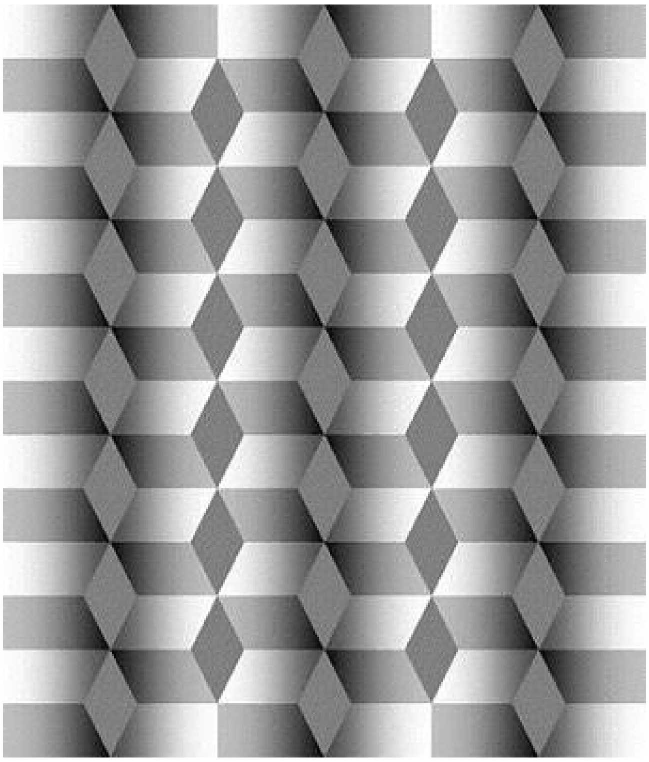
Paradox

Issue 1, 2009

THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY



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Words from the Editor

Welcome to the first 2009 edition of Paradox, the magazine produced by the maths and stats society (MUMS). Here is a little random history:

MUMS has been around since at least 1930, though no one really knows the origins of this mystic society. There are copies of Paradox in the maths library dating back to the 80's; alternatively, you can find past issues on our website.

For years Paradox had only maths problems, some of which contained witty phrases such as "an urn, nay, a jug (I'm utterly fed up with urns)..." and "an urn-like container (but not an urn)..." It was self described as the mathematical equivalent of Farrago, minus most of the interesting bits.

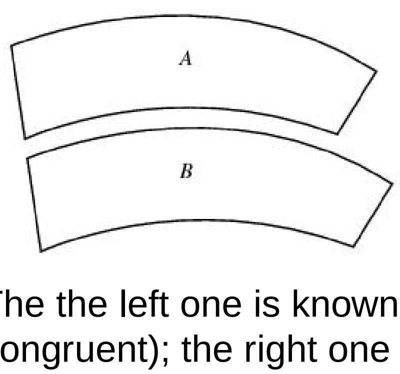
As you will find out, Paradox has changed; now it contains mostly bad maths jokes. Apart from that, you can read our articles on: ways to improve traffic flow, the probability of a digit appearing in a year number, functionals, funny stuff lecturers from this department said, and unorthodox book reviews.

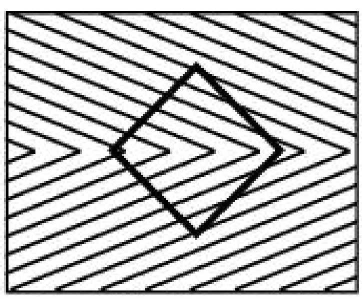
This shall be my last issue of Paradox. Over the years I have enjoyed support from our readers and I thank you all. The magazine will be put into the capable hands of some unknown person.

In 1981 the MUMS committee consisted of 2 people. Back then Paradox advertised, "If you want a society all to yourself, I'm giving this one away." It has grown a lot since then. I shall leave it to our president to entertain you about the benefits of MUMS on the next page.

— James Wan

Three optical illusions:





The the left one is known as the Jastrow illusion (A and B are congruent); the right one is known as the Orbison illusion.

Words from the President

Congratulations! You are among the elite five individuals who have decided, perhaps against your better judgement, to peruse the President's Words section of the Paradox. You have won one copy of:

The Words of El Presidente – the Choose Your Own Adventure Edition

START HERE.

You nearly saunter past the hidden corridor opposite the maths department main office. Your progress is halted only by a trolley full of A5 sized magazines. You pick one up, attracted by its simple and minimalistic cover art. You flip through the pages before looking up and noticing the board of photographs on a nearby wall. As your eyes travel to the right, you notice an invitingly open door with laughter and light emanating from within. You decide :

- To put down the Paradox back in the trolley. (Go to 1)
- To go to the library/first year learning centre to do some studying prior to a fairly full on day with four lectures and a three hour lab. (Go to 2)
- That it's a beautiful day outside and you're going to spend your three hour gap lying on the South Lawn grass embracing the Australian sunshine. (Go to 3)
- That it's a shoddy day outside and you're going to try to get to Union House before the rain gets too heavy. (Go to 4)
- To go home because you only had the one lecture at 10 in the morning. (Go to 5)
- That you're going to be adventurous and go into that room to see what it's all about. (Go to 6)

1) In hurriedly placing down the Paradox, you receive a nasty little paper-cut on your pinkie. Contrary to choose-your-own-adventure expectations, you do not later die/lose limbs due to the wound becoming infected by a particularly

virulent strain of necrosis causing bacteria. However, the stinging pain from the wound does distract you long enough for a ninja to backstab you. As your life slowly ebbs away, you realise that you should have kept the Paradox and bolted into the safety of the ninja-free MUMS room (the one in the wall).

