths Olympics prize. Admittedly the winning team, 'The Contravariant Funksters', weren't all from Monash, but enough were that we will certainly be licking our wounds for some time.

Thankfully, MUMS came to the rescue, and gave all students at this university the chance for redemption at the trivia night on the last day of semester! 'Where the Hell is Bill?' ended up winning once Bill eventually arrived. In order to help train you all to defeat any future onslaught, should it come from another university, I have devised a series of canonical practice questions, each more canonical than the last:

- What was noticeably absent from every Target solved in the MUMS room for the first three months of this year?
- What colour is the new carpet in the MUMS room? (Yes, that's right, we have a new carpet.)
- According to the new Star Trek movie (but not the *lnto Darkness* one) what would Spock have said after "the unthinkable happened"?
- Who originally suggested the name *lnto darkness* and why? (That's a lower case L, in case you didn't notice).
- Are any of these questions actually canonical?

If you can answer all of these you'll surely have no problems dealing with whatever trivia will be thrown at you next time. While the MUMS year might appear to be over for you... it doesn't have to be! You can still come into the MUMS room whenever it's open (usually around lunch) and, if you're really keen, we'll soon be planning our 2014 Puzzlehunt. Email `mums@ms.unimelb.edu.au` if you'd like to be involved!

— Andrew Elvey Price

Fashionability and Conjectures

Suppose you were sent to an alternate universe and you were interested in its chemistry. Using the most up-to-date scanner, you manage to find and identify two substances nearby: water and ethanol.

You already know, from previous work, that the periodic table for this universe has only three elements: carbon, oxygen and hydrogen. Your scanner says that water and ethanol are both molecules, consisting of many atoms. You might ask: how many carbon atoms, oxygen atoms, and hydrogen atoms are there in each molecule of water? How about each molecule of ethanol? How would you find that ethanol consists of two carbon atoms, six hydrogen atoms, and one oxygen atom?

The above discussion describes the chemistry analogue of the sorts of questions that pop up in a field of mathematics called *representation theory*: the study of mathematical objects by representing them as matrices. Let's push the analogue through to mathematics. In representation theory, *modules* are the molecules and *simple modules* are the atoms, while a complete list of all possible simple modules is the periodic table of elements.

The questions then become: Given a module M, what are its constituent simple modules, and how many of each simple module are in M?

During the second half of the twentieth century, representation theorists were interested in special types of modules, called the *Verma modules*. Two mathematicians, David Kazhdan and George Lusztig, were interested in how simple modules came together to form these Verma modules. In their 1979 paper [1], they proposed a way to calculate how many of each simple module were in a given Verma module. They believed that the calculation could be done by fiddling around in an algebraic object called the *Hecke algebra*. But for them to be sure that the calculation would work, they needed to prove a certain conjecture. This came to be known as the *Kazhdan-Lusztig conjecture*.

In our analogy, it's as if Kazhdan and Lusztig developed a method to find, given a molecule M, how many oxygen, carbon, and hydrogen atoms are in M. If the Kazhdan-Lusztig conjecture was true, it would mean that this method worked 100% of the time.

My first encounter with the Kazhdan-Lusztig conjecture came when I referenced their 1979 paper in the final page of my Honours thesis, in mid 2012:

> "Naturally, this leads to an investigation of the relationship between representation theory of the Hecke algebra of type W and the group algebra $\mathbb{C}W$. Kazhdan and Lusztig investigated this in their seminal paper [1]."

Embarassingly, it turned out that they weren't investigating that stuff at all.

In late 2012, as I started my postgraduate studies, I mentioned to my supervisor, Arun, that I wanted to understand Kazhdan and Lusztig's 1979 paper:

"Why are you interested in reading the paper?"

"It links the representation theory of the Hecke algebra and the group algebra."

"That's not what it does."

At this point there was a pause in the dialogue, and I couldn't think of what to say. Confused, I muttered something about going away and thinking about it again. I returned the next day and asked him again:

"If the paper's not doing what I think it's doing, what is it doing then?"

"The paper shows the existence of the Kazhdan-Lusztig basis, which might help us compute the decomposition numbers of Verma modules."

This was gibberish to me. Sometime later in the week, I asked him again:

"What are Kazhdan and Lusztig doing in their paper?"

"The paper shows the existence of the Kazhdan-Lusztig basis, which might help us compute the decomposition numbers of Verma modules."

Still gibberish. For a while, I kept hassling him about Kazhdan and Lusztig's paper. Then he asked:

"Why are you interested in it? Is it because it's fashionable?"

I was taken aback—no teacher had ever asked me a confronting question like this before, and I didn't know whether he was annoyed or worried or both! In hindsight, probably both. I thought about it for about ten seconds, and then said "I don't know, I'll get back to you."

"If it's because it's fashionable, then that's a mistake. A lot of students do something because it's fashionable and that's a mistake..."

I got annoyed. He was right though. I've since found out that Arun's always right. I soon learnt how to not take good criticisms personally.

So I realized that a reason I was interested in the paper was because it was the work of two big name mathematicians: Kazhdan and Lusztig. The important question was: did I care about the work itself? Looking at the paper, and seeing words that I liked (*Hecke algebras, Coxeter groups*) made me realize that it still might be worth pursuing.

