PARADOX

Welcome to the second issue of *Paradox* for 1996. I am still optimistic that there will be enough time for one more issue before the year's end. This time round we have our regular *Paradox* competition (with greater rewards by popular demand!), information about the much-anticipated Maths Olympics, more jokes, brainteasers, a short story by Lewis Carroll and more.

As usual, the *Paradox* box (near the southern entrance of the Richard Berry building) is there for solutions to problems, comments, or suggestions. Alternatively you can send email to `paradox@maths.mu.oz.au` . Many thanks to people who gave positive feedback about the last issue and to those who have helped with this one.

Chaitanya Rao, *Paradox* Editor.

MATHS OLYMPICS!

For those who haven't heard about it, the Maths Olympics is the largest annual event that the Melbourne University Mathematical Society conducts. It involves forty-five minutes of fun and chaos as thirty teams of five attempt to solve as many problems as they can in the restricted time. The contest is a mathematical relay – to receive the next problem, a team member has to run down to the front of the lecture theatre after the team has correctly solved (or given up on) the previous one. There will also be a free competition for spectators – tasty treats will be on offer! Below are the main details:

When?	Thursday 26th September, 1996, 1pm to 2:15pm.
Where?	Theatre A, Richard Berry Building (Maths).
Prizes (per team)	1st : $200
	2nd : $150
	3rd : $100
	4th : $ 50
Team Entry Fee	$5 per team of five (payable on the day).

Entry forms are available from the Maths office and further details (including rules and sample questions) can be found in CG53. But be quick as numbers are limited!

PARADOX COMPETITION

A number of solutions were received for the *Paradox* competition of the previous issue, but unfortunately no winners could be found for the first two problems. So this time round, we have included these problems again (because they are interesting and worth trying!), plus two more. Solutions can once again be placed in the *Paradox* box. Note the more attractive rewards for the best solutions!

<u>Deadline for submissions:</u> Friday 20th September

The old problems again:

Problem 1 ($15): Given 13 distinct real numbers, show there exist two, say x and y, satisfying:

$$\frac{xy+1}{x-y} > \frac{7}{2}$$

Hint: It may help to reciprocate both sides first.

Problem 2 (\$10): The numbers $0, 1, 0, 1, 0$ and 0 are written clockwise around the circumference of a circle. It is possible to make "moves" in each of which we add 1 to each number of a certain pair of adjacent numbers. Is it possible by means of finitely many such moves to make all the numbers on the circumference equal?

And here are the new problems:

Problem 3 (\$15): A student is shocked to find that he only has 37 days to prepare for the upcoming Maths Olympics. From past experience he knows that he will require no more than 60 hours of "training" (problem-solving, running up and down stairs, who knows?). He also wishes to train at least 1 hour per day. Show that no matter how he organises his schedule, there is a succession of days during which he trains exactly 13 hours (here we are assuming that he is training for a whole number of hours per day).

By the way, **nobody** would be doing this in reality, would they?!

Problem 4 (\$10): Evaluate the sum

$$\sum_{n=1}^{1996} \frac{n^2+n+1}{n!}$$

<u>Solution to problem 3 from the last issue:</u>

Let $ABCD$ be a tetrahedron with edge lengths $AB = 41, AC = 7, AD = 18, BC = 36, BD = 27$ and $CD = 13$. Find the distance between the mid-points of AB and CD.

(Winner: Dave Coulson – the following is only a slight adaptation of the solution he submitted):

The solution makes use of Apollonius' theorem which states:

If $\triangle ABC$ is a triangle and M is the mid-point of BC then

$$\begin{aligned} AB^2 + AC^2 &= 2AM^2 + 2MC^2 \\ \Rightarrow AM^2 &= \frac{1}{2}(AB^2 + AC^2 - \frac{BC^2}{2}) \end{aligned}$$

(see Figure 1)

One way of proving this is by application of the cosine rule twice – on $\triangle ABM$ and $\triangle ACM$ – then using that $\cos\angle AMB = -\cos\angle AMC$ and $BM = MC$.

