Issue 2, 1998

Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society

Illustration from Through the Looking-Glass, Sir John Tenniel.

President:	Chaitanya Rao kumar@ms.unimelb.edu.au
Vice-President:	Vanessa Teague vjt@ms.unimelb.edu.au
Treasurer:	Adam Cagliarini aac@ms.unimelb.edu.au
Secretary:	Tony Wirth awirth@ms.unimelb.edu.au
Education Officer:	Kathryn Turkovic kmturk@ecr.mu.oz.au
Publicity Officer:	Andrew Oppenheim oppy@ms.unimelb.edu.au
Editor of Paradox:	Jeremy Glick jglick@ecr.mu.oz.au
1st Year Representative:	Geoffrey Kong g.kong1@ugrad.unimelb.edu.au
2nd Year Representative:	Desmond Lun dsml@ecr.mu.oz.au
3rd Year Representative:	John Dethridge jcd@students.cs.mu.oz.au
Honours Representative:	Sally Miller smm@ms.unimelb.edu.au
Postgraduate Representative:	Dugal Ure ure@ms.unimelb.edu.au
Cybernetic Consultant:	Ben Burton benb@acm.org
Web Page:	http://baasil.stats.mu.oz.au/Mums

Paradox

Editor:	Jeremy Glick
Layout:	Desmond Lun
Sub-Editor:	Kuhn Ip
Problems:	Jian He
Contributors:	John Dethridge, Lawrence Ip
Web Page:	http://baasil.stats.mu.oz.au/Mums/Paradox
Email:	paradox@ms.unimelb.edu.au
Printed:	August, 1998

The excitement never stops

We are now in the midst of the most exciting time of year for MUMS. The Maths Olympics are around the corner (see below for more information), and there is a jam-packed schedule of MUMS events coming up (the LaTeX seminar, for example). To top it all off, we have a brand new committee, headed by Chaitanya Rao, which promises to take the Maths and Stats Society into a new era of splendour. A new MUMS president became required when Lawrence Ip, our former commander-in-chief, left for the U.S.A. to study for a Ph.D.

We received so many entries for the various competitions that we ran last edition that we've decided to include many more. Also, we've included a variety of articles and jokes (of a mathematical nature), so there should be something for everyone.

— Jeremy Glick, *Paradox* Editor

The *Paradox* team would like to thank Chaitanya Rao, Vanessa Teague, and Andrew Oppenheim for assisting with this issue.

The Maths Olympics

For those who haven't heard about it, the Maths Olympics is the largest annual event that the Melbourne University Mathematics and Statistics Society conducts. It involves forty-five minutes of fun and chaos as twenty-eight teams of five attempt to solve as many problems as they can in the restricted time. The contest is a mathematical relay — to receive the next problem, a team member has to run down to the front of the lecture theatre after the team has correctly solved (or given up on) the previous one. There will also be a free competition for spectators — tasty treats will be on offer! Below are the main details:

When?	Friday 28th August, 1998, 1:00 – 2:15p.m.
Where?	Theatre A, Richard Berry Building.
Prizes (per team)	1st : $150
	2nd : $125
	3rd : $100
	4th : $ 75
	5th : $ 50
	6th : $ 25
Team Entry Fee	$10 per team of five (payable on the day).

Entry forms are available from the Maths and Stats office and further details (including rules and sample questions) can be found in G53. The MUMS web page is also a useful resource, with sample questions and last year's problems and their solutions. In addition, a training guide is provided on page 7. Be quick as team numbers are limited!

Maths jokes

In the last issue of *Paradox*, prize-money was offered for the submission of maths jokes. For his efforts, $10 was awarded to Doron Boltin. The competition has been continued for another issue for anyone else who would like to earn the easiest $10 ever.

Three people were lost in a valley. 'We don't know where we are! Can anyone please tell us!' they called out, desperately hoping that the echo would send their message to someone who could answer. To their surprise, they heard, 'You're lost!' echoing through the hills some ten minutes later. 'That must have been a mathematician,' commented one of the company. The others were puzzled and asked him to explain. 'For three simple reasons,' he replied, 'firstly, the answer took too long to arrive; secondly, it was completely true; and thirdly, it's totally useless.'

A constant function and an e^x were walking down the street. Suddenly, the constant function saw a differential operator approaching and fled. Confused by this rash reaction, e^x decided to follow her and ask for an explanation. 'Well, you see,' she explained, 'there's a differential operator coming this way, and when we meet, he'll differentiate me, and then there will be nothing left of me!' 'That doesn't bother me,' declared e^x, 'I'm e to the x!' e^x continued to stroll on, and after some time, met the differential operator. 'Hi,' e^x said cheerfully, 'I'm e to the x.' 'Hello, I'm d/dy.'

A physicist, an engineer, and a statistician were hunting ducks. The physicist saw a duck and shot, but the shot missed wildly, flying five metres to the left of the duck. The engineer also saw the duck and shot — but her shot went five metres to the *right* of the duck. 'Nailed 'im!' rejoiced the statistician.

The bottom of the barrel

Though admittedly the winning jokes were somewhat below expectations, it is definitely possible to get much worse, as the following submissions from Jon Faulkner succinctly illustrate:

Q: What is the difference between a mathematician performing complex number arithmetic and one making mad passionate love with hundreds of beautiful babes?
A: None. Both are imaginary.