In March 2013, I traveled to Denmark, where Geordie Williamson and Ben Elias gave a week-long intensive class about their work on Kazhdan-Lusztig theory (see [2]). Their work proved Soergel's conjecture, which not only proved the Kazhdan-Lusztig conjecture, but also gave a first proof of another conjecture, called the *Kazhdan-Lusztig positivity conjecture*. From their work we can get a 'holistic' perspective on Kazhdan-Lusztig theory. But this is a subject for another article.

To finish off, I've written the statement of the Kazhdan-Lusztig conjecture, in its 'easy to digest' form. Also, for those who are interested, I've written the statement of both the Kazhdan-Lusztig conjecture and the Kazhdan-Lusztig positivity conjecture in full, gory detail.

Conjecture. *The decomposition numbers of Verma modules can be calculated by fiddling around with the Hecke algebra.*

The Kazhdan-Lusztig conjectures (in gory detail)

Let $W = \langle s_1, s_2, \dots s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$ be a Coxeter group. The *Hecke algebra* $\mathcal{H}$ of W is the $\mathbb{Z}[v, v^{-1}]$-algebra generated by $T_{s_1}, T_{s_2}, \dots, T_{s_n}$ with relations

$$T_{s_i}^2 = (v^{-1} - v)T_{s_i} + 1 \qquad \text{and} \qquad \underbrace{T_{s_i}T_{s_j}T_{s_i}\dots}_{m_{ij}\text{ factors}} = \underbrace{T_{s_j}T_{s_i}T_{s_j}\dots}_{m_{ij}\text{ factors}}$$

Let $- : \mathcal{H} \longrightarrow \mathcal{H}$ be the $\mathbb{Z}$-algebra automorphism given by

$$\bar{T_{s_i}} = T_{s_i}^{-1} \qquad \text{and} \qquad \bar{v} = v^{-1}$$

The *Kazhdan-Lusztig basis* $\{C_x \mid x \in W\}$ of $\mathcal{H}$ is characterized by

$$\bar{C}_x = C_x \qquad \text{and} \qquad C_x = T_x + \sum_{y<x} p_{y,x} T_y$$

where $p_{y,x} \in v\mathbb{Z}[v]$. The polynomials $p_{y,x}$ are called the *Kazhdan-Lusztig polynomials*.

Conjecture 1 (The Kazhdan-Lusztig conjecture)**.** *For each $w \in W$, denote by M_w the Verma module of highest weight $-w(\rho) - \rho$, where ρ is the half-sum of the positive roots, and let L_w be its irreducible quotient, the simple highest weight module of highest weight $-w(\rho) - \rho$. Then*

$$\mathrm{ch}(L_w) = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} p_{y,w}(1)\mathrm{ch}(M_y)$$
$$\mathrm{ch}(M_w) = \sum_{y \leq w} p_{w_0 w, w_0 y}(1)\mathrm{ch}(L_y)$$

where w_0 is the element of maximal length in W.

Conjecture 2 (The Kazhdan-Lusztig positivity conjecture)**.** *The coefficients of the polynomials $p_{y,x}$ are non-negative.*

Conjecture 1 is Conjecture 1.5 of [1], and was first proven independently by Beilinson and Bernstein [4] and Brylinski and Kashiwara [3]. It was also later proven by Elias and Williamson in [2]. Conjecture 2 is found just before Definition 1.2 of [1], and was first proven by Elias and Williamson in [2].

— Jon Xu

References

[1] David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*. Inventiones Mathematicae, 1979.

[2] Ben Elias and Geordie Williamson, *The Hodge theory of Soergel bimodules*. Preprint, http://arxiv.org/abs/1212.0791, 2012.

[3] J.L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*. Inventiones mathematicae, 1981.

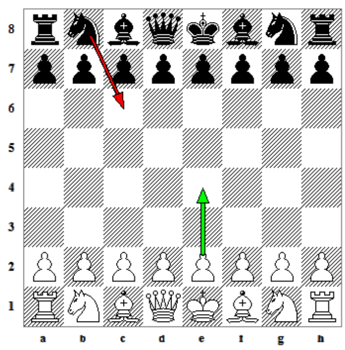
[4] Alexandre Beilinson and Joseph Bernstein, *Localisation de g-modules*. Ser. I. Math, 1981.

The Beauty of Mathematical Chess

The typical chess puzzle is presented as a position together with a caption such as 'mate in 2'. Unfortunately, these kinds of puzzles tend to impress only two groups of people: serious players who want to take their chess to a high level, and babies who use the pawns as dummies. But in fact, the geometry of the chess board and the way the pieces move gives rise to some quite beautiful mathematical problems which can be tried and appreciated by even non-pure mathematicians (aka 'lesser mortals'). Only a basic knowledge of chess moves provided below is needed to follow this discussion.