Turning to Figure 2 (with X and Y the mid-points of AB and CD respectively) we apply the theorem to each of $\triangle ADB$ and $\triangle ABC$ to obtain:

$$\begin{aligned} DX^2 &= \tfrac{1}{2}(18^2 + 27^2 - \tfrac{41^2}{2}) &= 106\tfrac{1}{4} \\ CX^2 &= \tfrac{1}{2}(7^2 + 36^2 - \tfrac{41^2}{2}) &= 252\tfrac{1}{4} \end{aligned}$$

From $\triangle XDC$ we finally obtain

$$\begin{aligned} XY^2 &= \frac{1}{2}(106\frac{1}{4} + 252\frac{1}{4} - \frac{13^2}{2}) \\ &= 137 \end{aligned}$$

Figure 1

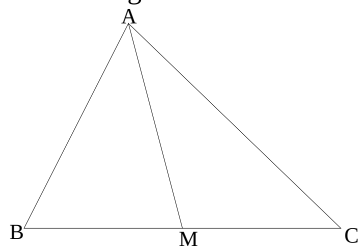
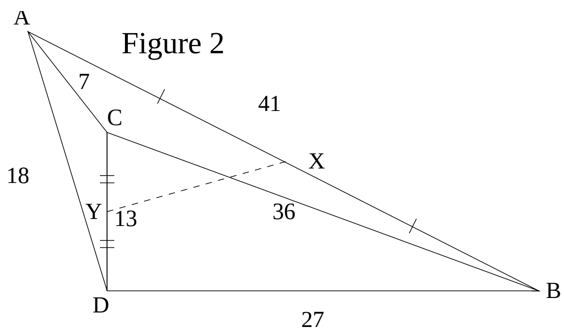


Figure 2



So the distance between the mid-points of AB and CD is $\sqrt{137}$. Note that this same answer can be reached by consideration of other triangles in the tetrahedron.

MORE PROBLEMS

Here are some more problems, not up for any money, but they can be regarded as a warm-up for the (slightly?) more challenging ones preceding!

1. Prove that at your next dinner party with six people, there will be either three people all of whom know each other, or three none of whom know each other. Is this true with five people at the party?

2. Show that if each of two integers is a sum of two perfect squares, then so is their product.

3. You are a participant on “Let’s Make a Deal” (remember that show?). The host shows you three closed doors. He tells you that two of the closed doors have a goat behind them and that one of the doors has a new car behind it. You pick one door, but before you open it, the host opens one of the two remaining doors and shows that it hides a goat. He then offers you a chance to switch doors with the remaining closed door. Is it to your advantage to do so?

4. Take the digits $1, 2, 3, 4, 5, 6, 7, 8, 9, 0$, and compose two fractions that, when added together, will be equal to 1. Each number is to be used once, and only once.

IMO NEWS

While our attention was focussed on the Olympics in Atlanta, another more cerebral contest was underway. The heat was just as intense in Mumbai (formerly Bombay), India, as 426 of the world's most talented high school students gathered for the 37th International Mathematical Olympiad. The contestants sat two gruelling $4\frac{1}{2}$ hour papers, with 3 questions on each paper.

The top half of the field receive medals, with gold, silver and bronze medals awarded in the ratio 1 : 2 : 3. Australia acquitted itself admirably, coming 23rd out of 75 nations. Two of the team members won silver medals and three others were each awarded a bronze medal.

Australia first competed in 1981, and since then has come in the top 25 countries each year. Next year's IMO is in Mar del Plata, Argentina.

Test your problem solving skills on question 4 of this year's IMO:

The positive integers a and b are such that the numbers $15a+16b$ and $16a-15b$ are both squares of positive integers. Find the least possible value that can be taken by the minimum of these two squares.

For more details about the Olympiad see `http://www.amt.canberra.edu.au/`.

Newcomb's Paradox

The following problem was posed by a Professor of Philosophy some years ago:

A 'being' put one thousand dollars in box A and either zero or one million dollars in box B, and presents you with two choices:

1. Open box B only.
2. Open both box A and B.

The being put money in box B only if it predicted you will choose option 1. The being put nothing in box B if it predicted you will do anything other than choose option 1 (including choosing option 2, flipping a coin, etc.).

Assuming that you have never known the being to be wrong in predicting your actions, which option should you choose to maximize the amount of money you get?

(Responses to this problem are most welcome for possible publication next time!)

hunting an elephant

MATHEMATICIANS hunt elephants by going to Africa, throwing out everything that is not an elephant, and catching one of whatever is left.

EXPERIENCED MATHEMATICIANS will attempt to prove the existence of at least one unique elephant before proceeding to step 1 as a subordinate exercise.