Q: What do you call a mathematician having sex?
A: A square root.

A flaw of democracy: Ostrogorski's paradox[1]

Completely unrelated to the Liberal Party, Ostrogorski's paradox shows that a party can win an election even though its opponent's positions are preferred by a majority of voters on every issue in question.

Consider an election disputed by party X and party Y, whose positions on the issues are x and y respectively. The following chart tabulates the results, where V_1–V_5 are the five voters of the constituency and I_1–I_3 are the three issues involved:

		Issues			
		I_1	I_2	I_3	Ballot
Voters	V_1	x	x	y	X
	V_2	x	y	x	X
	V_3	y	x	x	X
	V_4	y	y	y	Y
	V_5	y	y	y	Y
Preferred Position		y	y	y	

Each row of the chart records a voter's preference on each issue and the ballot cast by the voter. V_1, for example, prefers position x on issues I_1 and I_2 and position y on I_3; he consequently votes for party X. The chart shows that a majority of voters prefers position y on each of the three issues. Nevertheless, party X wins the election by a vote of 3 to 2. Clearly, V_1–V_5 may be five (equinumerous, or nearly so) groups of voters instead of five individuals.

All proper fractions have the same value![2]

Let m and n be any two integers such that n is less than m. Then by ordinary long division,

$$\frac{1-x^n}{1-x^m} = 1 - x^n + x^m - x^{n+m} + x^{2m} - \cdots. \tag{1}$$

Now let x have the value 1. The left-hand side of (1) assumes the indeterminate form 0/0. Applying L'Hôpital's rule, we have

$$\lim_{x\to 1} \frac{1-x^n}{1-x^m} = \lim_{x\to 1} \frac{-nx^{n-1}}{-mx^{m-1}} = \frac{n}{m}.$$

But the limit, as x approaches 1, of the right-hand side of (1) is $1-1+1-1+1-1+\cdots$. Therefore n/m, being equal to an expression which is independent of m and n, must always have the same value.

[1] Glenn W. Erickson and John A. Fossa, *Dictionary of Paradox*, University Press of America, Inc., Lanham, 1998, p. 145.

[2] Eugene P. Northrop, *Riddles in Mathematics*, Pelican Books, Harmondsworth, 1964, p. 207.

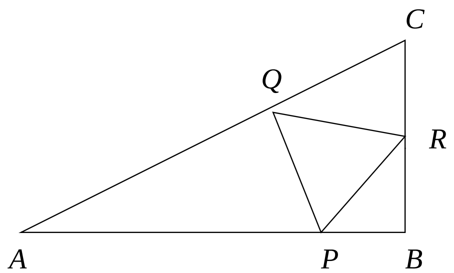
Problems

The following are some problems for prize-money. The person who submits the best (i.e. clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number, and will have their solution published in the next edition of *Paradox*. Solutions may be emailed to `paradox@ms.unimelb.edu.au`. (LaTeX format would be appreciated though not demanded.) If you do not have access to email then drop in a hard copy of your solution to the MUMS pigeon-hole near the Maths and Stats Office in the Richard Berry building.

1. ($10) We are given tiles in the form of right angled triangles having perpendicular sides of length 1cm and 2cm. Form a square from 20 such tiles. (Only solutions from first year students will be accepted for this problem.)

2. ($10) On the island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colours meet, they both simultaneously change colour to the third colour (e.g. if a grey and brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be same colour?

3. ($10) In $\triangle ABC$, $AB = \sqrt{3}$. $AC = 2$ and $BC = 1$. What is the smallest equilateral triangle, $\triangle PQR$, that can be inscribed (one vertex on each side) in $\triangle ABC$?



4. ($5) Show that for all distinct real numbers a, b and c,

$$\left(\frac{1}{b-c}\right)^2 + \left(\frac{1}{c-a}\right)^2 + \left(\frac{1}{a-b}\right)^2 = \left(\frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b}\right)^2.$$

5. ($15) For the sequence $a_n = [n\sqrt{2}]$, show that there are infinitely many a_n which are powers of 2. ($[x]$ is the largest integer $\leq x$.)

From beyond the wardrobe doors ...

Never the ones to waste an opportunity, Flip and Flop were at it again.

'I've got infinity,' said Flip.
'I've got infinity plus 1!!' replied Flop, quick as a snap.
'But you can't have that. Nothing's bigger than infinity.'

Suddenly, there was a crash in the wardrobe. Opening it cautiously, a person tumbled out followed by a cloud of dust. They introduced themselves.

