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Articles & Special Features

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COVER:	A scene from the 2009 film *Agora* in which the Greek mathematician Hypatia teaches her pupils about gravity by dropping a handkerchief. Find out more on page 10.

Words from the Editor

Congratulations on making it to the end of the semester (well, almost) and welcome to the second issue of Paradox for this year! There's plenty of procrastination material here to distract you from your upcoming exams or assignments, but there also many other (and better) reasons to read Paradox.

Have you ever wondered how a career in mathematics could lead to medicine? Perhaps you've been wondering about who introduced rigour into mathematics, or maybe you just want to know more about our beloved Pythagorean theorem and Pascal's triangle. There's plenty of that and more in this issue, plus some more puzzles for you to get tied up with, what with there being knots in the puzzles and all.

Perhaps you were looking for none of this and are about to put Paradox down disappointedly. But you would then miss out on the rest of the many other joys contained within this issue, or at least the jokes that stubbornly continue to populate it. If even that isn't enough, then you must either be a very demanding person or you seriously ought to consider joining the team and making your own contribution—there are many ways to get involved in Paradox so just get in touch by email.

While this would almost have been a no-surprises Paradox (unlike the federal budget), I had to go out of my way to make sure that I didn't leave you totally unsurprised. Hence, it is with a tinge of sadness that I note this shall be my final issue as Editor before someone takes over from me at long last. The past three years have been an incredibly fun learning experience as many records have been broken and many changes have been made to the content of Paradox whilst its format of old has been faithfully maintained.

Many thanks to all who have contributed to Paradox over the past three years: without you, it could not have become the entertaining yet accessible publication that it is today. I wish the best of luck to the new editorial team as they begin their own journey and lead Paradox towards an even more exciting future. As always, please email `paradox.editor@gmail.com` with new contributions and all the best for the many years ahead!

— Kristijan Jovanoski

Words from the President

Well, it looks like my glorious reign as supreme leader of MUMS is finally coming to an end. Soon you will all have a new (probably slightly inferior) President to write these words which you have accidentally glanced at on the way to the good bit of Paradox. Yes, that's right, after this Friday's AGM with free pizza[1] there will be a new person clogging up Paradox with their scrawl.

So, you've made it all the way past the first paragraph, well I guess you must must be stuck on long train trip or something with only this issue of Paradox to occupy you, so here's a quick puzzle to keep you occupied:

itdk/ u[mt k[,tfu P kdnt ,[ukp,j p,utytfup,j u[ryput p, 'p;t h[y. tpukty

Ah, right, I guess you don't have access to a computer though. . . which could make that tricky to decode. Anyway, speaking of puzzles, the Wild Goldfish recently found the Philosopher's Stone and saved the world from the evil Tsar and, more importantly, they won the Puzzlehunt for their troubles. The rest of you should be proud too; collectively, you smashed the record for most correct answers in a single Puzzlehunt, with a total of 2031.

Looking forward for a minute now, assuming that MUMS somehow survives for a week without yours truly at the helm, you have an extremely exciting Trivia Night to look forward to; as always it will be on the last day of semester. I can only speculate on what will happen, but I would assume that the night will involve a good deal of dirt holding (which the Flowerpots should be good at), colouring in (which a Whole Table of Arts Students should be good at) and of course some good old rock paper scissors (you might want to read the complete strategy guide in the MUMS room to prepare).

Just a warning though, I hear that contestants may be required to breathe without the aid of water, so, unfortunately, the Wild Goldfish may have difficulty pulling off the win to add to their Puzzlehunt success.

— Andrew Elvey Price

A statistician can have his head in an oven and his feet in ice, and he will say that on the average he feels fine.

[1] That's the official title.

Interview with Terry Speed: Bioinformatician and Medical Researcher

Professor Terry Speed is a world-renowned statistician, currently working in the field of bioinformatics. Among many accolades, he was recently awarded the 2013 Prime Minister's Prize for Science in recognition for his ongoing work in genomics. He generously gave up some of his time to talk with me in his office at the Walter and Eliza Hall Institute (WEHI).

As the Head of the Division of Bioinformatics at WEHI, how would you explain bioinformatics to someone new to the field?