2) Later that day, you amble on to your evening train, fatigued by a day of hard work. You push your way through the masses and score the last free seat in the whole carriage. As the noise of the train rolling away from the platform roars in your ears, you feel yourself strangely light-headed.

The next thing you remember, you're in a cushy bed at hospital, with an ambulance charge of $5000 and a hand written letter:

"Dear passenger,

In collapsing on one of our peak-hour trains, you have delayed (on average) 32 other trains and approximately 32000 other passengers. Some of those, indubitably, were desperately trying to save their marriages by rushing home to prove to their spouses that family comes first. WELL DONE, you selfish selfish tool.

– Connex."

If only, you think to yourself, if only you'd taken a power-nap in the comforts of the MUMS room.

3) Totally forgetting that you're albino, you enjoy a tanning session courtesy of the Sun – a session that will, months down the track, lead to a melanoma.

4) Owing to rainy weather being a highly non-linear dynamical system, the clouds soon gather and coalesce into one epic thunderstorm. Having consumed an inordinate amount of red meat the previous night and thus having a high iron content in your blood, you are struck by lightning repeatedly. As your central nervous system fries, you make one final logical line of thought: if you had gone into the MUMS room and waited out the rain whilst learning about the non-linearity of dynamical system (or just playing scrabble/chess/go/card games/chatting), you wouldn't be feeling quite so shocking right now.

5) Years later, on your death bed, you realise that you squandered your university days by not joining in more student societies and their awesome activities.

6) You are welcomed by the Melbourne University Mathematics and Statistics

society (MUMS) and their terrible puns. You also learn why 6 is a perfect number, whilst having fun playing *insert game of your choice*. YOU WIN!

Moral: the MUMS room is a panacea for all things non-financial crisis.

END HERE.

Okay, on a more serious note, MUMS has had a pretty good start to our semester, we've had record number of new members and we've been trying out new activities such as the Games Nights and a new seminar series. This is of course in addition to our regular plethora of seminars and competitions such as the Puzzle Hunt (go Google it!), trivia nights and the Maths Olympics. But at the end of the day, MUMS is a student society run for and by students, and we really need you guys to help us out by:

a) coming to MUMS events,

b) dropping by the MUMS room to say hi, and

c) writing articles/questions/puzzles/stories for the many competitions and activities MUMS organises throughout the year.

And what do you get out of it? Well, I can certainly tell you that I've learnt a lot from meeting some of the friendliest and brightest students on campus. I've learned the meaning of excellence; I've gained invaluable organisational skills, friends and a much better looking CV; I've absorbed quite a bit of maths along the way. So, drop by our next games night, or come to our seminars and stay a bit for snacks, we look forward to meeting you.

— Yi Huang

A magic ambigram (like a magic square, but it still works when you rotate it 180 degrees; both orientations give a sum of 264).

96	11	89	68
88	69	91	16
61	86	18	99
19	98	66	81

Fun with Functionals

Some of you may be familiar with the so called delta function. It was invented by Paul Dirac in order to facilitate his construction of a theory of relativistic quantum mechanics in the late 1920s, a work that was praised by many of his contemporaries, and was perhaps what was most responsible for his elevation to the higher ranks of the profession. However, the construction of the functional would remain ad hoc and non-rigorous until the mathematicians of his time started to take it apart and analyse it to pieces.

Regardless, it is a relatively easy object to define in a naive fashion. The *Dirac delta function* $\delta(x-y)$ is defined so as to have the following property: for smooth functions f,

$$\int_{-\infty}^{\infty} \delta(x-y)f(x)dx = f(y).$$

Note that as an easy consequence of this definition we have

$$\int_{-\infty}^{\infty} \delta(x-y)dx = 1.$$

In short, we observe that $\delta(x)$ is not really a function in the strictest sense of the word, since if we try to write it in the standard form we have that $\delta(y) = 0$ for $y \neq 0$, and $\delta(y) = \infty$ for $y = 0$. It turns out in fact that the delta function is part of a class of mathematical objects known as *functionals,* which one can think of as limits of sequences of functions. This is part of a more general picture in mathematics where we take limits of sequences of objects and do not necessarily get the same type of object back.

To give an explicit example of this sort of thing, observe that we can think of a function geometrically by thinking of its graph as the local representation of some sort of manifold. Now, if we were to think of an n-dimensional graph we would have what is known as a local representation of the same, or a *chart,*of our manifold M. We can then construct a sequence of manifolds M_n whose behaviour is bounded in some way – say their volume is bounded – which without loss of generality we can require to be a smooth sequence, and then ask what happens in the limit as $n \to \infty$. Then it turns out that we do not always get a manifold back – we actually will often get more pathological objects, known as Alexandrov spaces, which could be roughly thought of as "a manifold with kinks and collapsed parts".

In particular, we can construct sequences of functions that converge to the delta functional. Perhaps the most popular is the convergence of a sequence of peaked Gaussians,

$$f_a(x) = \frac{1}{\sqrt{a\pi}} e^{-(x/a)^2},$$

and we are interested in the limit $a \to \infty$.

Note that via the standard trick of squaring the function above and taking polar coordinates we can show that the integral of each f_a is one, as required. In particular $\lim_{a\to\infty} f_a(x)$ has exactly the same properties as the Dirac delta functional, so we conclude that these must be one and the same.

