

PARADOX

Welcome to the second and final edition of *Paradox* for 1997. There are some more problems and puzzles to keep you busy over the summer holidays, along with more mathematical humour. Those who are still recovering from the Maths Olympics may like to turn to the team rankings on page three. We have even included an article about a paradox.

Readers are invited to make contributions or comments for next year, which can be e-mailed to us at paradox@ms.unimelb.edu.au.

Thanks to all the people who helped with this edition, especially Chaitanya Rao, Stephan Tillmann, Adam Cagliarini and Lawrence Ip.

Vanessa Teague, *Paradox* editor.

There are three kinds of people in the world: those who can count and those who can't.

PROBLEMS, PROBLEMS, PROBLEMS

Here are a few challenging problems in case you miss maths during summer!

1. In triangle ABC , $AB = AC$ and $\angle BAC = 20^\circ$. P is a point on AC such that $AP = BC$. Find $\angle PBC$.
2. Find the maximal integer M with non-zero last digit (in its decimal representation) such that, after crossing out one of its digits (not the first one) we can get an integer that divides M .
3. Each of the 450 members of a parliament slaps the face of exactly one of his or her colleagues. Prove that after that they can choose a committee consisting of 150 members, none of whom has been slapped in the face by any other member of the committee.
4. Let S be a subset of $1, 2, 3, \dots, 1989$ such that no two members of S differ by 4 or 7. What is the largest number of elements that S can have?

SOME ACTUAL HEADLINES(!)

These have nothing at all in common with mathematics (except that they caused great amusement among the MUMS committee), but we decided to include them anyway.

- Grandmother of eight makes hole in one
- Police begin campaign to run down jaywalkers
- Two convicts evade noose, jury hung
- Squad helps dog bite victim
- Autos killing 110 a day, let's resolve to do better

- If strike isn't settled quickly it may last a while
- War dims hope for peace
- Smokers are productive, but death cuts efficiency
- Cold wave linked to temperatures
- Something went wrong in jet crash, experts say

DARTH VADER STRIKES DURING MATHS OLYMPICS!

In the year of the re-release of the Star Wars trilogy, the '94 and '95 winners, "The Empire Strikes Back", struck back, complete with a Darth Vader helmet and lightsabre.

There were many more entries than available places this year. With only 28 places, over 10 teams were left disappointed. Get in early next year!

Scores follow below. The first number is the aggregate score, the second number is the number of the last question that was solved. In the event of equal scores, the second number was used to resolve ties. If this number was the same, the previous question solved was used, and so on. e.g. 100/16 indicates that the team scored 100 points and that the last question they solved was question 16.

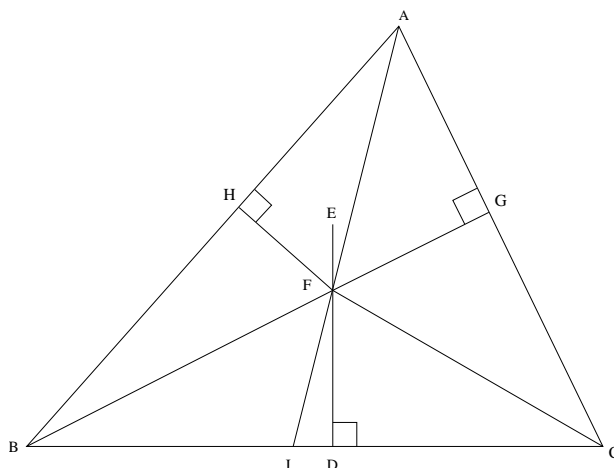
(1)	The Empire Strikes Back	(past IMO team members)	100/16
(2)	Been There, Done That	(Ex-Melbourne Grammar '96)	90/18
(3)	3 Lawyers, an Engineer and a Pom	(Eng/Law/Arts)	90/18
(4)	Polytopes	(Pure Maths Lecturers)	90/15
(5)	Meds on Prozac	(Med)	90/14
(6)	Liquid Helium	(Maths/Eng/Optom/Arts)	75/15
(7)	Franciscan Monks	(Eclectic)	75/14
(8)	Division by Zero	(Sci/Eng)	70/18
(9)	MGS	(Melbourne Grammar)	65/15
(10)	Black Tigers	(Eng)	65/14
(11)	Scotch College		65/14
(12)	Fat Chance	(Stats Lecturers)	65/13
(13)	Red Pen Specials	(Tutors)	65/13
(14)	Whatever!	(Med/Dental/Eng)	55/13
(15)	Firemaster	(Applied Maths Lecturers)	55/13
(16)	Just Guess	(Sci/Eng)	45/14
(17)	Linear Progression	(Sci/Eng/Maths)	45/13
(18)	Newton's Nymphos	(Sci)	45/12
(19)	Drunk and Disorderly	(1st Years)	40/13
(20)	MACROB	(MacRobertson Girls' High School)	40/11