Algebraic chess notation in 60 seconds

The purpose of chess notation is to record chess moves in a way which is simple to understand, easy to notate, and unambiguous. In algebraic notation, we record the piece that moved (K = king, Q = queen, R = rook, B = bishop and N = knight; no letter is required if the move was made by a pawn) and its destination square, with the files of the chessboard being labelled from a to h, and the ranks from 1 to 8, so that white's bottom left corner is square a1 and the opposite corner is h8:

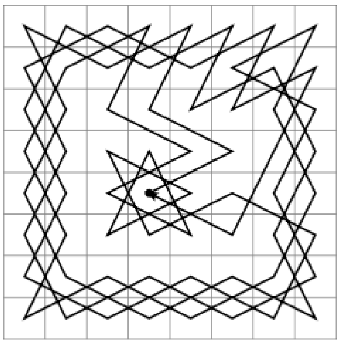


For example, the white pawn move indicated above is written as e4, and black's knight move is written Nc6. Captures are indicated with an x (e.g. Nxd8 indicates that a piece on d8 was taken by a knight). There are many other conventions in place to deal with checks, castling, and move ambiguity (which occurs when two pieces can move to the same square), but they are not needed to follow this article so we will not discuss them here.[1]

[1]For a full explanation of algebraic notation, read the Wikipedia page at http://en.wikipedia.org/wiki/Algebraic_notation_(chess).

The knight's tour problem

This problem is concerned with finding a sequence of knight moves on a chessboard such that the knight visits every square exactly once. Here is an example of a knight's tour, first recording the moves, then the visited squares:



6	19	58	37	4	17	60	47
57	38	5	18	59	48	15	62
20	7	36	3	16	61	46	49
39	56	29	32	35	2	63	14
8	21	34	1	30	27	50	45
55	40	31	28	33	64	13	26
22	9	42	53	24	11	44	51
41	54	23	10	43	52	25	12

Unless you are either very smart or very lucky—or both—you will probably run into a dead end or discover an inaccessible square on trying to do a knight's tour for the first time. This is because the chessboard (as seen from the knight's perspective) is a complicated network of squares and connections between squares. Here is a simple conceptualization:

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

The numbers represent the degree of each square, i.e. how many legal moves a knight has from that square. When you move the knight to a new square on the chessboard, you not only make that square unusable for the rest of the knight's tour, but you also decrement the degree of the neighbouring squares by 1. Running into a dead end is equivalent to stepping on a square with degree 0. Furthermore, leaving a square on the board with degree 1 is risky, since aside from the possibility of cutting off the remaining vertex it also forces the knight's tour to end on that square. Here are some strategies for solving the knight's tour, in ascending order of merit:

- **Brute Force:** Good luck with this one. Writing a search algorithm isn't too difficult, but exhausting all of the roughly 4×10^{51} possibilities for move sequences might take a little while.

- **Divide and Conquer:** Divide the board into smaller squares, find knight's tours for the smaller squares, and then piece up the tours to solve the larger board. Unfortunately the natural partition of the 8×8 board into smaller 4×4 boards fails, as it has been shown that no knight's tour for the 4×4 board exists.[2] Nevertheless this strategy can be employed for boards of other sizes (e.g. 12×12).

- **Border/Corner:** Since knights are more likely to get stuck on the edge of the board ("Knights on the rim are dim"), this strategy is based on using up the edge and corner squares first. In practice, this usually involves making the knight run laps around the outside edge of the chessboard. Since Warnsdorff's Rule below is time-consuming for humans, the border/corner strategy is probably the most effective heuristic to use when armed with only pen and paper.

- **Warnsdorff's Rule:** An extension of the border/corner strategy, Warnsdorff's Rule states that one should always move to the square which has the minimal number of remaining vertices. Where two squares have the same degree, one may be chosen at random. This heuristic is remarkably successful for generating valid knight's tours, even for large boards.

Problem: Why is there no knight's tour of a 5×5 board starting from a2?

The eight queens problem

Can you place eight queens on a board so that no two attack each other? Clearly no two queens can share a rank or file. Checking for queens attacking each other along diagonals is more difficult. The main frustration for a human trying to solve the problem is that when one attempt is found to be illegal, at least three queens have to be moved to produce a new configuration.

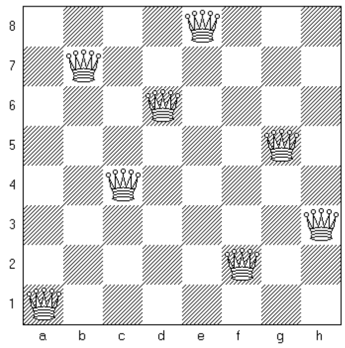
Fortunately, the eight queens problem can be efficiently solved by a straightforward 'generate and test' algorithm, due to the rapidity with which squares become unavailable, as well as the ease of checking whether or not a given configuration is valid. In fact, when we consider the rank/file constraint, there

[2] Luis Paris, *Heuristic Strategies for the Knight Tour Problem*, p3 `http://faculty.harrisburgu.net/~paris/papers/ppr_icai_knight_tour.pdf`.

remain only $8! = 40320$ configurations which need to be checked for diagonal attacks. The outline of the algorithm is as follows:

1. Start with an empty board.
2. On the first/next available row, place a queen on the first legal square (e.g. a1). Repeat for the next few rows.
3. If unable to place a queen, then go to the previous queen and try the next available square.
4. If eight queens are legally on the board, record the configuration.

It's easy to see that this search will reliably identify all possible configurations of eight queens. In fact, there are 92 solutions to the problem, but due to rotational and reflectional symmetry, we need to reduce this by a factor of $4 \times 2 = 8$ to get 12 fundamental solutions. Here is one of them:



There are several variations of the basic eight queens problem:

- A solution to the n queens problem exists for every board size except for $n = 2$ and $n = 3$.
- Four queens can be placed on a $3 \times 3 \times 3$ cube so that no two attack each other.[3]
- The maximum numbers of other pieces that can be placed on the chessboard subject to the same constraints are 8 rooks, 16 kings, 14 bishops, or 32 knights (e.g. all on white squares). Interestingly, one proof that 32 is the maximal number of knights presupposes a knight's tour.