'We're Flip, Flop. Who are you?'
'Cantor, Georg Cantor. But what am I doing here? I was about to take lunch.'
'That doesn't matter. What you can do though, is help us settle our argument.'
'What is the problem?'
'Infinity.'
'Ahhh ... ' came his response, with a hint of a smile.
'Why the knowing look?'
'Well, infinity is a tricky subject. Firstly, you need to consider what infinity actually is.'
'That's easy. It's a really, really big number.'
'Not quite. To understand the idea of infinity and what it *equals* to, we'll go back to fundamentals — counting. If you had sixteen members in a finite set, for example, how would you convey the sense of these sixteen objects to someone else?'
'Point to each one in the set, and recite — one, two three ... '
'But what if they only knew up to ten, with anything above ten being "many"?'
'I suppose you could put sixteen rocks in a basket, or make sixteen marks on the wall.'
'That's right. Or in other words, you set up a one-to-one correspondence between the rocks in the basket, and the objects whose number is sixteen. Counting in its simplest form. For every object there is one rock, and for every rock there is just one object.'
'I see what you're getting at. Now we can define equality between two sets as being able to set up a one-to-one correspondence between the two sets.'
'Exactly, and because the actual number of items doesn't enter into the notion of equality, we can extend it to sets with an infinite number of members. If two infinite sets are such that a one-to-one correspondence can be set up between their members, then we can consider them to have the *same transfinite number* ... '
'But wait,' interrupted Flip. 'What's an infinite set?'
'Well, I suppose we can determine whether a set is infinite by counting them,' countered Flop.
Cantor smiled another knowing smile. 'Not quite, because you can have infinite sets which can't be counted. I'll show you later. Instead, you might like to think of an infinite set as one which can be put into one-to-one correspondence with part (or a subset) of itself.'
'But wouldn't that mean that you would have part of a set being equal to the whole set?'

'Ahh ... but you are applying a concept associated with *finite* magnitude. We're dealing with the infinite at the moment, and you must always keep that in mind. Now, the "lowest" transfinite number is for the set of natural numbers, and I like to refer to it as $\aleph_0$ (pronounced "aleph" zero). A set which has the transfinite number $\aleph_0$ can be said to be *countable.*'

'I suppose you could think about that in another way. If a one-to-one correspondence between the set of natural numbers and a given set can be set up, then that set is countable and has transfinite number $\aleph_0$.'

'Exactly! Let's take a look at the set of positive rational numbers then ... ' said Cantor, proceeding to scribble on some paper. (The brackets indicate repeats.)

$$\begin{array}{cccccccccc}
\frac{1}{1} & & \frac{1}{2} & \to & \frac{1}{3} & & \frac{1}{4} & \to & \frac{1}{5} & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & & \\
\frac{2}{1} & & \left(\frac{2}{2}\right) & & \frac{2}{3} & & \left(\frac{2}{4}\right) & & \frac{2}{5} & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & & & \\
\frac{3}{1} & & \frac{3}{2} & & \left(\frac{3}{3}\right) & & \frac{3}{4} & & \frac{3}{5} & \cdots \\
\downarrow & \nearrow & & \swarrow & & & & & & \\
\frac{4}{1} & & \left(\frac{4}{2}\right) & & \frac{4}{3} & & \left(\frac{4}{4}\right) & & \frac{4}{5} & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \ddots
\end{array}$$

'If we follow the arrows, we see that we will eventually include every positive rational number in our ordering, which is equivalent to counting the positive rational numbers. While not exactly easy to work out where a given rational number fits in, it does demonstrate that the set of rational numbers is countable, and has the same transfinite number as the set of natural numbers.'

$$\begin{array}{ccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
\frac{1}{1} & \frac{2}{1} & \frac{1}{2} & \frac{1}{3} & \frac{3}{1} & \frac{4}{1} & \frac{3}{2} & \frac{2}{3} & \frac{1}{4} & \frac{1}{5} & \frac{5}{1} & \frac{6}{1} & \cdots
\end{array}$$

'Great, but what about the uncountable set?' asked Flip impatiently.

'Hold your horses ... for that, we will consider the real numbers between 0 and 1. All real numbers have a decimal representation, even if it is an infinite decimal representation. Now supposing we could order *all* them ... '

'... and if we could show that one was missing, then it would indicate the set was uncountable, right?'

'Bravo. Any idea what that number might be?'

'Well ... we could construct it: If the nth decimal place of the nth number was one of 0, 1, 2, 3 or 4, we could make the nth decimal place of our number be 7. Otherwise, make it 3 ... '

'Well done again. That would make our number different from all the other numbers we had already ordered, by at least one decimal place.'

'Q.E.D.'

'Notice too that the set of rational numbers is a subset of the set of real numbers. Therefore, the set of real numbers has transfinite number greater than $\aleph_0$.'

'That means one is more infinite than the other one!!'

'Ahhh ... ' began Cantor. But it was too late. Flip and Flop were off again.

'My infinity is more infinite than yours ... '

Meanwhile Cantor's thoughts turned back towards his lunch ... and he wandered back into the wardrobe.

FURTHER READING

Joseph W. Dauben, *Georg Cantor: his mathematics and philosophy of the infinite*, Harvard University Press, Cambridge, 1979.

N. IA. Vilenkin, *In search of infinity*, Birkhäuser, Boston, 1995.

— Kuhn Ip

A training guide for the Maths Olympics

The highlight of the MUMS calendar is the annual Maths Olympics. One of the best things you can do in the lead-up to this great event is to engage in the training that up until now has only been practised by one or two teams every year. I do not pretend to offer the final word when it comes to training methods for the Maths Olympics, but I will try to give a glimpse into the methods my team has used in the past seven years we have competed. What worked for us may not necessarily work for your team. Try them and see!

Firstly, a few words about the Maths Olympics itself. It's a maths relay with teams of five people. The team is split up into two groups on opposite sides of Theatre A with a runner relaying a question from the marker to the groups. Only one side may work on any one question and you can't work on the next question until you've either solved the previous question or given up on it. There's no penalty for wrong answers so you can guess as much as you want. Calculators aren't permitted for this contest — you're going to have to use old-fashioned pencil and paper!