(21)	Virtual Knowledge	(Sci/Eng)	35/11
(22)	University High School		35/7
(23)	The Golden Ratio	(122 students)	30/11
(24)	Gottim–Yes	(Monash Sci)	30/11
(25)	East Don Yum Cha	(Eng/Sci)	25/12
(26)	PLC	(Presbyterian Ladies' College)	25/8
(27)	Cross Products	(MUPHAS Maths 121)	20/6
	MLC	(Methodist Ladies' College)	20/6

Many thanks to Frank Calegari, who set the questions, and Dugal Ure and Adam Cagliarini, who did most of the organisation. Comments and suggestions can be sent to Lawrence Ip at lip@ms.unimelb.edu.au.

ALL TRIANGLES ARE ISOSCELES!

No prizes for guessing that there's a flaw in this proof somewhere. Of course, since you know that it exists, you won't have any trouble finding it... or will you?



Let $\triangle ABC$ be any triangle. Bisect BC at D , and from D draw DE at right angles to BC . Bisect the angle $\angle BAC$, and let the bisector meet BC at J .

1. If AJ does not meet DE , they are parallel. Therefore, $\triangle AJB$ is congruent to $\triangle AJC$, since they have side AJ in common and $\angle AJB = 90^\circ$, $\angle JAB$ are equal to $\angle AJC = 90^\circ$, $\angle JAC$ respectively. Therefore $AB = AC$, i.e., $\triangle ABC$ is isosceles.
2. If the bisector meets DE , let them meet at F . Draw FB , FC , and from F draw FG , FH , at right angles to AC , AB respectively.

Then the triangles $\triangle AFG$, $\triangle AFH$ are congruent, because they have the side AF common, and the angles $\angle FAG$, $\angle AGF$ are equal to $\angle FAH$, $\angle AHF$ respectively. Therefore $AH = AG$ and $FH = FG$.

Again, the triangles $\triangle BDF$ and $\triangle CDF$ are congruent, because $BD = DC$, DF is common, and the angles at D are equal. Therefore, $FB = FC$.

Again, the triangles $\triangle FHB$, $\triangle FGC$ are right-angled. Therefore, $FB^2 = FH^2 + HB^2$ and $FC^2 = FG^2 + GC^2$. But $FB = FC$, and $FH = FG$. Therefore, the square on HB is equal to the square on GC . Therefore, $HB = GC$. Also, AH has been proved to be equal to AG . Therefore, $AB = AC$, i.e., $\triangle ABC$ is isosceles.

Therefore, all triangles are isosceles, Q.E.D.

Also, by a simple corollary, all triangles are equilateral (since we could similarly have shown that $AB = BC$).

This puzzle is taken from "Rediscovered Lewis Carroll Puzzles", edited by Edward Wakeling, Dover publications, 1995.

HOW TO AMUSE YOUR FRIENDS AT MATHS PARTIES

A physicist, a biologist and a mathematician were having a leisurely cup of coffee in Lygon Street, idly watching an empty café across the street. After a while, a couple walked in to this café. The scientists talked about other things, until they were surprised to see three people walking out of said café.

Both the physicist and the biologist found these observations hard to understand. The biologist supposed that they must have reproduced, whilst the physicist asserted that there had been an experimental error in one of the observations. The mathematician had no problems with explaining the data: "If one more person were now to walk into the café, there would then be no people in the café."

A physicist, a mathematician and a businessman were told to build a fence around some cows, using as little fence wire as possible.

The physicist went first, and plotted the position of each cow in his lab notebook, drew a minimal convex polygon, and constructed a fence of that shape.

The businessman was horrified, and, with an assistant, herded the cows close together, wound the fence wire around them, and pulled tightly until the cows were bunched together and the wire was cutting into their skin.

"Get shorter than that!" he sneered at the mathematician.