Firstly you need an idea about DNA—the genetic material of life that is within cells—long molecules, long strings of nucleotides which we abbreviate by letters A, T, C, and G. If you start off with that, you have a string of letters in its most rudimentary form: that's data that encodes the genetic information we pass on to our offspring and receive from our parents. Just dealing with that is a pretty large fraction of what bioinformatics is about, but if you add in a couple of extra details, such as DNA being transcribed into RNA and genes embedded in the DNA sequence needing to be switched on. If there's a molecule that you need to translate the DNA into, then that's a molecule that we deal with in bioinformatics.

The RNA then gets translated into proteins, which are sometimes called the 'workhorses of cells', since they do stuff. DNA doesn't really do stuff, it's a passive repository of information like a library, whereas RNA is a mediator between the library and the action, where proteins are the action. So if you talk about DNA, RNA and proteins, that in a certain sense is the raw material for bioinformatics. Although it is often interpreted much more widely now, bioinformatics is principally about these molecules of life.

Can you tell us a bit more about what you're currently working on?

So I've mentioned DNA and RNA, and whenever you think about life and cells, you have to think about who is causing what to happen. If I say that DNA is a repository of genetic information, and that we have genes embedded in that DNA and they get switched on and then things get expressed, you might ask, who turns it on? Who stops it? Who's running the show? And if you start thinking like that it's almost like you go back to God. Imagine a cell, created by a sperm and an egg, and now it's a zygote that will start dividing and eventually grow up to be a person. Who causes that to happen? What are the devices that nature has evolved to regulate the development of, say, a single cell to a whole human being?

One of the terms we have now for these processes is *epigenetics*. '*Epi-*' is a Latin prefix meaning 'on top of' or 'above', so it's something over and above the DNA which makes things happen. Sometimes people say it's like punctuation marks: how do you read the DNA? But it's a little more dynamic than that. So there's a whole layer of, as it were, biological structure on top of DNA, RNA and protein, that governs processes. Hence epigenetics can be considered to be the step between the genotype (the genetic makeup) and the phenotype (the expressed features) of an individual, and that's what I'm very interested in. It's rather subtle and yet complex, but it gets to the heart of life in a way that DNA, RNA, and protein alone don't quite.

Where did your interest in medicine stem from, given your mathematical background?

Very simple. I went to school there [points out of his office window towards University High School], and at the time I was doing sixth form, or Year 12 nowadays. In that year Sir Macfarlane Burnet won a Nobel Prize, and he was working here [WEHI] and I thought, "I'd love to do that!" No, I didn't think that I'd like to win a Nobel Prize; rather, I thought I like to work in medical research [laughs heartily]. So I went across the road to Melbourne Uni, thinking I'd do medicine and science and then get qualified to come back across the road to work at this institute. That was 1961. I started work here in 1997, so there's a *small* gap of 36 years where I did several other things. I did one term of medicine, then I switched to maths, but I never lost my interest in medicine or genetics, which I kept up throughout my career. So, it was inspired by the medical research in this very institute, 53 years ago.

Going back to your Bachelor's where you studied maths and statistics, what intrigues you the most about these fields?

It's hard to say but I guess people—and I would be one of them—like maths because it is efficient. It sort of appeals to reasoning. Compared to the biology that was given to me at the start of my course, where there was a lot to remember, you could kind of reason things out with maths: you didn't need to remember it all. You don't 'remember' calculus, but rather reason through it. That's appealing, that there's not a lot of baggage. Of course it does help to know a lot of mathematics, but on the other hand it helps to develop the capacity to think things through from first principles.

And of course there's the beauty in mathematics as well. Obviously I like all of the aspects of mathematics we normally consider to be beautiful, like geometry, algebra, and the beauty in theorems, but also I like the fact that it does stuff. Newton's laws tell you about planets, and it can help you shoot up satellites, and land something on the moon. It was really good to have an understanding of, as it were, the laws of nature that enable things to happen. So, beauty, logic, applications, all of those come together. Not many fields where you get that level of satisfaction, I don't think.

The application of mathematics to the medical field is very powerful—what made you realise that you could connect these two fields of interest?

Bioinformatics has its origins in evolution and genetics, and I got interested in both of those during my first year at Melbourne Uni. As I began doing a combined medicine/science degree, I learnt more and more about these fields, and by the time I was more or less a matured statistician, I was still inclined to apply statistics to areas of biology, particularly genetics and evolution. Well in a sense, the application of mathematics, statistics and algorithms to genetics and evolution: that's what bioinformatics became. Twenty years ago, you would have said you were doing statistics, genetics and molecular biology. These days, you say that you are doing bioinformatics. So the name just came to describe what I was interested in, and what I was doing at the time!