Why should you participate in the Maths Olympics? Because it's a fun thing to do. In fact one of the rules explicitly says that "You shall have fun!" What other event at uni allows you to combine the joy of solving interesting problems, team work and physical activity all at the same time? You will have the chance to watch your lecturers embarrass themselves. The lecturer teams have often fared poorly (they're a bit too old!). It's probably the only chance you'll have to get physical with them — one year the then head of maths was taken out with bruised ribs. Listen out too for the witty comments by the compère.

Even if you don't feel like competing, come along and spectate. The start has always been a great sight — all runners simultaneously hustling for that first question. It can also be really funny seeing friends of yours being really pumped up after solving questions, or really flustered while attempting them! If that's not enough there's a spectator competition

Continued page 10

Paradox August centrefold: The four colour theorem

The four colour theorem states that any map drawn on the plane can be coloured using only four colours so that no two regions sharing a border have the same colour.

The theorem was thought of by a British university student, Francis Guthrie, while he was colouring a map of England in 1852 (possibly during a boring mathematics lecture), but he could find no way to prove that it was true. It was passed on by his brother to Augustus de Morgan, who asked a number of his colleagues if they could find a proof, but none could.

The theorem was published in the *Proceedings of the London Mathematical Society* in 1878 in an article by Cayley, which inspired several mathematicians to try and prove it. A few submitted their attempts, but they were all later shown to be incorrect.

Herman Minkowski once told his students that the four colour theorem had not been proved because only "third-rate mathematicians" had attempted to do so. After a long attempt to prove it himself, he finally admitted, 'Heaven is angered by my arrogance. My proof is also defective.'

After nearly 100 years of work related to the theorem performed by various mathematicians, it was finally proven by Kenneth Appel and Wolfgang Haken in 1976. They relied on a computer to show that nearly 2000 special maps all had a particular property that allowed their proof to work.

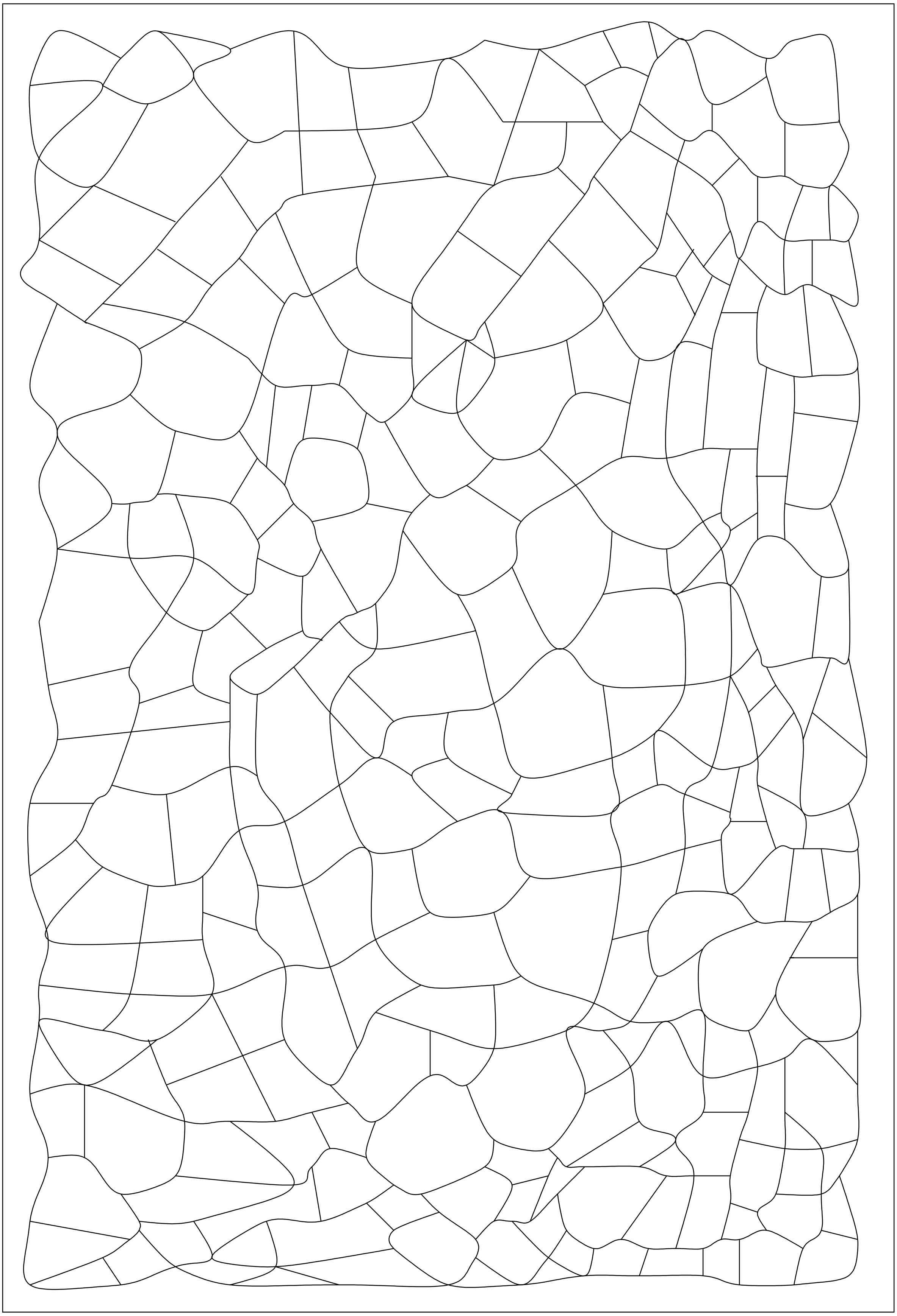
On the opposite page is a map which, by the four colour theorem, can be coloured using only four colours. We would like to confirm that this is indeed the case, and as such, a prize of $10 will be awarded to the person who submits the best colouring of the map. To enter, simply fill in the following details, and deposit your coloured centrefold in the MUMS pigeon-hole. Entries will be judged on correctness, artistic merit and originality. Winners will be announced in the next edition.