Smiling, the mathematician loosely wound a strand of the fence wire around her waist, and said, "I am on the outside."

THE UNREALITY OF MOTION: ZENO'S PARADOX

by Tillus

It is a sunny and peaceful day in ancient Elea. We are sitting on the steps of a temple, when Zeno comes along. Yesterday, we were chatting about properties of lines and points (topics of conversation are not so easy to find in these days without a bad program on the TV), and while considering geometric diagrams we got the impression that points are extensionless and indivisible, while lines are continuous linear sets of these points (i.e. something like the real line). Zeno seems to be puzzled by this fact, but its analytical truth cannot be denied. Pointing to our juggling balls flying through the air, he now starts the conversation.

Z: So, you really think that these balls are moving? (Moving as in traversing space in time. As one believes when juggling...)

W: Sure!

Z: Now these balls, going from here to there, first they have to reach halfway, then cover half of the remaining distance, then half of the rest and so forth. There is no last moment when they arrive at the other side, so they'll never arrive in finite time. - And don't tell me anything about convergence, I want to know how and when they arrive, i.e. how and when the repetitive division ends!

W: It's easier if you don't accept our sophistication: we see that they arrive!

Z: Observation... You cannot even start, for the same reason.

W: Hmm.

Z: O.K., even if you could get going, you would never be able to overtake someone, since you always have to go to the place from which they already started.

W: But this also happens!

Z: Look. If one could overtake, which would appear faster: an arrow or Achilles?

W: Achilles is fast, but the arrow is even faster. (So we expect the arrow to overtake Achilles).

Z: See, the arrow is always in a present moment during his flight. But in a moment, there is no motion to observe (since it is just an extensionless point on the timeline). This means that at each moment of its flight, the arrow is at rest, and between the moments there is no time when the arrow could move. Therefore, the arrow is always at rest. Isn't that funny, the arrow appears to be faster and is always at rest.

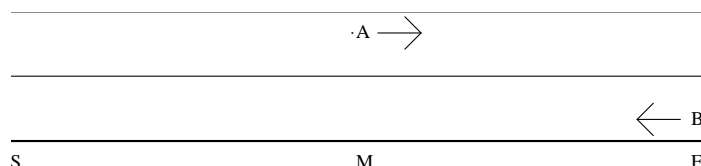
W: But we DO see it moving!

Z: Then the arrow has to be able, for some reason, to travel along a continuous linear set, as in point by point.

W: Why not?

Z: Let's think about it. Imagine a stadium, where one runner is coming from the end, the other from the middle and both are moving with equal speed in opposite directions.

[Zeno draws a diagram in the sand in front of us, and places two balls as runners on the track.]



Z: Now, what do we expect when they start (or continue) moving?

W: A takes half the time to get to the end that it takes B to reach the start.

Z: Sure. The arrow moves along a linear set of time points, and if this happens, the runners do the same in space (which seems ridiculous to me, since they don't even start in the first place). But the amount of points between middle and end and end and start is the same! A and B run with the same speed, so that they traverse the same amount of points in the same time. But now it takes them the same time to get to their respective goals. This means, considering the conclusion that you pointed out earlier, that half of the time is equal to the whole time! Beautiful! A nice concept of motion is this! So there you are: four nonos only because of one assumption based on observation!

LAST ISSUE'S PROBLEMS FOR PRIZEMONEY

Question 1 Solution by John Dethridge (Special mention also goes to Chin Chuan Ong)

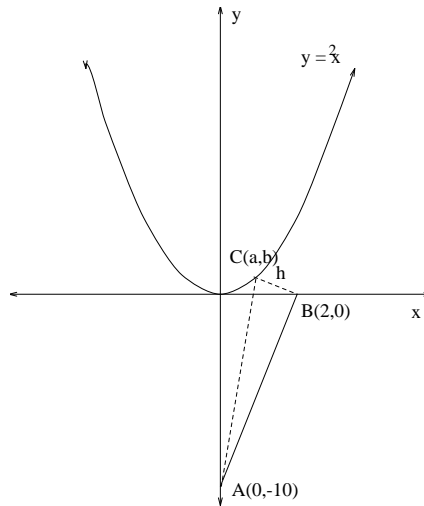
(\$10) Points A and B have Cartesian coordinates $(0, -10)$ and $(2, 0)$ respectively. Find the point C on the parabola $y = x^2$ which minimises the area of $\triangle ABC$ and determine this area.