Related to the Dirac delta is the so called "Heaviside functional" $H(x)$, which is obtained naively through the integration of the Dirac delta. In particular, we define

$$H(y) = \int_{-\infty}^{y} \delta(x)dx.$$

The behaviour of the Heaviside is relatively easy to describe: for $y < 0$, $H(y) = 0$; for $y > 0$, $H(y) = 1$. For $y = 0$, $H(y)$ is, again, undefined, just like the Dirac delta. And, just like the Dirac delta, it is possible to construct the Heaviside as a limit of a sequence of functions.

I will now conclude this short piece by mentioning a curious identity which will hopefully amuse the reader:

$$H(x) = \exp(-\frac{H(-x)}{\delta(-x)}),$$

where here $\exp$ is the standard exponential function. This can be seen by checking that $H(x) = 1$ for $x > 0$ since $H(-x) = 0$ and $\delta(-x) = 0$, but $\delta = H'$ so $H(-x)/\delta(-x) = 0$ and $\exp(0) = 1$. For $x < 0, H(-x) = 1, \delta(-x) = 0$ and $H(-x)/\delta(-x) = \infty$. Then $\exp(-\infty) = 0$ as required. However, a proper proof should show that these considerations are independent of the choice of sequence of functions converging to the functionals in question.

— Chris Goddard

Puzzle 1:
You have a regular octahedron and a regular tetrahedron, both with edge length 1. If you glue them together along two of the faces, how many faces does the resulting solid have?

Some Hilarious Book Reviews

A while back, I was lucky enough to stumble across a collection of short book reviews (Advances in Mathematics, volume 23 (1977), p223) at the end of an article I was reading. They were mostly written by the late great Gian-Carlo Rota (1932-99). Rota was a mathematician and philosopher, and at the time of his passing held the post of professor at the Massachusetts Institute of Technology. He was noted by Zeilberger in his 95th opinion[1] as being *purely a bird* according to Dyson's classification scheme for mathematicians (Notices of the AMS, volume 26(2) (2009), p212) – and a pure bird is surely something that should be looked up to. I hope you all enjoy the reviews as much as I did.

∞

Reviews by Rota:

Linear Operators and Approximation Theory. By P. P. Korovkin. Hindustan Publishing Company, 1959.

This is a very well put together account of the basic facts about approximations of continuous functions of one real variable by sequences of positive operators, the typical case being approximations by Bernstein polynomials. . . . The translator has found it prudent to remain incognito, probably in view of the enormous mass of errors of interpretation and assorted misprints which overwhelm the reader at every page. It is high time that firms such as publishers of this book should seriously reconsider their editorial policies.

Language Learning in Wittgenstein's Later Philosophy. By C. S. Hardwick, Mouton, 1971.

On reading this thorough description of one theme in Wittgenstein's "Philosophical Investigations", the conclusion . . . becomes inescapable that the philosophies of the late Wittgenstein and the early Heidegger are basically identical. Unfortunately, Wittgenstein gives examples of an unstated theory, and Heidegger develops a theory without examples.

Automaten und Funktoren. By L. Budach and H. J. Hoehnke, Akademie-Verlag, 1975.

Warning: Automata theorists have discovered category theory.

The Philosophy of Quantum Mechanics. By M. Jammer, Wiley, 1974.

[1] http://www.math.rutgers.edu/~zeilberg/Opinion95.html

Most books with titles like this one – and they are legion – are pathetic repetitions of trite platitudes. Not this one; if you want a book on the foundations of quantum mechanics for your shelves, this is it.

The Paradox of the Liar. By R. L. Martin, Yale University Press, 1970.

Anyone who wants ample proof of the sterility and isolation from live contemporary problems of today's analytic philosophy should read this book.

Challenging Mazes. By L. D. Quinn, Dover, 1975.

Variation of a Knossian[2] theme.

Cohomology Theory of Topological Transformation Groups. By W. Y. Hsiang, Springer, 1975.

Up-to-date, courteously and clearly written, and giving background material to the average reader. Unusual behavior in this field.

Optimization Theory. By M. R. Hestenes, Wiley, 1975.

It used to be called the calculus of variations, and everybody thought it was as dull as night. Now it is called optimization theory, and everybody thinks it is red hot.

∞

More review by R. E. Kalman, Stanford University:

Scientific Method: Optimizing Applied Research Decisions. By R. L Ackoff. Wiley, 1962.

One might as well begin by saying that this book is useless to the mathematician in his professional capacity. But he will find it rewarding extracurricular reading and a bridge to the "other cultures" inhabited by his scientific colleagues.

The book is written from a modern, but today already rather orthodox, point of view. We might call it the statistics-operations-research/mathematical economics school. As in other writings of this school, this book may be commended for its breadth, imagination, and worldliness, but it is tainted with a slight suspicion of being shallow...

[2] The Labyrinth of Knossos.

The following quotations, taken from summaries at the end of chapters seem to me to be typical of the author's attitudes: "But most problems of optimization are characterized by some incompatibility (conflict) between objectives." "Measurement, perhaps more than any other activity, has been the principal stimulus to progress in both pure and applied science." "To a large extent the distinction (between pure and applied research) was made in order to provide status to scientists who consider themselves 'pure' at a time when it was thought that pure scientific problems required greater knowledge and ability for their solution than did applied problems."

...The last chapter, especially, contains common sense in abundance, and should be required reading for everyone who has to write a "proposal".