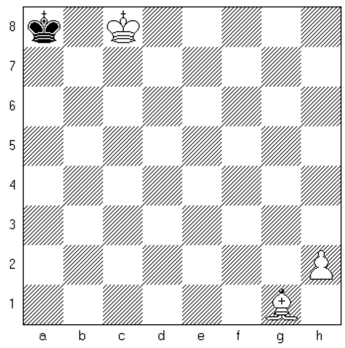
[3] J. Barr and S. Rao, *The n-Queens Problem in Higher Dimensions*, http://arxiv.org/pdf/0712.2309v1.pdf.

- The *minimum* number of queens that need to be placed on a chessboard to ensure that every square is attacked is called the *domination number*. For an 8×8 chessboard, the queen has a domination number of five,[4] although it takes some effort to find such an arrangement.

The maximum number of queens you can legally obtain in a regular chess game is nine. In fact, identifying promoted pieces is a key theme in retrograde analysis, which leads to our next topic of discussion.

Retrograde analysis

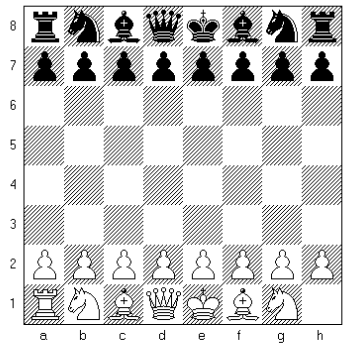
Since chess players generally analyse possibilities which may occur *from* a given chess position, retrograde analysis is a rebellious category of chess problem which instead focuses attention on the moves *leading up to* a particular position. Retrograde analysis may ask questions about castling legality, whether a position is reachable via a legal series of moves, or whether a particular piece is the result of a promotion. Here is a simple illustration:



White to play. What was black's last move?

We can quickly deduce that black's last move must have been a king move from a7, since moving from b8 or b7 would mean a situation with adjacent kings, which is not possible in chess. But on a7 it was in check from the bishop on g1. What did white play to bring about this check? It could not have been Bh2-g1+, since the pawn is on h2. The answer is that white played a discovered check with a piece no longer on the board, namely Nb6-a8+, after which black responded with Kxa8, reaching the diagrammed position above.

[4] P.B. Gibbons and J.A. Webb, *Some New Results for the Queens Domination Problem*, p148 `http://ajc.maths.uq.edu.au/pdf/15/ocr-ajc-v15-p145.pdf`.



Black to play. State one move that *must* have happened in the game.

It is very easy to reach the second position above with white to play: one simply needs to direct a black knight to capture the rook on h1 and move it back, and get white to waste moves with Nb1-c3-b1 in the meantime. The stipulation that black is to play causes problems however—it seems that we are always off by half a move. While the unenlightened may try knight manoeuvres (in vain) of ever-growing complexity, the pure mathematician notices that the 'round trip' of a knight always takes an even number of moves, so that there is no way to resolve the move difference with the knights.

What then? The only other piece that we can move is the rook, but that seems to be of no help either—each rook similarly needs to move an even number of times to return to its original position (e.g. Ra8-b8-a8). The only possibility is that the white rook wasted one move with Rh1-g1 and was then captured on g1. And since a black knight on f3 would force white to capture it due to the check, the knight must have arrived and left via h3 (i.e. Nh3xg1 and Ng1-h3). In fact, this is easily constructible and works out perfectly:

1.Nf3 Nf6 2.Ng1 Nh5 3. Nf3 Nf4 4.Nd4 Nh3 5.Rg1 Nxg1 6.Nf3 Nh3 7.Ng1 Nf4 8.Nf3 Nh5 9.Ng1 Nf6 10.Nf3 Ng8 11.Ng1, reaching the diagrammed position.

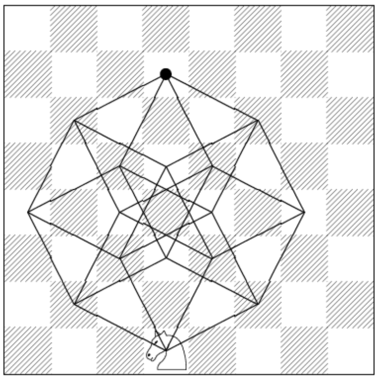
Path-related problems

Another category of problem deals with enumerating the number of shortest paths from one square to another. Many such problems can be solved using combinatorial principles. In addition, the paths often trace out nice geometric patterns. For example, in how many different ways can a knight move:

1. From d1 to c4 in two moves?
2. From d1 to e5 in three moves?

3. From d1 to d7 in four moves?

The paths for 3) are shown here:[5]



It turns out that the two (2!) paths for 1) trace out a square, the six (3!) paths for 2) trace out a cube, and the 24 (4!) paths for 3) trace out a tesseract (the 4-dimensional analogue of the cube). Why this happens, I have no idea, but it is undoubtedly a beautiful piece of geometry.