PROFESSORS OF MATHEMATICS will prove the existence of at least one unique elephant and then leave the detection and capture of an actual elephant as an exercise for their graduate students.

COMPUTER SCIENTISTS hunt elephants by exercising Algorithm A:

1. Go to Africa.
2. Start at the Cape of Good Hope.

3. Work northward in an orderly manner, traversing the continent alternately east and west.

4. During each traverse pass,

    a. catch each animal seen

    b. compare each animal caught to a known elephant

    c. stop when a match is detected

EXPERIENCED COMPUTER PROGRAMMERS modify Algorithm A by placing a known elephant in Cairo to ensure that the algorithm will terminate.

ASSEMBLY LANGUAGE PROGRAMMERS prefer to execute Algorithm A on their hands and knees.

ENGINEERS hunt elephants by going to Africa, catching grey animals at random, and stopping when any one of them weighs within plus or minus 15 percent of any previously observed elephant.

ECONOMISTS don't hunt elephants, but they believe that if elephants are paid enough, they will hunt themselves.

STATISTICIANS hunt the first animal they see N times and call it an elephant.

OPERATIONS RESEARCH CONSULTANTS can also measure the correlation of hat size and bullet colour to efficiency of elephant-hunting strategies, if someone else will only identify the elephants.

POLITICIANS don't hunt elephants, but they will share the elephants you catch with the people who voted for them.

SOFTWARE SALES PEOPLE ship the first thing they catch and write up an invoice for an elephant.

HARDWARE SALES PEOPLE catch rabbits, paint them grey, and sell them as desktop elephants.

WHAT THE TORTOISE SAID TO ACHILLES - BY LEWIS CARROLL

Lewis Carroll is probably best known for having written "Alice in Wonderland" and "Through the Looking Glass." His fascination with mathematics, in particular logical problems and paradoxes, motivated these stories and the one below. Below is an excerpt from his story of Achilles and the Tortoise. The philosopher Zeno had challenged Achilles and the Tortoise to a race to demonstrate his belief that it would be impossible for the Tortoise to be overtaken by Achilles if given a head start, as an infinite series of distances would need to be covered. This part of the story is set during that race and demonstrates the wit of Lewis Carroll.

Achilles had overtaken the Tortoise, and had seated himself comfortably on its back.

"So you've got to the end of our race-course?" said the Tortoise. "Even though it does consist of an infinite series of distances? I thought some wise acre or other had proved that the thing couldn't be done?"

"It *can* be done," said Achilles. "It *has* been done! *Solvitur ambulando.* You see the distances were constantly *diminishing*: and so—"

"But if they had been constantly *increasing*?" the Tortoise interrupted. "How then?"

"Then I shouldn't be *here*," Achilles modestly replied; "and *you* would have got several times round the world, by this time!"

"You flatter me—*flatten*, I mean," said the Tortoise; "for you *are* a heavy weight, and *no* mistake! Well now, would you like to hear of a race-course, that most people fancy they can get to the end of in two or three steps, while it *really* consists of an infinite number of distances, each one longer than the previous one?"

"Very much indeed!" said the Grecian warrior, as he drew from his helmet (few Grecian warriors possessed *pockets* in those days) an enormous note-book and a pencil. "Proceed! And speak *slowly*, please! *Short-hand* isn't invented yet!"

"That beautiful First Proposition of Euclid!" the Tortoise murmured dreamily. "You admire Euclid?"

"Passionately! So far, at least, as one *can* admire a treatise that won't be published for some centuries to come!"

"Well, now, let's take a little bit of the argument in that First Proposition— just *two* steps, and the conclusion drawn from them. Kindly enter them in your note-book. And, in order to refer to them conveniently, let's call them A, B, and Z:

(A) Things that are equal to the same are equal to each other.

(B) The two sides of this Triangle are things that are equal to the same.

(Z) The two sides of this Triangle are equal to each other.

"Readers of Euclid will grant, I suppose, that Z follows logically from A and B, so that any one who accepts A and B as true, *must* accept Z as true?"

"Undoubtedly! The youngest child in a High School—as soon as High Schools are invented, which will not be until some two thousand years later—will grant *that*."

"And if some reader had *not* yet accepted A and B as true, he might still accept the *Sequence* as a *valid* one, I suppose?"