Finally, is there any advice you would give to students in their final year of study, or students that are hoping to apply their knowledge from one field to another?

Well. . . my advice is to never take advice! I really don't like giving advice, but what I'd much prefer is if somebody needs advice, that probably means they need discussion, to throw some ideas around, and that's a very individual thing, as advice tends to not take into account personal situations and context.

So, avoiding giving advice makes it difficult to answer your question! But I suppose I can say some of the things you need to consider. If your circumstances permit it, I would suggest travelling to gain different experiences. I've worked in the UK and in America as well as extended visits to China and India, and I think this is great for meeting different people and learning different aspects of one's own subject. How you do things at your home institution will vary to other places you visit. We tend to hold dear to things that in other places they might not really care about, and you think, "Gee I thought this was terribly important—and they don't even know about it!" Yet they seem to survive; in fact, they're flourishing! So, it prompts you to think: "What's going on?"

It's good to have these varied experiences, bearing in mind that I'm not saying that you have to. You can have an *unvaried* experience, and you can read about these people who do brilliant work. There was an amazing case a couple of years ago of a guy, who I think dropped out of his Ph.D. program, and just went and lived in some part of his city, and 30 years later he wrote this amazing piece of mathematics, which got him all sorts of prizes, even though he had been almost a total recluse for that period. Though he had limited interactions with people, he plugged away at his topic and did brilliant work.

As for interdisciplinary work, well, I just find it fun! I like talking to people from other parts of science, engineering and technology; I like seeing the relevance of what I do broadly. Rather than wanting to chisel away at a very narrow area, I like breadth and relevance and applications. So the interdisciplinary side of what I do follows from how I view my discipline. I mean, if you're doing number theory, you'd look around and see that you can apply it to coding and so on, but I don't think a number theorist has to feel interdisciplinary in the same way I do as a statistician, because statistics is somewhat transparently interdisciplinary right from the word go.

If there is some implicit advice there, then that is to find a place where it is easy to be interdisciplinary, and yet be aware that it is a two-edged sword: it is possible to be *too* interdisciplinary and not be, say, 'deep' in any particular discipline. I do like the idea of depth as well, and it almost necessarily requires specialization. Interdisciplinary depth... it's sort of hard to put your finger on it, but it's one of those things that you know when you see it!

— Ruwan Devasurendra

The generation of random numbers is too important to be left to chance.
— Robert Coveyou

Film Review: *Agora* (2009)

Agora is a 2009 Spanish English-language historical drama directed by Alejandro Amenabar (*The Others*, *The Sea Inside*) set in Alexandria, Egypt in 391 AD at the height of the Roman Empire in the region. It depicts an epic struggle between mathematics and astronomy, religion, politics, and moral philosophy, as well as the role they played in the 4th Century and continue to play today.

The film is divided into two halves, with both halves concerning the intellectual struggle of Hypatia (Rachel Weisz), a mathematician, astronomer, philosopher and teacher, and the battle between Christians, Jews, and polytheistic Pagans for power in Alexandria. The two halves are separated by approximately 10 years chronologically: the first half centres on the context of Hypatia teaching her pupils, and the second half around the lives of influence that her students lead upon graduation.

Hypatia contrasts and debates (with herself mainly, but also with her slave and assistant, Aspasius) the Ptolemaic model of the solar system with the heliocentric theories of Aristarchus, as well as the "perfection" of the circle. She also explains an Apollonian cone and conducts an experiment where she shows that the landing position of an item dropped from the top of a mast of a moving boat is not affected by the fact the boat is moving.

Much of the film involves the juxtaposition of "peddling faith" and science with its atheist "belief in philosophy" rather than a prescribed religion, and draws similarities in conflicting religious ("Jesus was a Jew") and scientific views ("heavens should be simple") of the day. It also highlights that Euclid's

notion of equality (from *Elements*: "things which equal the same thing also equal one another") can be applied to contexts outside mathematics.

Agora also delves into the role of women in Alexandria, and is made strikingly clear by the director by an almost-complete absence of women in the film, with the exception of Hypatia, who is in a privileged position being the daughter of Theon of Alexandria (Michael Lonsdale), a respected academic and teacher.

The film is well-shot and the scale of the set is impressive (which the many wide-angled shots allow the viewer to admire); its large budget (the largest in Spanish history) has been well spent on costumes and props. Although filmed in Malta, it presents a believable likeness to how I imagined Alexandria.