∞

In 1973, Associate Professor Michael Mahoney of the History of Science Department at Princeton University wrote a biography of Pierre de Fermat. At that time, Andre Weil had been studying the history of Fermat in great detail for some time. It was arranged for Weil to review the book for the Bulletin of the AMS. He began: "In order to write even a tolerable good book about Fermat, a modicum of abilities is required." He listed these abilities:

- Ordinary accuracy
- The ability to express simple ideas in plain English.
- Some knowledge of French.
- Some knowledge of Latin.
- Some historical sense.
- Some familiarity with the work of Fermat's contemporaries and of Fermat's own mathematics.
- Knowledge of and sensitivity to mathematics.

Weil then demonstrated, via annotated quotations from the book under review, that the author possessed none of them.

— Stephen McAteer

Quotes from Lecturers

The following are gems of humour, wisdom, and occasionally even cluelessness coming from our very own lecturers, gleaned from the editor's four years' worth of maths classes at the University of Melbourne. Enjoy!

Associate Professor Iain Aitchison:

Anything that allows the university to count the same work as many times as it can should be encouraged.

(In response to "when are you normally in your office?") I'm not usually there; when I am, I'm not normal.

Those who have done your cultural education... would know about Pacman.

If you think you can prove this in this course, then you are even more foolish.

Hopefully I'll see you relatively soon, if not then I'll see you a bit later than that.

Associate Professor John Groves:

Real numbers aren't real.

I can't show you an example here, as it doesn't happen.

The continued fraction converges in the sense that your most recent analysis course would approve of.

Even arts students can do this subject... well not all of them.

You are probably tired of theorems on isomorphic things, as they are all the same.

We'd like to prove uniqueness, but we're not going to, because it's false.

Associate Professor Barry Hughes:

There aren't many good maths jokes.

Real analysis is a disgusting subject. It has very poor feng shui.

If you remember what a residue is, many wonderful things will happen in your life.

For us – I'm using the royal plural – for me...

It's like doing push-ups, but problems are good for you.

Professor Hyam Rubinstein:

Total boundedness is boundedness on steroids.

Let's draw a solid figure (chalk breaks)... well, it's too solid.

We can go back to the real world (we are in the complex world).

My proof is a little bad, but the standard books' are even worse.

Oh dear, I forgot to prove this theorem.

There's chaos in this course.

You never make sign mistakes in $\mathbb{Z}_2$.

Dr Lawrence Reeves:

Why do we want to understand polynomials? Well, who knows.

These problems come from the actual antiquity, as opposed to the 70's.

Associate Professor Jerry Koliha:

You should wake up at night hearing a voice telling you, "don't subtract measures..."

It is a linearly ordered webpage, that's why we call it linear analysis. (On the webpage: "Want to make a million bucks?" – a link to the Clay Institute millennium problems.)

A mathematician doesn't have to be crazy, but it helps.

(While handing out lollies, which he did in many classes) The cardinality of this bag of lollies is smallish. There is a bijection of this into a small subset of you. If you take 2, you ruin the function.

Can you read this? I have a wonderful optometrist, if you want... we split the profits.

There are people who call this the Koliha-Drazin inverse. I like these people. They are my friends. If you use that name, your score will go up by 20 points.

These will go into your bag of tools of an operator theorist. Which now you are becoming. The operator theorist, not the tools.

Professor Greg Hjorth:

This is very concrete, it's the dual of the space of continuous functions on X.

The pronunciation of my surname is impossible.

We all think "clopen" is a very groovy word.

It's so much fun doing the pictures with coloured chalk, do you mind if I finish?

I hope you will awaken and find this very convincing.

Dr Craig Westerland:

If you can draw in 4D, you should be teaching this.

"Totally normal neighbourhood" sounds like sociology.

Apparently this idea is so cool that the board can't sustain anything else being written on it.

Ortho means perpendicular, normal means perpendicular, orthonormal means perpendicular.

(When his phone rang) Oh, I thought you were snoring.

Associate Professor Craig Hodgson:

(On the hairy-ball theorem) This only applies if you have hair all over your head, so it doesn't apply to me for instance.

Dr Paul Norbury:

What do you call the circumference of a rectangle?

If you can't differentiate from first principles, then what has become of you.

My son's weight, funnily enough, was π kg.

Maths Jokes

An optimist believes we live in the best of all possible worlds. A pessimist fears this is true.

∞

Q: What turns theorems into coffee?

A: A co-mathematician.

Q: What does a mermaid wear?

A: An algae-bra.

Q: How did your parrot die?

A: Polynomial. Polygon.

Q: Can an English major learn maths?

A: Cosecant!

∞

I've memorised 100000 digits of π. They're all 3's. Of course I haven't memorized exactly where they occur.

∞

In theory, there is no difference between theory and practice. In practice, there is no relationship between theory and practice.

∞

Actual book title:

Sex, Crime, and Functional Analysis

Part I: Functional Analysis

By J. D. Stein

∞

Old academics never die, they just lose their faculties.

Old accountants never die, they just lose their balance.

Old astronauts never die, they just go to another world.

∞

Definitions:

x-bar: where statisticians go when their work drives them to drink.

One to one correspondence: love letter.

True Stories

Many years ago, one assessment at the University of Melbourne took the form of a true or false test of 30 questions. One mark was awarded for each correct answer, and one was deducted for each incorrect answer. One particular student scored an incredible -30. The examiners cogitated on this curiosity and decided to award him 29 out of 30 instead, subtracting only one mark for confusing true and false.