Conclusion

There is an unfortunate stereotype that chess is a game for people who love analysis and calculation. While that may be true of competitive chess, it does not preclude the rest of humanity from appreciating the game. I have presented here an introduction to some interesting chess problems in the hope that readers might share in my enthusiasm. Naturally, there are many more interesting problems which I have not been able to list here. I have added links to some other famous problems below:

- Mutilated chessboard tiling `http://tinyurl.com/5w5tov`
- Euler's 36 officers problem `http://tinyurl.com/kudtqom`
- Helpmates `http://en.wikipedia.org/wiki/Helpmate`
- Kim Yong Woo *Chess and Mathematics* `http://profstewart.org/pm1/talks/Chess.pdf`

— Jason Tang

[5] Diagram taken from Noam Elkies, *The mathematical knight*, p10 `http://www.math.harvard.edu/~elkies/knight.pdf`.

Flashback: Bad Proofs

Several years ago, A/Prof Barry Hughes set a first-year exam question which asked for a proof that there is no largest prime number.[1] The following was sent to him by one of the graduate students who helped with the marking:

Hi Barry, just for educational value, I compiled a list of the most incorrect answers to question B4(b). It is entitled "15 good reasons why Pure Mathematics is not taught to first year students."

1. **Proof by example:** "Let x be the largest prime. Then $x = 91$ but $91 + 6 = 97$, which is prime. Therefore, 91 cannot be the largest prime number. Therefore there is no largest prime number."
2. **Proof by oddness:** "If n is the largest prime number, then n is odd. Then $(n + 1)/2$ is even. Therefore, $(n + 1)/2 + n$ is odd. But $(n + 1)/2 + n$ is not divisible by any number except itself. As it is bigger than n, the assumption is wrong, by contradiction."
3. **Proof by intuition:** "Prime numbers are integers that can be divided by themselves only; prime numbers are odd with the exception of 2. By intuition as $n \to \infty$, there will always be an odd number that cannot be divided by any other number besides itself."
4. **Proof by $\sqrt{2}$:** "that p is divisible by q, i.e. $p/q = 2r$, where r is an even number. Then $p = 2qr$ so $p^2 = 4q^2r^2$. But r^2 does not exist and $q! = 1$. Therefore, q must exist. Since q exists, p must be divisible. Therefore, by contrary, there is no largest prime number."
5. **Proof by superinduction:** "2 is a prime number. Now assume N is the largest prime. But then $N + 1$ exists and is also prime. Therefore, by induction there is no highest prime number.
6. **Proof by the previous question:** "Suppose N is the largest prime. Then let $N = (n^2)/2$. Therefore $n = \sqrt{2N}$. But from above $1 + 2 + 3 + \ldots + n > N$. Hence, there is a larger prime number than N."
7. **Proof by tutorial question:** "Let m, n be two integers with $m > n + 1$. If k is even, $m^k + n^k$ cannot be expressed in terms of $(m + n)$(polynomial in m and n) and so is prime. Therefore, as m and n can be any numbers, there is obviously no largest prime number."

[1]This article originally appeared in Issue 2 of Paradox in 2000.

8. **Proof by having no idea what a prime is:** "Say the largest prime possible is x, then $2x$ is also a prime since the statement is true for all natural numbers."

9. **Proof by experimental data:** "Suppose n is the highest prime. Then $2n-1$ is also prime. But $2n-1 > n$ so there is no highest prime (Check: $2 \times 2 - 1 = 3$, $2 \times 3 - 1 = 5$, $2 \times 5 - 1 = 11$, $2 \times 11 - 1 = 23$, so true)."

10. **Proof by subscript:** "If there is a highest prime, we can number all the primes p_1, p_2, ..., p_n. But as there is no highest natural number, there is always an $n+1$ so there must be a p_{n+1}. Therefore, there is no highest prime."

11. **Proof by infinity:** "Let n be the highest prime number. But ∞ is greater than all numbers so $\infty > n$. If n is the highest prime this would mean ∞ has factors. Therefore, we have a contradiction."

12. **Proof by reverse logic:** "All prime numbers are odd. Suppose there were a highest prime. Then we have a highest odd number. But if $2k+1$ is the highest odd number, then $2k+3 = 2(k+1)+1$ is also an odd number. Therefore, we have contradiction and therefore we have a contradiction."

13. **Proof by denial:** "Assume there is a largest prime M. We can add 1 to M until we get another prime number N $(M+1+1+1+\ldots+1 = N)$. But then $N > M$. Therefore, M is not the largest prime number, so there is no largest prime number."

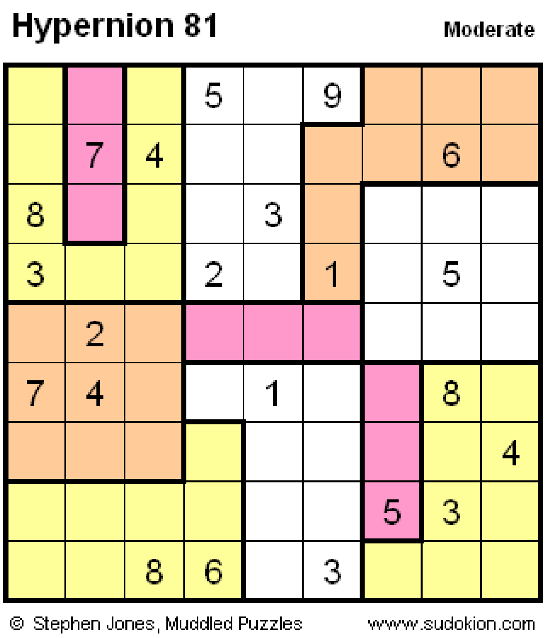
14. **Proof by formula:** "As prime numbers are derived via the formula, we can assume it works for $n = k$ giving the highest prime number. But then it also works for $n = k+1$, so there is no highest prime number."