Name: ______________________________

Email address: ______________________________

RESULTS OF LAST ISSUE'S CENTREFOLD

In the last issue, a decimal expansion of π with four intentional errors was printed, and our readers' task was to spot and report these errors. Congratulations to Malcolm Davey and James Ong who have been awarded $8.10 for finding all four errors (and an unintentional missing line) within hours of the release of that issue, and thereby showing conclusively that honours students have too much spare time. Thanks to everyone else who submitted claims.



From page 7

as well — questions already attempted by all the teams are put up on an overhead and spectators who solve them win chocolates for their efforts. For those who prefer to just sit back and watch, you'll find that the compère hurls sweets into the crowd at regular intervals.

SOME TIPS

The following tips offer some ways to improve your performance and also enjoyment in the Maths Olympics (to a level even higher than what it already would be!).

- *Enter early.* You won't enjoy it as much if you're not competing! Due to the finite size of the lecture theatre, only about 28 teams can enter. In some years, almost 50 teams have submitted entries, only for many to be turned away.
- *Bring along lots of pens and paper.* You'll need it!
- Have a look at sample questions to get a feel for the type of questions posed. Questions tend to be similar in style to the ones in the Australian Mathematics Competition (formerly sponsored by Westpac). As a result, it's not necessary to have done any maths at uni, in fact this may even be a handicap!
- The runner can and should help. In years gone past, the runner was forbidden from helping with the question solving. However, due to blatant violations of this rule and the difficulty of enforcement, it was repealed. Keep working when the runner goes off, as the answer may have been wrong. One worthwhile tactic is to have the runner running back and forth guessing answers while the rest of the side tries to actually work out the answer.
- If you get stuck, read the question carefully, even if you have no idea, spend a minute as it may be easier than it first looks. Questions are usually designed to have a neat solution that doesn't require too much calculation. If you still have no idea, don't be afraid to abandon the question. One easy trap to fall into is to keep on investing more time in a question because it is psychologically difficult to give up on a question once you've spent a significant amount of time on it.
- *Guess.* Always look for the opportunity to guess the answer.

  For example, consider the following question:

$$\begin{aligned} f(1) &= 1 \\ f(2n) &= f(n) \\ f(2n+1) &= 1 - f(n) \end{aligned}$$

  What is $f(98)$?

At first you might think,

$$\begin{aligned} f(1) &= 1, \\ f(2) &= f(1) = 1, \\ f(3) &= 1 - f(1) = 0, \\ f(4) &= f(2) = 1, \\ f(5) &= 1 - f(2) = 0, \\ f(6) &= f(3) = 0. \end{aligned}$$

Hmmm, no obvious pattern.

Then you might say,

$$\begin{aligned} f(98) &= f(49) = 1 - f(24) = 1 - f(12) = 1 - f(6) = 1 - f(3) \\ &= 1 - (1 - f(1)) = 1 - (1 - 1) = 1. \end{aligned}$$

That wasn't too bad was it?

However, another possibly quicker approach would be to notice that the rules imply that $f(n)$ must be either 1, or $1 - 1 = 0$. Thus a quick way to do this question would be to *guess* 0 and when that fails, try 1.

Remember that there is no penalty for guessing, you can do it as many times as you like (provided the runner doesn't mind!). So let your legs do the thinking for you.

- A vital part of the Maths Olympics is the ingenious team names that teams come up with every year. Some team names from the past have included, "No Real Solutions", "The Return of the Lemma", "Pythagoras was Wrong!", "We Don't Count", "Meds on Prozac", and "Gottim — Yes". Start thinking now! For those who really want to get into the spirit of things, team mascots, music, costumes and make-up are encouraged. Bring along a cheer squad too.

- Lastly, don't take it too seriously. Have fun! Unless you're one of those people who can't stand coming 2nd (a link from the MUMS web page is devoted to them!), just sit back and enjoy the spectacle when the other half of your team is working on a problem. Yell and scream a bit! At the end, even if you receive no monetary reward, I guarantee you will have had one of your most entertaining hours at uni.

— Lawrence Ip

A word about the author:

Lawrence competed in the last seven Maths Olympics, first competing when he was in year 11 at school. He was a member of the winning team in 1992, 1994, 1995 and 1997. He will not be competing this year because he has gone to the U.S.A. to do postgraduate work. This is a good thing because he has decided that he is getting too old to compete successfully and would have retired anyway.

Solutions to last issue's problems

Only solutions to selected problems have been printed. If you are interested in any of those which have been omitted, please email us and we will gladly send the solution to you.