The basic plot of the film is largely historically true (dramatized obviously) but the names and roles of the characters (particularly in the second half) align with what history tells us about them. But don't do too much Googling before watching the film as you may accidentally spoil the suspense the film creates as well as its ending!

This film has something for everyone: mathematics, romance, a great score by Dario Marianelli (*Atonement*, *V for Vendetta*, *Anna Karenina*), great cinematography, violence, and enough tomato sauce for a large BBQ! It is available on iTunes and DVD; it was on special for $10 at a popular Melbourne DVD retailer rather recently! Highly recommended.

— Ben Hague

Augustin-Louis Cauchy (1789–1857)

As to the methods, I have sought to endow them with all the rigour that is required in geometry…

This quotation,[1] taken from Cauchy's *Cours d'Analyse de l'ecole royale polytechnique*, exemplifies his greatest contribution to analysis: the introduction of rigour. This article will examine how this rigour changed analysis, as well as how Cauchy gave analysis new, stronger foundations, concluding with how these ideas continued after his death.

The introduction of rigour

The 19th Century principle of rigour was indeed a mighty one: intuition could not be included in a proof, and each step in a proof would have to be justified by a definition, theorem or axiom.[2] This ideal seems largely absent from 18th Century analysis; the focus on using the powerful tool of calculus for applications spawned the conclusion that results in analysis were true if they held in empirical tests.[3] But in 1821 Cauchy changed this, introducing rigour and the ideal just spoken of. Predominantly, he did this during his years teaching at the *École Polytechnique*, from his appointment in 1815 until his self-imposed exile from France in 1830.[4]

Cauchy's rigour allowed him to make the ideas and proofs in analysis much more precise. While his definitions lacked the ε-δ notation were so familiar with, Cauchy would often translate his definitions into this notation when he used them in his proofs.[5] Cauchy's rather wordy definition of the derivative, for instance, (which is too long to include here verbatim) centres on the limit of the difference quotient, where the two terms are infinitesimals, $\Delta x = i$, f is continuous over an interval of x, and the quotient limit is the derivative:[6]

[1]Grabiner notes that Cauchy is speaking about the logical structure of Greek geometry in Judith Grabiner, *The Origins of Cauchy's Rigorous Calculus* (Cambridge: The MIT Press, 1981)., 6.

[2]Ibid., 5.

[3]Ibid., 17.

[4]For a highly-detailed biography explaining the complexities of Cauchy's life, both political and otherwise, see Bruno Belhoste, *Augustin-Louis Cauchy: A Biography*, trans. Frank Ragland (New York: Springer-Verlag, 1991).

[5]Grabiner, *The Origins of Cauchy's Rigorous Calculus*., 115.

[6]For Cauchy's full definition see Jesper Lützen, "The Foundation of Analysis in the 19th Century," in *A History of Analysis*, ed. Hans Niels Jahnke (USA: American Mathematical Society, 2003)., 158–9.

$$\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i}$$

However, he translates this definition into ε-δ notation when he proves that

$$A \leq \frac{f(X) - f(x_0)}{X - x_0} \leq B$$

where $f(x)$ is continuous on the interval $x \in [x_0, X]$ and $f'(x)$ has local minimum A and maximum B:

Proof. Designate by δ and ε two very small numbers; the first being chosen in such a way that, for numerical values of i less than δ, and for any value of x between x_0 and X, the ratio $\frac{f(x+i)-f(x)}{i}$ always remains greater than $f'(x) - \varepsilon$ and less than $f'(x) + \varepsilon$.[7]

□

A new foundation for analysis

Cauchy applied his rigour to his definitions to give analysis a stronger foundation, but his new definitions also gave analysis a completely new foundation. In line with some before him, Cauchy based his calculus on limits; however, his definition of a limit removed some of the restrictions of the 18th Century concept.[8] For example, his definition permitted that a variable could achieve a value greater than its limit, hence allowing variables such as convergent alternating series to oscillate about their limit.[9]

But his definition of continuity—which was central to many of his proofs—differed greatly from the widely-accepted view of continuity in the 18th Century, that of continuity in the Eulerian sense. Euler, characteristic of the 18th Century attempt to base analysis on algebra, defined a continuous function as one which was described with one analytic expression. For instance, $y = x^2 + x^3$ was deemed continuous, but, as it was defined as two analytic expressions,