A wealthy 15th Century German merchant, seeking to provide his son with a good business education, consulted a learned man as to which European institution offered the best training. "If you only want him to be able to cope with addition and subtraction," the expert replied, "then any French or German university will do. But if you are intent on your son going on to multiplication and division – assuming that he has sufficient gifts – then you will have to send him to Italy."

In an interview, Paul Erdös was asked why he never married. He replied, "This may sound very strange to you, but I cannot tolerate sexual pleasure."

Einstein was once asked where he kept his laboratory. He smiled and took a fountain pen out of his pocket, and said, "here."

One of J. Littlewood's papers ended with "Thus σ should be made as small as possible." However, the sentence did not appear in print; in its place was the tiniest σ ever.

It is said that G. H. Hardy had four ambitions in life, none of which he achieved:

- To prove the Riemann hypothesis;
- To score the winning play in an importance cricket match;
- To kill Mussolini;
- To prove the non-existence of God.

Hardy could not stand having his pictures taken. He also hated mirrors, and would cover any mirrors in a hotel with towels.

A story told in this maths department goes that a particular mathematician would insist to drink a "Hanh-Banach beer" (just like the theorem) in the pub, as opposed to the more trivial Hanh beer.

Edifying Quotes

"Only two things are infinite, the universe and human stupidity, and I'm not sure about the former." – Albert Einstein

"The only thing that interferes with my learning is my education." – Albert Einstein

"A certain impression I had of mathematicians was that they spent immoderate amounts of time declaring each other's work trivial." – Richard Preston

"How is an error possible in mathematics? A sane mind should not be guilty of a logical fallacy, yet there are very fine minds incapable of following mathematical demonstrations." – Henri Poincaré

"The ultimate goal of mathematics is to eliminate any need for intelligent thought." – Alfred N. Whitehead

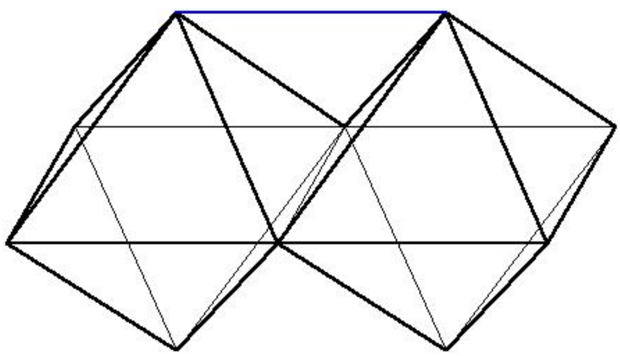
"Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." – Bertrand Russell

"It is never worth a first class man's time to express a majority opinion. By definition, there are plenty of others to do that." – G. H. Hardy

"Caution, skepticism, scorn, distrust and entitlement seem to be intrinsic to many of us because of our training as scientists…These qualities hinder your job search and career change." – Stephen Rosen

"It has been said that World War I was a chemist's war and World War II was a physicist's war. There are those who say that the next World War, if one should occur, will be a mathematician's war." – John Curtiss

Solution to puzzle 1:
Surprisingly, the answer is 7. As the octahedron and tetrahedron glued this way is space-filling, six of the original faces merges into three rhombi. To see this, consider two octahedra lying side by side as in the figure below. Then when we join up the top two vertices, a tetrahedron appears in the empty space above the octahedra. This can also be seen as the dihedral angle of a regular octahedron ($\cos^{-1} -\frac{1}{3}$) and the dihedral angle of a regular tetrahedron ($\cos^{-1} \frac{1}{3}$) add up to π.



Puzzle 2:
Plant 9 trees on flat ground such that there are 10 rows of 3 trees each.

Is 8 Your Favourite Number?

One day, I was passengering along (what you do in a car when you're not driving), thinking about how great it was that there was an '8' in the year. So, yes, I was thinking that 8's are pretty cool, cause they're loopy and kinda look like ∞ on its side. But this got me thinking – just how cool is it that 2008 had an 8 in it? There certainly hadn't been a year with one for a while, and since New Year's I have been staring at dates and sighing, but just how special was 2008? I hear that in the eighties such occurrences were commonplace.

Knowing that the essence of mathematics is assigning exotic letters to otherwise simple concepts, I decided that ϖ would indicate how common such an occurrence was: let ϖ be the number of years with the digit 8 in them divided by the total number of years we are considering. And let's start at 1AD because calculating the precise age of the universe is problematic. Well, this is easy. You just need to count them, for example you can write a simple computer program to do that.

Of course you can look for an exact formula. Between 1 and 10 there is 1 year with an 8. Between 1 and 100 there are 10 in the eighties and 10 ending in 8 minus 1 overlapping = 19. Between 1 and 1000 there are 100 in the 800's, 90 in the 80's but not in the 800's and $9 \times 9 = 81$ ending in 8 but not in the previous two categories. You can generalise this and see that for any power n of 10 ($n > 0$), the number of years from 1 to 10^n with an 8 in them is $\sum_{i=0}^{n-1} 10^i \times 9^{n-i-1}$.

It has been popularly declared that $\varpi = 0.269880716$ by many New Age proponents, although since they use a base-20 calendar starting at 3114BC it should probably be 0.219423077. William Miller[3] incorrectly stated that $\varpi = 0.253253796$, Jesus said that ϖ was indeterminate, and if it's $\frac{272}{2009}$ then it's really not worth finishing that assignment.