"I doubt such a reader might exist. He might say 'I accept as true the Hypothetical Proposition that, if A and B be true, Z must be true; but I *don't* accept A and B as true.' Such a reader would do wisely in abandoning Euclid, and taking to football."

"And might there not *also* be some reader who would say 'I accept A and B as true, but I *don't* accept the Hypothetical'?"

"Certainly there might. *He*, also, had better take to football."

"And *neither* of these readers," the Tortoise continued, "is *as yet* under any logical necessity to accept Z as true?"

"Quite so," Achilles assented.

"Well, now, I want you to consider *me* as a reader of the *second* kind, and to force me, logically, to accept Z as true."

"A tortoise playing football would be—" Achilles was beginning.

"—an anomaly, of course," the Tortoise hastily interrupted. "Don't wander from the point. Let's have Z first, and football afterwards!"

"I'm to force you to accept Z, am I?" Achilles said musingly. "And your present position is that you accept A and B, but you *don't* accept the Hypothetical—"

"Let's call it C," said the Tortoise.

"—but you don't accept:

(C) If A and B are true, Z must be true."

"That is my present position," said the Tortoise.

"Then I must ask you to accept C."

"I'll do so," said the Tortoise, "as soon as you've entered it in that note-book of yours. What else have you got in it?"

"Only a few memoranda," said Achilles, nervously fluttering the leaves: "a few memoranda of—of the battles in which I have distinguished myself!"

"Plenty of blank leaves, I see!" the Tortoise clearly remarked. "We shall need them *all*!" (Achilles shuddered.) "Now write as I dictate:

(A) Things that are equal to the same are equal to each other.

(B) The two sides of this Triangle are things that are equal to the same.

(C) If A and B are true, Z must be true.

(Z) The two sides of this Triangle are equal to each other."

"You should call it D, not Z," said Achilles. "It comes right *next* to the other three. If you accept A and B and C, you *must* accept Z."

"And why *must I*?"

"Because it follows *logically* from them. If A and B and C are true, Z *must* be true. You don't dispute *that*, I imagine?"

"If A and B and C are true, Z *must* be true," the Tortoise thoughtfully repeated. "That's *another* Hypothetical, isn't it? And, if I failed to see its truth, I might accept A and B and C, and *still* not accept Z, mightn't I?"

"You might," the candid hero admitted, "though such obtuseness would certainly be phenomenal. Still, the event is *possible*. So I must ask you to grant one more Hypothetical."

"Very good. I'm quite willing to grant it, as soon as you've written it down. We will call it

(D) If A and B and C are true, Z must be true.

"Have you entered that in your note-book?"

"I *have*!" Achilles joyfully exclaimed, as he ran the pencil into its sheath. "And at last we've got to the end of this ideal race-course! Now that you accept A and B and C and D, *of course* you accept Z."

"Do I?" said the Tortoise innocently. "Let's make that quite clear. I accept A and B and C and D. Suppose I *still* refuse to accept Z?"

"Then Logic would take you by the throat, and *force* you to do it!" Achilles triumphantly replied. "Logic would tell you, 'You can't help yourself. Now that you've accepted A and B and C and D, you *must* accept Z!' So you've no choice, you see."

"Whatever *Logic* is good enough to tell me is worth *writing down*," said the Tortoise. "So enter it in your book, please. We will call it

(E) If A and B and C and D are true, Z must be true.

Until I've granted *that*, of course, I needn't grant Z. So it's quite a *necessary* step, you see?"

"I see," said Achilles, and there was a touch of sadness in his tone.

Here the narrator, having pressing business at the Bank, was obliged to leave the happy pair, and did not again pass the spot until some months afterwards. When he did so, Achilles was still seated on the back of the much-enduring Tortoise, and was writing in his notebook, which appeared to be nearly full. The Tortoise was saying "Have you got that last step written down? Unless I've lost count, that makes a thousand and one. There are several millions more to come. And *would* you mind, as a personal favour—considering what a lot of instruction this colloquy of ours will provide for the Logicians of the Nineteenth Century—*would* you mind adopting a pun that my cousin the Mock-Turtle will then make, and allowing yourself to be renamed Taught-Us?"

"As you please!" replied the weary warrior, in the hollow tones of depair, as he buried his face in his hands. "Provided that *you*, for *your* part, will adopt a pun the Mock-Turtle never made, and allow yourself to be renamed A Kill-Ease!"