UMO Questions 2021

MUMS

1. Alice is playing a game with Xavier and Yasmin. Alice thinks of a pair of integers (x, y). Separately, she tells Xavier the value of x, and tells Yasmine the value of y.

   She then tells Xavier and Yasmine the possible values of (x, y) she could have thought of in this table.

$y \backslash x$	1	2	3	4	5	6
1	○	○				
2		○	○			
3			○	○		
4				○	○	
5					○	○

   The following conversation ensues:

   - Xavier: "I don't know what x and y are"
   - Yasmine: "I don't know what x and y are"
   - Xavier: "I don't know what x and y are"
   - Yasmine: "I don't know what x and y are"
   - Xavier: "I now know what x and y are"

   What is the value of y?

2. Evaluate

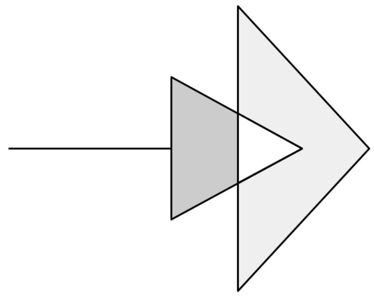
   $$\int_1^{e^{\frac{\pi}{4}}} \frac{1}{x^2} \cos(\log(x)) dx$$

   Give your answer as a simplified fraction a/b.

3. Let $\lfloor x \rfloor$ denote the largest integer less than or equal to some real number x. If n is a positive integer such that the sum

   $$\lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \lfloor\sqrt{3}\rfloor + \cdots + \lfloor\sqrt{n}\rfloor = P,$$

   where P is prime, then what is the largest possible value of P?



4. The arrow shown in the diagram is made from two overlapping triangles. The lighter shaded area comprises $\frac{13}{15}$ of the larger triangle and the darker area $\frac{4}{5}$ of the smaller triangle. What is the ratio of the shaded area of the smaller triangle to the shaded area of the larger? Express your answer as "a:b", where a and b are coprime positive integers.

5. Define
$$f(n) = \sum_{k=1}^{n} k \binom{n}{k} n^{n-k}$$
Find the sum of the distinct prime divisors of $f(50)$.

6. Consider 13 points placed on the circumference of a circle in such a way that when every possible pair of points among these 13 points is joined by a line segment, no three line segments intersect at a single point. Into how many regions do these line segments divide the circle?

7. Find the 9^{th} derivative of $\sin(2x)$ evaluated at $x = 0$.

8. Alice is playing a game with Xavier and Yasmin. Alice thinks of a pair of integers (x, y). Separately, she tells Xavier the value of x, and tells Yasmine the value of y.

   She then tells Xavier and Yasmine the possible values of (x, y) she could have thought of in this table.

$y \backslash x$	1	2	3	4	5	6	7	8	9
1					○				
2				○					
3			○				○		
4		○		○		○			
5	○				○				○
6				○					
7			○				○		
8						○			
9					○				
10									
11							○		

Xavier then truthfully proclaims: "I know that Yasmine does not know the value of x, even after Yasmine has heard this sentence".

What must be the value of y?

9. A set of positive integers has the properties that

   - Every member in the set, apart from 1, is divisible by at least one of 2, 3 or 5.
   - If the set contains $2n, 3n$ or $5n$ for some integer n, then it contains all three and n as well.

   The set contains between 300 and 400 numbers. Exactly how many does it contain?

10. At how many points in the real number line is the following expression not differentiable in x?

$$\left|\left|\left||x|-1\right|-\frac{1}{2}\right|-\frac{1}{4}\right|-\frac{1}{8}\right|$$

11. A triangle initially has a base length of 3cm and a height of 4cm. Then, the triangle spontaneously starts growing. After t seconds, the height grows at a rate of $4t$ cm/s and the base grows at $2t + 3t^2$ cm/s. What is the rate of change of the area of the triangle at $t = 2$? Give your answer in cm^2/s, and do not include units.

12. Let f be a function on the interval $[1, 5]$ satisfying $-1 \le f(x) \le 1$ for all x, and $\int_1^5 f(x)dx = 0$. If M is the maximum value of $\int_1^5 \frac{f(x)}{x}dx$, find the value of e^M. Give your answer as a simplified fraction a/b.

13. Let $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x - g(y)) = 1 - x - y$ for all pairs of real numbers (x, y).

    Find $f(2021) + g(2021)$.

14. Given $x > 1$, consider the following inequality.

$$\log_x(7x) + \log_7(x) - \log_7(e)\log_x(e) \ge 3$$

   The inequality has a solution of the form $[a, +\infty)$. Find the value of $\frac{a}{e}$.

15. Evaluate

$$\frac{1}{\pi}\int_0^\infty e^{-x}\frac{\sin(x)}{x}\,dx.$$

   Give your answer as a simplified fraction a/b.

16. Evaluate

$$\lim_{n\to\infty}\left(\int_0^2 x^n dx\right)^{\frac{1}{n}}.$$

17. Compute

$$\prod_{k=1}^{12}\sin\frac{k\pi}{25}.$$

   Give your answer as a simplified fraction a/b.

18. Miles finds a coin that comes up tails with probability $5/12$. Miles starts tossing the coin and only stops if he has flipped at least as many tails as heads. What is the probability that Miles keeps tossing the coin forever? Give your answer as a simplified fraction a/b.

19. Let ABC be a triangle whose area is 1. Let G be the centroid of ABC (the centroid is the intersection of the three medians, i.e. the lines joining each vertex to the midpoint of the opposite side). A line passing through G cuts the triangle ABC into two regions with areas A_1 and A_2. What is the largest possible value of $A_1 - A_2$? Give your answer as a simplified fraction a/b.

20. Consider the 11×11 matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Find $(A^{10})_{6,6}$.

21. Evaluate

$$\lim_{n\to\infty} \int_0^\pi \sqrt{1 + n^2(\sin(x)^{2n-2} - \sin(x)^{2n})}\, dx$$

Give your answer as a decimal number rounded to the nearest tenth.

22. Barry and Quang are playing a game. Barry thinks of a number y, satisfying $1 \le y \le M$ for some integer M. Three times in total, Quang is allowed to give Barry a number x such that $1 \le x \le 16$, and Barry must truthfully respond with the remainder of y divided by x. Given that Quang employs an optimal strategy, what is the largest possible M such that Quang is guaranteed to always be able to unambiguously determine the value of y?

23. Compute

$$\lim_{n\to\infty} \left[\int_0^2 (2021 + 2x - x^2)^n \, dx \right]^{\frac{1}{n}}.$$

24. Let V be the set of all positive integers less than 2021 that are coprime to 2021. Let U be a subset of V with the following properties:

- For the product of any two (possibly identical) elements in U, its remainder on division by 2021 is an element of U.
- For the product of all elements in U, its remainder on division by 2021 is 1.
- $|U| = 7$.

How many such subsets U exist?