To give a value for ϖ, even assuming we start counting at 1AD, requires a year to count to. 2000 seemed an obvious choice ($\varpi = 0.271$ precisely), since all the computers at the time allegedly couldn't count past it and it was probably going to bring down civilisation as we knew it. But not even a single plane fell out of the sky on 01/01/00, thus proving that the Y2K bug was little more than a brilliant marketing tool for the IT companies ("upgrade or die!"). We could use the year of your death, however that's the domain of actuarial students and you might be dead by the time I'd finished with all the statistics.

So let's use 2012. Not arbitrarily, mind you. The Mayan Long Count calendar starts in 3114BC (on the day of creation, but we are currently in the 4th

[3] American Baptist who thought that Jesus would return in 1844.

creation), and expresses dates in base-20, except that the second last digit is base-18 to give a year 360 days (they also had religious and civil calendars, the latter having a year of 365 days). With 5 digits in total, 12.19.19.17.19 will take place in December 2012. So you'd better watch out. Multiple New Age figures have independently arrived at the conclusion that 2012 will be a year to not remember (we'll all be dead) due to the end of this creation, a magnetic pole shift or possibly just a life-ending natural disaster. Either way, it settles our question. With life ending in 2012, $\varpi = 0.269880716$ or if you've been using the Long Count calendar since 0.0.0.0.0 then we need to revise our calculations to base-20 to arrive at $\frac{1141}{5200}$.

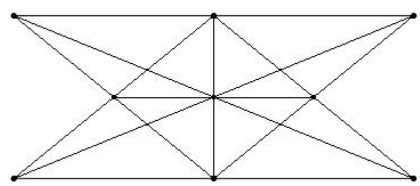
More interesting is if the universe was to last forever, preferably with someone left to maintain the calendar. Let ϖ be the probability that a year, chosen at random, has the digit 8 in its decimal representation. This is consistent with the previous definition in the finite case. Consider the probability p that a number picked at random does not contain an 8. Each digit has a one in ten chance of being an eight, so a $\frac{9}{10}$ chance of not being eight. Hence an n digit number not containing an 8 is $(\frac{9}{10})^n$. As n approaches infinity, p approaches 0. Thus $\varpi = 1 - p = 1$.

Another funky result is Jesus' value for ϖ. Jesus said that "Heaven and earth will pass away, but my words will not pass away. But no one knows of that day and hour, not even the angels of heaven, but my Father only." Such statements have not stopped many people, such as Miller (who tried twice), from making predictions, supposedly on Biblical evidence, of when Jesus will return. Conclusion: ϖ is indeterminate.

So there you have it. If you don't like 8's, you can always rub them all out and replace them with your favourite digit. Only the numerical results will change, not the theory. See you in 2018.

— Matthew Baxter

Solution to puzzle 2:



Transportation Networks: Coordination vs Anarchy

In the "real" world, many systems can be characterised by a network with nodes and paths joining them. Here, we will consider traffic flows of a decentralised transport system for personalised vehicles. It is natural to ask ourselves whether or not this network system is the most efficient one, or alternatively, on average, does this road network allows commuters to get from A to B most efficiently?

On the surface, this question seems easy to answer; one can

- set up a model of the system;
- find the global minimum using the convex minimum cost flow algorithm.

In reality, individual commuters do not collectively opt for the most optimal strategy, but their own optimal strategy. Hence, the actual performance of the network is often far from its best, even if all individuals choose the quickest route and all information is available to them. The key question here is to understand how far the actual performance is from the most optimal one.

To illustrate a point, we will consider the following simple network:

> Let c and d be the two routes available for commuting between A and B. c is a short but narrow road, where the effective speed become slower as traffic volume increases. On another hand, d is a long freeway, where the traffic volume has a negligible effect on the speed.

Now, we need a way to measure efficiency of the system. Here, we will use the notion of delay, the time needed to get from A to B. In real systems, delay will be associated with the traffic flow; the higher flow results in longer delay. This intuition coincides with the fact that more cars travelling on the same road leads to traffic congestion. This means that one commuter's choice of route can potentially conflict with other commuters' goal to reduce the travelling time. Over a period of time, this system will stabilise at equilibrium point(s) called the Nash equilibrium. The point here is that this Nash equilibrium may not be the social optimum.

Back to the example, suppose that the delay on c is proportional to the flow, while the delay on d is independent of the flow. So mathematically, we may have the following equations (ignoring units here) that describe the system:

$$l_c(f_c) = f_c, \quad l_d(f_d) = 10,$$

where l_i is the delay on i, and f_i is the flow on i. Here we will assume that there is a constant total flow F. That is, $F = f_c + f_d$. Then, the total expected delay can be calculated by

$$\mathcal{C} = l_c(f_c) \cdot f_c + l_d(f_d) \cdot f_d.$$

Suppose that the total flow $F > 5$. Then it is an easy secondary school exercise to show that $\mathcal{C}$ achieves the minimum with $f_c = 5$. So, if $F = 10$, then $\mathcal{C} = 75$. From now, we will fix $F = 10$.

Note that in this setup, it requires half of the total traffic flow to use the route c and the other half to use the route d. So, a user on link d – let's say Bob – can expect a delay of 10. However, if he switches to the link c, then he effectively reduces his travelling time from 10 to 6 (since he increases the flow in the link c). So, in order to achieve the social optimum, it requires altruism on part of the commuters, which is unlikely. This creates a conflict between social efficiency and personal efficiency.