Solution: Let h be the distance from the line AB to the point C . Then

$$|\triangle ABC| = \frac{1}{2}|AB|h = \frac{1}{2}\sqrt{2^2 + 10^2}h = \sqrt{26}h$$

Let $C = (a, b)$. The line AB has equation $5x - y - 10 = 0$, and in general the perpendicular distance from the point (a, b) to a line $Fx + Ey + D = 0$ is $\frac{|Fa + Eb + D|}{\sqrt{F^2 + E^2}}$, so

$$\begin{aligned} h &= \frac{|5a - b - 10|}{\sqrt{5^2 + 1^2}} \\ &= \frac{|5a - b - 10|}{\sqrt{26}} \end{aligned}$$



$$\begin{aligned}
 &= \frac{|5a - a^2 - 10|}{\sqrt{26}} \quad \text{since } C \text{ is on the parabola } y = x^2. \\
 &= \frac{|a^2 - 5a + 10|}{\sqrt{26}} \\
 &= \frac{(a - \frac{5}{2})^2 + \frac{15}{4}}{\sqrt{26}}
 \end{aligned}$$

Therefore, the minimum value of h and the minimum area of $\triangle ABC$, occur when $c = (\frac{5}{2}, \frac{25}{4})$, and the corresponding area is $\frac{15}{4}$.

Question 2 Solution by Jeremy Glick.

(\$15) How many six digit numbers (base 10 of course) contain each of their digits more than once? For example, 234342 and 334343 are counted but 232331 is not.

Solution: The problem can be split into several groups:

- Three groups of two identical numbers $\frac{6!}{2!^3} \times \frac{10!}{3!7!} = 10800$
- Two groups of three numbers $\frac{6!}{3!^2} \times \frac{10!}{2!8!} = 900$
- One group of two numbers and one of four $\frac{6!}{4!2!} \times \frac{10!}{8!} = 1350$
- One group of six numbers $\frac{6!}{6!} \times \frac{10!}{9!} = 10$

Therefore, the total is $10800 + 900 + 1350 + 10 = 13060$. If we assume that a six digit number cannot start with 0, then multiply the answer by $9/10$ to get 11754.

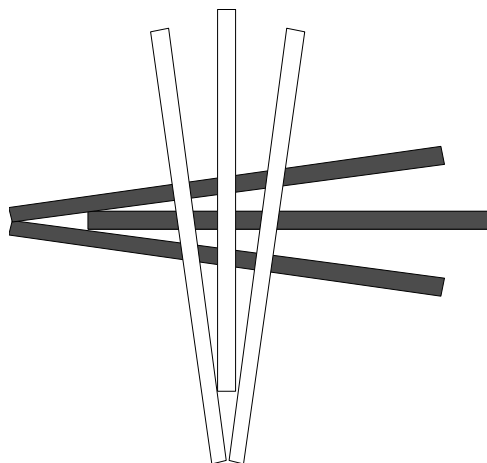
Q: Why do computer scientists confuse Christmas and Halloween?

A: Because Oct 31 = Dec 25.

Question 3 Solution by Nicholas Fone.

(\$15) How can one arrange six matchsticks so that each one is touching every other? The arrangement must be constructible and reasonably stable in reality.

Solution:



OUR FAVOURITE WEBSITES

MUMS homepage. This also contains links to all of the sites listed below.

<http://www.maths.unimelb.edu.au/~lip/mums/links.html>

“Mathematics: Ancient Science and Its Modern Fates”. Vatican library exhibition

<http://sunsite.unc.edu/expo/vatican.exhibit/exhibit/d-mathematics/Mathematics.html>

Interactive experiments with fractals

<http://reality.sgi.com/employees/rck/hydra>

CSIRO Division of Mathematics and Statistics

<http://www.dms.csiro.au/>

CSIRO research programs and projects

<http://www.csiro.au/csiro/csirores.htm>

About women in mathematics

<http://math.berkeley.edu/~nring/index.html>

<http://www.math.umd.edu/~wim/>

The Australian Bureau of Meteorology

<http://www.bom.gov.au>

Sites with links to lots of mathematical sites

<http://www.mth.uea.ac.uk/~h720/headings/maths.html>

<http://www.rose-hulman.edu/~swickape/math.html>