15. **Proof by continuity:** "Let x be the largest prime number. Then $x >$ all other primes. But then $(x+n)$, the next prime number, does not exist. However, numbers are continuous and so $(x+n)$ does exist. Therefore, there is no x."

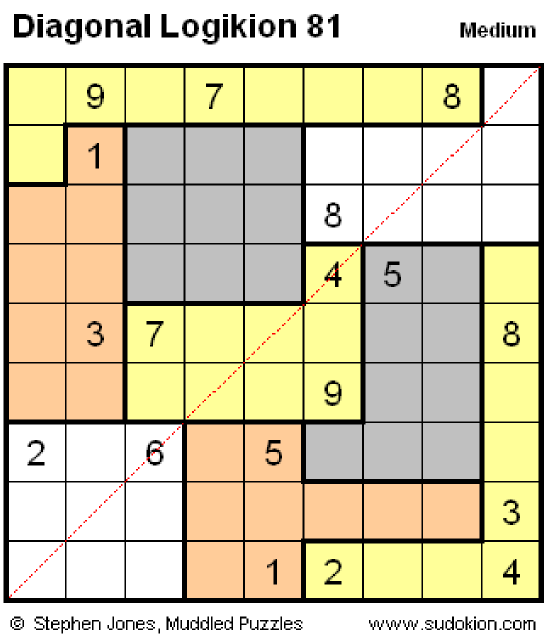
These are all verbatim answers from the 150-odd papers I marked, i.e. about one in ten. My favourite is definitely number 10; I think that is quite ingenious.

— George Doukas

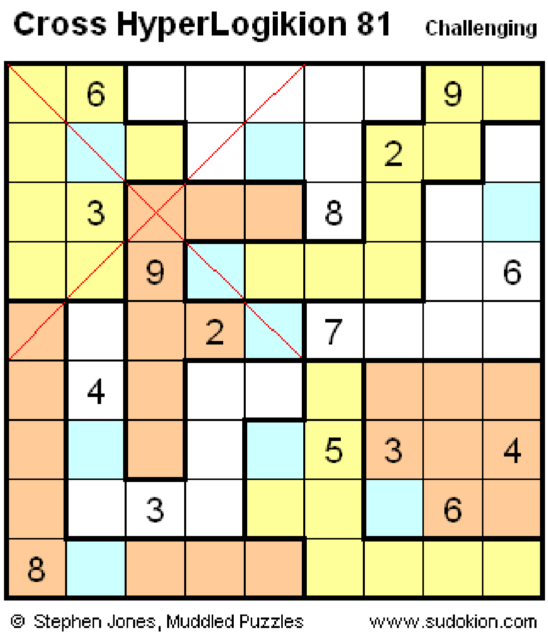
SUDOKION: Spatial-Logic Puzzles



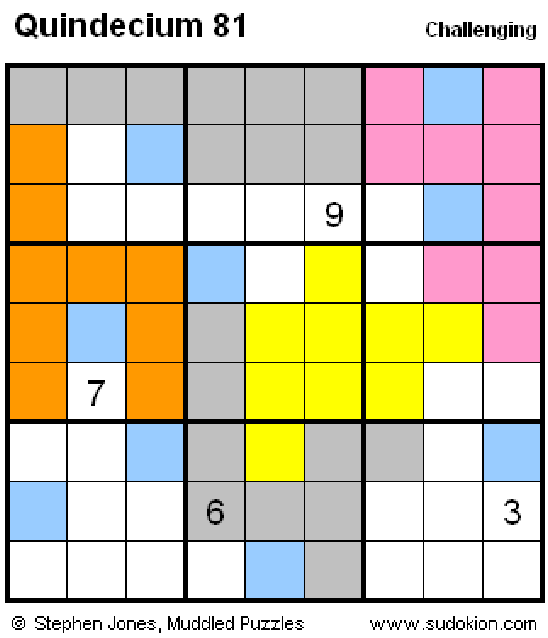
Every row, column, and cluster, including the fragmented pink cluster, must contain the numbers 1 to 9.



Every row, column, coloured cluster, and the red diagonal must contain the numbers 1 to 9.