In this scenario, as long as $l_c \neq l_d$, there will always be an incentive for Bob to switch to the other route. So, if all commuters put their own interest first (assuming that each Bob can decide perfectly and has perfect information), then a Nash equilibrium is established when $l_c = l_d = 10$. So, the flow $f_c = 10$ and at the total cost of $\mathcal{C} = 100$.

It should be noted that in the real network, if all functions $l_{ij}(f_{ij})$ are strictly increasing and the flows f_{ij} are continuous (as a good approximation), then there is always exactly one Nash equilibrium.

From here, we can use a similar technique to analyse real road networks. Youn et al. has studied Boston's road network. The common function l_{ij} used is based on the Bureau of Public Roads (BPR),

$$l_{ij}(f_{ij}) = \frac{d_{ij}}{v_{ij}} \left[1 + \alpha \left(\frac{f_{ij}}{p_{ij}} \right)^{\beta} \right],$$

where

d_{ij} is the distance of the link (i, j);

v_{ij} is the speed limit;

f_{ij} is the flow; and

p_{ij} is the capacity of the road segment.

The parameters α and β can be determined to fit the empirical data.

For various total traffic volumes F from Harvard Square to Boston Common, the socially optimum flows f_{ij}^{SO} are determined by minimising

$$\mathcal{C} = \sum_{seg(i,j)} l_{ij}(f_{ij}) \cdot f_{ij},$$

where $seg(i, j)$ is the road segment between i and j. For the Nash equilibrium, they used the (not obvious) fact the equilibrium flows f_{ij}^{NE} minimise the objective function

$$\widetilde{\mathcal{C}} = \sum_{seg(i,j)} \int_0^{f_{ij}} l_{ij}(f'_{ij}) df'_{ij}.$$

Then, the *price of anarchy* PoA is defined to be the ratio

$$PoA = \frac{\sum l_{ij}(f_{ij}^{NE}) f_{ij}^{NE}}{\sum l_{ij}(f_{ij}^{SO}) f_{ij}^{SO}}.$$

The common features in most networks that Youn and colleagues studied are the following:

PoA is approximately 1 at low traffic volume and very high traffic volume;

PoA can reach as high as 1.30. This means up to 30% of travelling time is wasted if not being coordinated.

The lesson here is that given a transportation system, if it is not coordinated, then its efficiency may be significantly reduced. So, the next natural question is: how can we coordinate the traffic flow?

A common solution normally involves taxation on certain sections of road ways. This is equivalent to adjusting the function $l_{ij}(f_{ij})$. However, this has its own problem. In order to reach the desired Nash equilibrium, taxation must be interactive with the non-constant term of $l_{ij}(f_{ij})$. On the contrary, closing roads to cars is easy to implement. But how will road closure improve traffic conditions?

Intuition tells us that closing roads will mean more congestion. On the contrary, Braess' Paradox suggests that road closures can potentially improve the efficiency of a network. When a road is closed, the users of that road must change their route and thus change the network dynamics. Most of the time, this results in reduced efficiency, but if the right road is selected, the opposite is possible. Youn and colleagues verified this idea by considering a closure of a single road section; they calculated the total expected delay at the new Nash equilibrium and then compared it with the social optimum of the original network.

So far, the idea seems good, but caution should be taken. There are many areas that need to be considered. In particular, the main factor that determines which road section should be closed is dependent on the choices of starting point and destination. Nevertheless, this should not limit the usefulness of this method in designing traffic network. It would certainly be nice if a few roads (e.g. Brunswick St, Fitzroy) in Melbourne are closed to car traffic.

— Tharatorn Supasiti

Solutions to Problems from Last Edition

We had a number of correct solutions to the problems from last issue. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

Josh Howie solved problem 3 and may collect $3.

Kate Mulcahy solved problem 3 and may collect $3.

Tharatorn Supasiti solved problem 5 and may collect $5.

Matthew Ng solved problems 1, 3, 5 and may collect $9.

Duc Truong solved problems 2, 3, 5 and may collect $11.

Adrian Khoo solved problems 2, 3, 5 and may collect \$11.

Jensen Lai solved problems 1, 2, 3, 4, 5 and may collect \$18.

1. Find the sum of squares of all entries in the nth row of Pascal's triangle. What about the sum of product of all adjacent pairs of entries?

   Solution: We want $\sum_i \binom{n}{i}^2 = \sum_i \binom{n}{i}\binom{n}{n-i}$. Combinatorially, this is the same as choosing i objects from a group of n objects, then $n-i$ objects from another group of n objects. The sum is thus all possible ways of choosing n objects from $2n$ objects, so the answer is $\binom{2n}{n}$.

   For the second sum, we have $\sum_i \binom{n}{i}\binom{n}{i+1} = \sum_i \binom{n}{i}\binom{n}{n-i-1}$. This is the same as choosing $n-1$ objects from $2n$ objects, so it equals $\binom{2n}{n-1}$.

2. Using a standard deck,

   a) find the probability of getting at least four cards of the same suit in a hand of 13 cards,

   b) find the probability of getting at least five cards of the same suit (i.e. a flush) in a hand of 13 cards.

   Solution: (a) Suppose we have at most 3 cards of each suit. Since there are 4 suits, that makes up 12 cards, not 13. Hence there must be a suit with at least 4 cards, i.e. the probability is 1.