Problem 2: ($5) Let $A_1A_2 \ldots A_n$ be a regular n-gon inscribed in the unit circle. Show that

$$|A_1A_2| \cdot |A_1A_3| \cdot \cdots \cdot |A_1A_n| = n.$$

Solution: Let ζ be a primitive n^{th} root of unity (for instance, $\zeta = e^{2\pi i/n}$). Then the powers of ζ form a regular n-gon within the complex plane. Moreover,

$$z^n - 1 = (z-1)(z-\zeta)(z-\zeta^2)\ldots(z-\zeta^{n-1}).$$

Let A_1 be the point 1. Let P be any point represented by the complex number z. Then

$$|PA_2| \cdot |PA_3| \ldots |PA_n| = |(z-\zeta)||(z-\zeta^2)| \ldots |(z-\zeta^{n-1})|.$$

Since $|a||b| = |ab|$, this implies

$$|PA_2| \cdot |PA_3| \ldots |PA_n| = \left|\frac{z^n-1}{z-1}\right|.$$

As $P \to A_1$, $z \to 1$, and

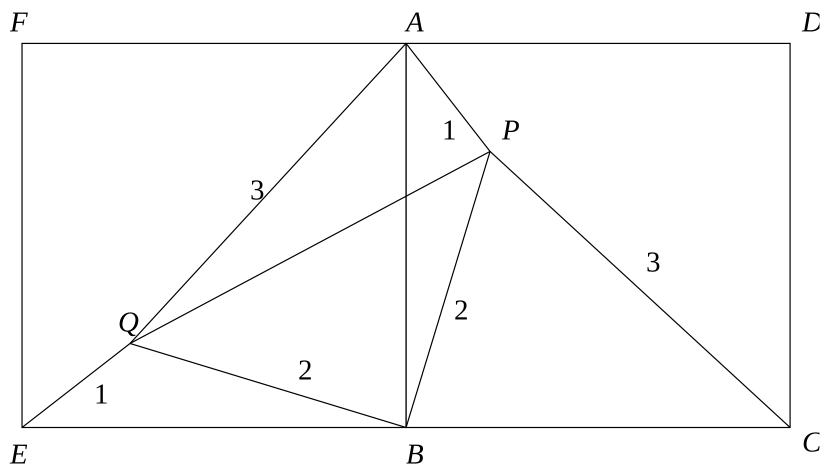
$$\lim_{z\to 1} \frac{z^n-1}{z-1} = n.$$

And we are done.

— Frank Calegari

Problem 3: ($10) P is a point inside a square $ABCD$ such that $PA = 1$, $PB = 2$, and $PC = 3$. How large is $\angle APB$?

Solution: Rotate the points A, P and D 90° anticlockwise about B to give the points E, Q and F respectively. Clearly, $\angle PBQ = 90°$ and $QA = PC = 3$.



By Pythagoras' theorem in $\triangle PBQ$,

$$PQ^2 = PB^2 + BQ^2 = 2^2 + 2^2 = 8 = QA^2 - AP^2.$$

Hence, by the converse of Pythagoras' theorem in $\triangle PQA$, we know that $\angle APQ = 90°$. However, as $\triangle PBQ$ is right angled and isosceles, $\angle BPQ = 45°$.

Therefore, $\angle APB = \angle APQ + \angle BPQ = 90° + 45° = 135°$

— Norman Do

Problem 4: (\$10) Using only sin, cos, tan, arcsin, arccos, and arctan keys on a calculator, show that starting from 0, pressing some finite sequence of buttons will yield any positive rational number a/b.

Solution: We prove the stronger claim that it is possible to construct $\sqrt{p/q}$ in this fashion. Firstly, let $\tan x = u$. Then

$$u^2 + 1 = \tan^2 x + 1 = \sec^2 x,$$

and hence

$$\cos x = \frac{\pm 1}{\sqrt{u^2+1}}.$$

Moreover, if u is positive, and we use the principal branch of arctan (as the calculator does), then

$$\cos(\arctan u) = \frac{1}{\sqrt{u^2+1}}$$

similarly,

$$\sin(\arctan u) = \frac{u}{\sqrt{u^2+1}}.$$

Now suppose that $\sin x = v$, with v positive. Then

$$\cos(\arcsin v) = \sqrt{1-v^2}$$

and hence

$$\tan(\arcsin v) = \frac{v}{\sqrt{1-v^2}}.$$

In particular, we find that

$$\tan(\arcsin(\cos(\arctan u))) = \frac{1/\sqrt{u^2+1}}{\sqrt{1-1/(u^2+1)}} = \frac{1}{u}.$$

We shall prove by induction that it is possible to construct $\sqrt{p/q}$ for all $p + q = n$ in a finite number of steps. Since $0/1 = 0$, the result is true for $n = 1$. In general, consider

$\sqrt{p/q}$. If $p = q$ then $1 = \cos(0)$. Otherwise, $p > q$, or $q > p$. Since we can always send $u \to 1/u$, we may assume that $q > p > 0$. Then

$$\sqrt{\frac{p}{q}} = \sin\left(\arctan\sqrt{\frac{p}{q-p}}\right).$$

Yet $\sqrt{p/(q-p)}$ is constructible by induction, since $p + q - p < p + q = n$. Hence $\sqrt{p/q}$ can also be constructed in a finite number of steps. To construct the rational number a/b, one merely constructs $\sqrt{a^2/b^2}$.