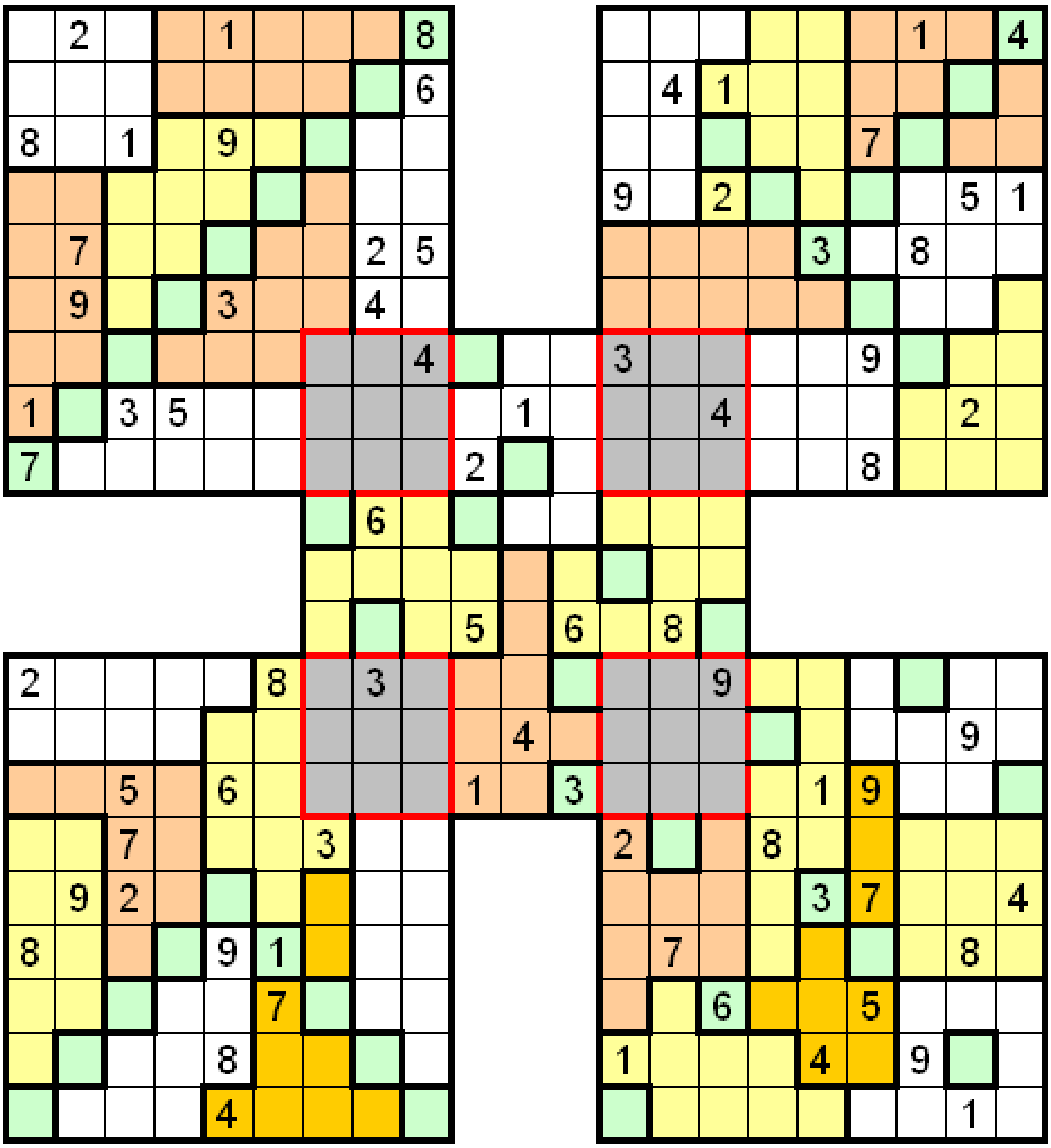
Every row, column, cluster, the red crossed lines, and a tenth cluster formed by the light-blue cells must each contain the numbers 1 to 9.



Every row, column, 3×3 box and colour cluster, including the fragmented blue cluster, must contain the numbers 1 to 9. The sum of every 3-cell row and column must be equal.

Super Pandemonion 369

Moderate



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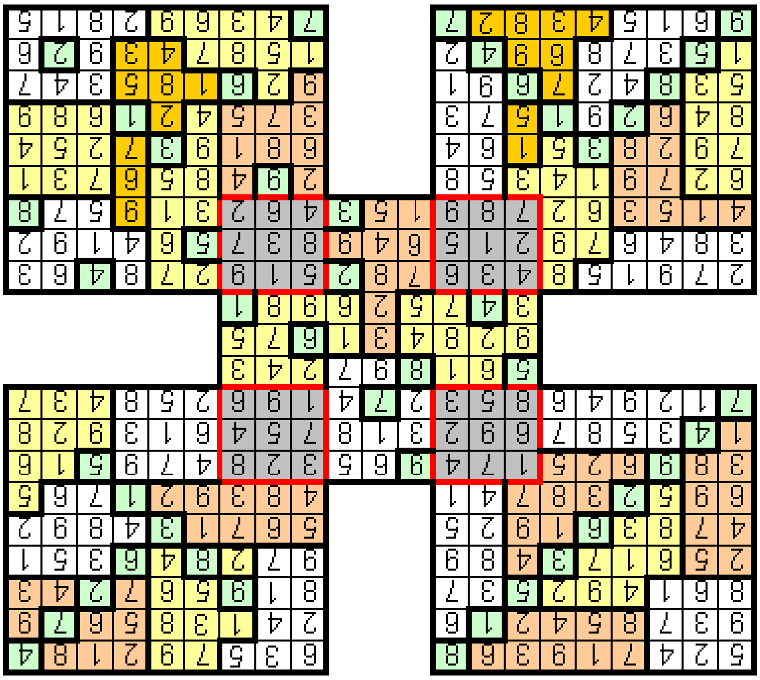
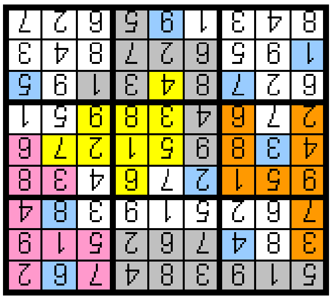
This puzzle has five interlocking sectors, with each grey cluster being common to the centre. Every row, column, and cluster, including the fragmented green cluster, of all five sectors must contain the numbers 1 to 9.