   (b) We first count the number of ways of not getting a flush: the number of cards in each suit mush be in one of the following configurations: $(4, 4, 4, 1), (4, 4, 3, 2), (4, 3, 3, 3)$.

   There're $4\binom{13}{4}^3\binom{13}{1}$ ways of getting the first configuration, $12\binom{13}{4}^2\binom{13}{3}\binom{13}{2}$ ways of getting the second configuration, and $4\binom{13}{4}\binom{13}{3}^3$ ways of getting the third configuration. Add them up, divide the answer by $\binom{52}{13}$ and subtract from 1, we get $20612373517/31750677980$, or around 0.65.

3. In an equilateral triangle ABC, P is on AB so that $AP = AB/3$, Q is on BC so that $BQ = BC/3$, and R is on CA so that $CR = CA/3$. The lines CP, AQ, BR enclose a triangle. Find the ratio of the area of this triangle to that of ABC.

   Solution: Let AQ and CP intersect at D, BR and AQ at E, and CP and BR at F. Then it is clear that BEQ is similar to BCR, and hence $BE = 3EQ$.

Next, construct G on BR such that QG is parallel to CP. Thus BQG is similar to BCF. Note that QEG is equilateral. It follows that $BE = EF$.

The area of a triangle is proportional to the product of two sides and the $\sin$ of the included angle. Hence the area of DEF is half that of BFC. As the area of ABC is the area of DEF plus 3 times the area of BFC, we get the ratio of area of DEF over area of $ABC = \frac{1}{7}$.

4. Find $\int_0^1 \frac{\log x}{x^2-1}\mathrm{d}x$.

   Solution: There are a few ways to do this problem. For instance, we can break it up into partial fractions ($\frac{1}{2}(\log x/(x-1) - \log x/(x+1))$) and do a series expansion on each term. With some justification, the series can be integrated term by term, and knowing $\sum_{i=0}^{\infty} 1/i^2 = \pi^2/6$, we can derive the answer.

   Another method, if you know about polylogs, is to compute the integral in terms of logs and polylogs and sub in the limits..

   We also describe a method using complex analysis. Make a change of variables $t = 1/x$, and we note that $\int_0^1 \frac{\log x}{x^2-1}\mathrm{d}x = \int_1^{\infty} \frac{\log t}{t^2-1}\mathrm{d}t$. Hence the integral is $\frac{1}{2}\int_0^{\infty} \frac{\log x}{x^2-1}\mathrm{d}x$.

   Now a result from complex analysis states that for a rational function $Q(x)$ with no poles at $x > 0$ and decays faster than $1/x$, $\int_0^{\infty} \log x Q(x)\mathrm{d}x = \frac{1}{2}\sum Res(Q(-z)\log^2 z)$. Note that our integral satisfies the above criteria (the "pole" at 1 is conveniently removed by the log).

   So we just need to calculate a quarter of the residue of $\log^2 z/(z^2-1)$ at -1. This is given by $\frac{1}{4}\log^2(-1)/-2 = \frac{\pi^2}{8}$.

5. In a triangle ABC, let a, b, c be the sides opposite angles A, B, C respectively. Show that

$$\frac{\cos(B/2)\sin(B/2+C)}{\cos(C/2)\sin(C/2+B)} = \frac{a+c}{a+b}.$$

   Solution: Applying $\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y)$ to the top and bottom of the left hand side, we get:

$$\frac{\sin(B+C)+\sin C}{\sin(B+C)+\sin B} = \frac{\sin A+\sin C}{\sin A+\sin B},$$

   as $A+B+C = \pi$.

   By the sine rule ($\sin C/\sin A = c/a$ etc), we may simplify it further to $\frac{1+c/a}{1+b/a} = \frac{a+c}{a+b}$, as required.

Paradox **Problems**

Below are some puzzles and problems for which cash prizes are awarded. Anyone who submits a clear and elegant solution may claim the indicated amount (unless two solutions are the same, in which case only the first submission will be rewarded). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

1. ($2) Solve $a^2 + b^2 + c^2 = a^2b^2$ in the integers.
2. ($2) In a triangle ABC, $\angle A = 120°$. Find the length of the angle bisector from A in terms of AB and AC.
3. ($3) If f is a function such that $f(ab) = \frac{1}{2}(f(a) + f(b))$, find $f(1234) - f(4321)$.
4. ($3) Solve the equations $9(x - y)(x^2 + y^2) = 1, 5(x + y)(x^2 - y^2) = 1$ in the reals.
5. ($5) You are in the centre of a circular pond 100 meters in radius. There is a zombie on the pond's edge that wants to eat you. The zombie can walk at 4 m/s and can't swim; you can swim at 1 m/s and run at 7 m/s. Can you escape the zombie?
6. ($5) Euler proved at least one new theorem every day. To conserve energy, however, he proved no more than 50 theorems in any month. Show that there is a succession of days in a year where Euler proves exactly 125 theorems.
7. ($6) Rumours say that this problem might appear in your next calculus exam.

   An object initially at $(0, 0)$ moves at all times toward another object initially at $(1, 0)$ and which is moving in the y direction. Both objects have the same constant speed. Find the path of the first object.

Paradox would like to thank Yi Huang, Stephen McAteer, Tharatorn Supasiti, Chris Goddard, Kate Mulcahy, James Saunderson, Adib Surani and Matthew Baxter for their contributions to this issue.

MUMS Puzzle Hunt 2009

30th March – 3rd April