— Frank Calegari

Problem 5: ($10) For each positive integer n, determine a set of n distinct positive integers such that no subset of them adds up to a perfect square.
Solution: Consider the set of integers

$$\{1, 2, 3, 4 \ldots n\},$$

any subset of this set adds up to at most

$$1 + 2 + \ldots + n = n(n+1)/2.$$

Let p be a prime bigger than $n(n+1)/2$. Then the set

$$\{p, 2p, 3p, 4p \ldots np\}$$

satisfies the required property. For clearly any subset of this set sums to a number divisible by p, and yet not divisible by p^2. Such a number cannot be square.

— Frank Calegari

Problem 6: ($10) Find the smallest integer $n > 4$ such that there exists a set of n people such that any two acquainted people have no common acquaintance and any pair of unacquainted people have exactly two common acquaintances. (Acquaintance is a symmetrical relation; if A knows B, then B knows A.)
Solution: Since either everyone is acquainted to someone, or is an acquaintance of an acquaintance, then everyone is acquainted to someone, assuming $n > 1$.

Take a person. Call this the primary person. Suppose that he/she is acquainted to p people. Call these the secondary people.

- These p people can't be acquainted, since no acquainted people can have acquaintances in common.
- If there are any other people, then they all must have two (no more) acquaintances in common with the first person. Call these the tertiary people. So they must all be acquainted to a secondary person.

- The secondary people must have two acquaintances in common with every other secondary person ($p > 1$ and $n > 2$). They all already have the primary person in common. So this means that they all must have one other acquaintance in common. Hence each pair of secondary people have one tertiary acquaintance in common. We can count the number of tertiary people from this. These are the only secondary acquaintances that a tertiary person may have.

- This means that the number of people is dependent on the number of acquaintances the first person has.

We have 1 primary person, p secondary people, and $p(p-1)/2$ tertiary people, since secondary people are acquainted with $p-1$ tertiary people, there are p secondary people, and we need to divide by two so that we don't count them twice. So

$$n = \frac{(p^2 + p + 2)}{2}.$$

For any n there is only one positive p.

Since our choice of person was arbitrary, then all people must have the same number of acquaintances, or else the conditions can't be satisfied. This is consistent with our secondary people.

If there are n people with p acquaintances each, then the number of "acquaintanceships" must be an integer, hence $np/2$ is an integer. This further narrows down the field, e.g. $p = 3 \Leftrightarrow n = 7$, but $(3)(7)/2$ isn't an integer so there is no solution for $n = 7$.

We can count the acquaintanceships between the tertiary people in the group if we define one person as the primary person.

- The primary person has p.

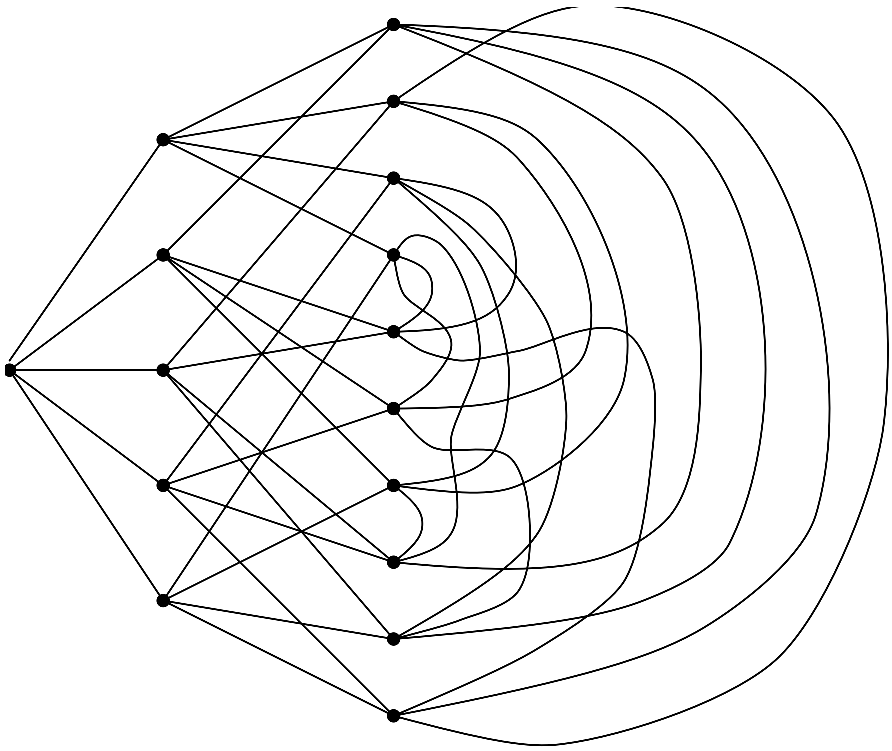
- The secondary people have $p-1$ each with tertiary people and there are p of them so we have $p(p-1)$.

- Each tertiary person has two secondary acquaintances. There are $p(p-1)/2$ of them, so they must each have $p-2$ aquaintances who are tertiary people. So there are $(p(p-1)/2) \times (p-2)/2$. (We divide by two so that we don't count things twice.)