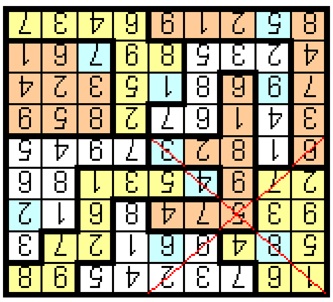
Solutions



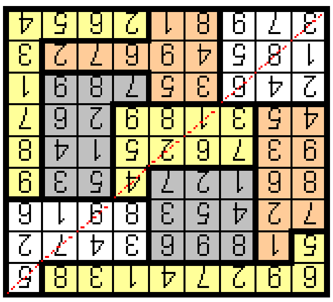
Super Pandemonion 369



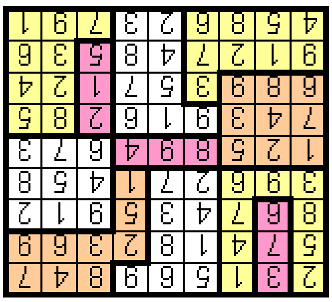
Quindecium 81



Cross HyperLogikon 81



Diagonal Logikon 81



Spirallium 81

The best of the best were asked: “What is 2 + 2 ?”
Engineer: “It’s approximately 3.99”
Physicist: “It lies between 3.98 and 4.02”
Mathematician: “I don’t know what the answer is, but I can prove an answer exists!”
Philosopher: “But what do you mean by 2 + 2?”
Logician: “Please define 2 + 2 more precisely.”
Accountant: *Closes all the doors and windows, looks around carefully then asks* “What do you want the answer to be?”

Paradox **Wants You!**

The highly-esteemed *Paradox* has been plunged into an existential crisis upon recent rumours that its long-time benevolent yet autocratic Editor, Kristijan 'I'm the only choice' Jovanoski, may no longer be a choice at all!

Rumour has it that he will flee the country sometime in 2014 once the MUMS Committee finally launches its long-overdue investigation into the shocking embezzlement of the famed *Paradox* staplers. While Kristijan 'You have no other choice' Jovanoski remains defiant, he has conceded that it may be a suitable time to look to the future and keep *Paradox* going should the unthinkable happen . . .

In an unprecedented move towards liberalization, he has announced several new positions in the incoming *Paradox* administration for 2014 that are now up for grabs:

- **Biographer:** You will revive and continue the recent tradition of supplying regular biographies of cool and famous mathematicians whose life stories will interest and entertain readers.
- **Book/Film Reviewer:** You will provide a regular review of a new book or film relating to mathematics or logic in each issue of of *Paradox* to entertain and inspire readers to digest its goodness as well.
- **Columnist:** You will write a regular column for the famed pages of *Para dox* on a particular theme or topic of your choosing within maths, stats, logic, puzzles, history, or anything else that is relevant.
- **Comic Designer:** You will bring the next hero of *Paradox* to life by designing a new comic series that will be a regular feature of the next several issues. A flair for drawing is not critical, since a camera, a Rubik's cube, and a gripping narrative have been enough in the past.
- **Content Sub-Editor:** You will look over articles to make sure that the maths used is correct and easy to follow. This is a maths (and stats) magazine after all!
- **Copy Sub-Editor:** You will look over submissions to check whether everything was written for humans rather than cyborgs. Spelling and grammar fanatics are highly desired but a keen eye for detail will do.

- **External Liaison**: You will seek to extend the boundaries of Paradox by expanding its outreach to interstate and international MUMS alumni as well as non-students by sourcing articles for Paradox written by individuals outside MUMS or even the University of Melbourne.
- **Historian:** You will embark on a journey of discovery through the archives of MUMS and the University of Melbourne to help restore ancient and valuable copies of Paradox as well as its mysterious predecessor ...
- **Interviewer:** You will interview MUMS alumni or mathematicians of interest to the MUMS community with the aim of providing readers of each Paradox with a valuable insight into what the future can hold.
- **Layout Manager:** You will use LaTeX typesetting to set out the final version of each Paradox issue elegantly and will convert submissions into LaTeX when necessary. Prior knowledge of LaTeX is useful but not critical; you will be taught how to use it and brought up to speed with the custom classes and styles used for Paradox in any case.
- **News Reporter:** You will report on the latest news happening within MUMS and the wider world at large (e.g. IMO, controversies, discoveries) that will be of interest to readers of Paradox and allow them to keep up with the times.
- **Puzzle Master:** You will revive the old tradition of supplying regular maths puzzles at the end of each issue of Paradox and will provide worked solutions in the following issue.
- **Strategy Master:** You will start a new tradition of supplying regular puzzles and/or articles concerning strategic board games (e.g. chess, backgammon, Go, *Risk*, *Monopoly*, *Settlers*, etc.) to provide interesting content for the non-purists.

To apply, just express your interest by emailing `paradox.editor@gmail.com` to discuss things further. You need not be a member of MUMS or even a student at the University of Melbourne to apply. First-come, first-served!

Paradox would like to thank George Doukas, Stephen Jones, Jason Tang, and Jon Xu for their contributions to this issue.