- A particular tertiary person, is an acquaintance to two secondary people, who are in turn, acquainted to $p-2$ other tertiary people each. Now this person can't be acquainted to any of these tertiary people. So for the conditions to be satisfied, the person must be aquainted to other tertiary people who are aquainted to these people.

  The acquaintances must be aquainted to $p-3$ more people. There are $p-2$ of them. So they can be acquianted to up to $(p-3)(p-2)$ people. They together must be aquainted at at least $2(p-2)$ people. so $2(p-2) \leq (p-3)(p-2) \Rightarrow p^2 - 7p + 10 \geq 0 \Rightarrow p \leq 2$ or $p \geq 5$. This equation only applies to $p \geq 3$ since if this isn't the case, then the condition doesn't need to be satisfied.

This says that the smallest p is 5. And hence the smallest n is 16. A solution exists in this case, and is illustrated below.



— Malcolm Davey

Those crazy physicists

The following are the results of a contest for "theories" sponsored by *Omni* magazine.

Grand prize winner: When a cat is dropped, it always lands on its feet. And when toast is dropped, it always lands with the buttered side facing down. I propose to strap buttered toast to the back of a cat; the two will hover, spinning inches above the ground. With a giant buttered cat array, a high-speed monorail could easily link New York with Chicago.

1st runner-up: If an infinite number of rednecks riding in an infinite number of pickup trucks fire an infinite number of shotgun rounds at an infinite number of highway signs, they will eventually produce all the world's great literary works in Braille.

2nd runner-up: Why yawning is contagious: you yawn to equalize the pressure on your eardrums. This pressure change outside your eardrums unbalances other people's ear pressures, so they must yawn to even it out.

3rd runner-up: Communist China is technologically underdeveloped because they have no alphabet and therefore cannot use acronyms to communicate ideas at a faster rate.

4th runner-up: The earth may spin faster on its axis due to deforestation. Just as a figure skater's rate of spin increases when the arms are brought in close to the body, the cutting of tall trees may cause our planet to spin dangerously fast.

Honourable mention: The quantity of consonants in the English language is constant. If omitted in one place, they turn up in another. When a Bostonian "pahks" his "cah," the lost r's migrate southwest, causing a Texan to "warsh" his car and invest in "erl wells."

Working space

You can't get too much of a good thing.

Paradox

Issue 2, 1998

THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY

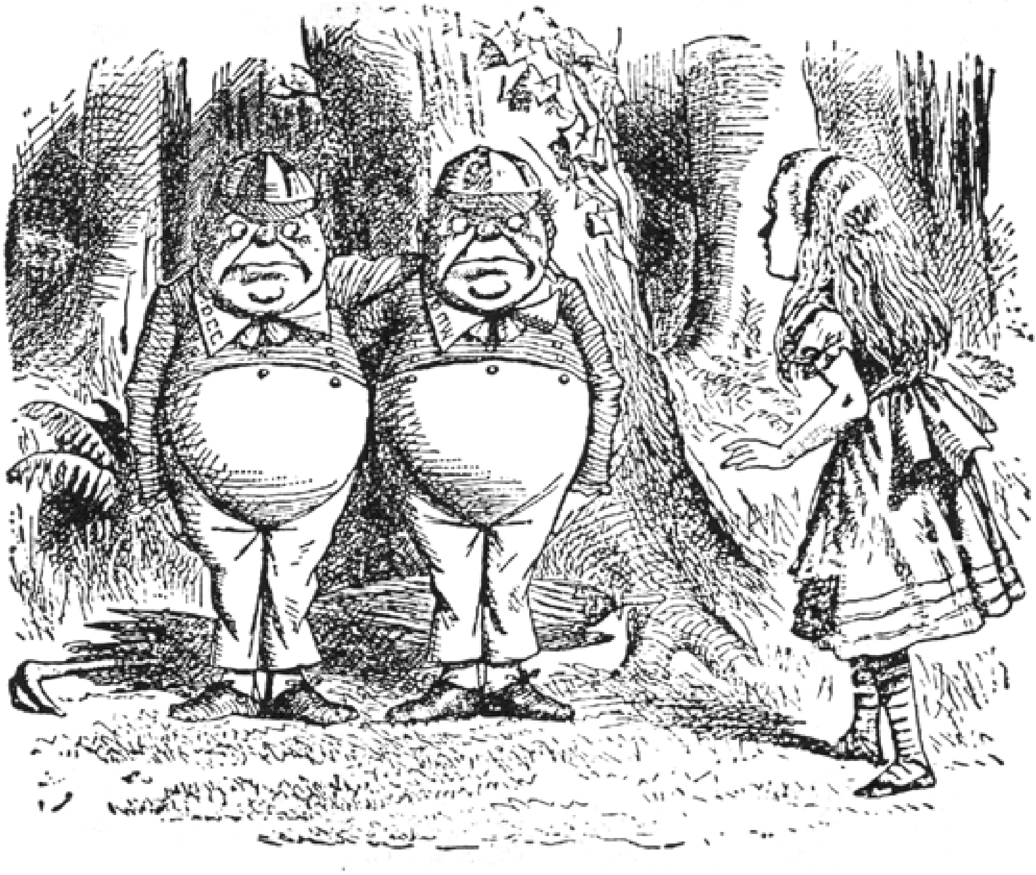


Illustration from Through the Looking-Glass, Sir John Tenniel.