[7] Cauchy, 1823 trans. Grabiner in Grabiner, *The Origins of Cauchy's Rigorous Calculus.*, 115.
[8] Ibid., 9.
[9] Ibid.

$$f(x) = \begin{cases} x, & x \geq 0 \\ x^2, & x \leq 0 \end{cases}$$

was considered discontinuous.[10] However, Cauchy noted that this definition was unclear, because a function could be both continuous and discontinuous, depending on how it was written. He gave the example of $|x|$:[11]

$$f(x) = \begin{cases} -x, & x \geq 0 \\ x, & x \leq 0 \end{cases} = |x| = \frac{2}{\pi} \int_0^\infty \frac{x^2 \, \mathrm{d}t}{t^2 + x^2}$$

Countering this definition of continuity, Cauchy gave his own, which is much closer to our modern one, and essentially states that a function $f(x)$ is continuous over a given interval if an infinitely small increase in x is associated with an infinitely small increase in f.[12]

His legacy

The most important person to continue Cauchy's ideas was Karl Weierstrass (1815–1897); in essence, he finished what Cauchy started. Most importantly, he made the ε-δ method ubiquitous, using it in his definitions and many proofs (compare Cauchy's and Weierstrass' respective definitions of continuity).[13] Furthermore, he solved the problems in some of Cauchy's proofs—like the existence of a definite integral for continuous functions—which were caused by a lack of understanding of the completeness of the real numbers, through his construction of these numbers.[14]

While this article has omitted many points (the most glaring of which is his contribution to Complex Analysis), I hope it has illuminated some of Cauchy's fundamental changes to analysis, especially through his introduction of rigour, using his definitions to set analysis on a new foundation, and the important point that his ideas continued after him.

— Sam Lyons

[10] Hans Niels Jahnke, "Algebraic Analysis in the 18th Century," in *A History of Analysis*, ed. Hans Niels Jahnke (USA: American Mathematical Society, 2003)., 124.

[11] Cauchy, 1844 in Lützen, "The Foundation of Analysis in the 19th Century.", 165.

[12] Cauchy, 1821, in ibid., 158–9. The full definition is found here.

[13] These definitions can be found in ibid., 158–9, 186 respectively.

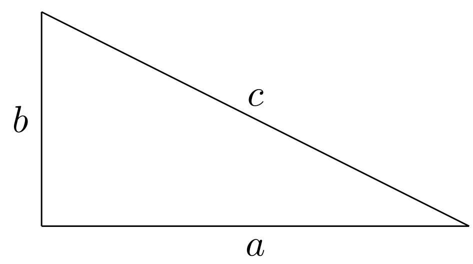
[14] Ibid., 185.

Pythagoras' Theorem and its Generalizations

Pythagoras' theorem is probably the most famous theorem in Euclidean geometry. In its most familiar form, it states that in a right-angled triangle, the square of the length of the hypotenuse is equal to the squares of the lengths of the other two sides. In other words:

$$a^2 + b^2 = c^2$$

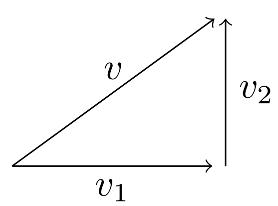
where a, b, and c are as in the picture below.



In this article, I'll describe two generalizations of this theorem in higher dimensions, and explain how the two statements are intimately linked via a kind of duality.

First generalization: lengths of vectors

Throughout this article, we'll be working a lot with the vector form of Pythagoras' theorem. Suppose we have two orthogonal vectors v_1 and v_2. Then, writing $v = v_1 + v_2$, we have a diagram as follows:



By Pythagoras' theorem, we therefore have:

$$\|v\|^2 = \|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2,$$

where $\|v\|$ denotes the length of the vector v. Our first generalization of Pythagoras' theorem is:

Theorem 1. *Let $v_1, v_2, \ldots, v_n$ be orthogonal vectors (possibly in a high-dimensional space). Then:*

$$\|v_1 + v_2 + \cdots + v_n\|^2 = \|v_1\|^2 + \|v_2\|^2 + \cdots + \|v_n\|^2.$$

Proof. Since $v_1, v_2, \ldots, v_n$ are all orthogonal to one another, it follows that $v_1 + v_2 + \cdots + v_{n-1}$ is orthogonal to v_n. Hence:

$$\|v_1 + \cdots + v_{n-1} + v_n\|^2 = \|v_1 + \cdots + v_{n-1}\|^2 + \|v_n\|^2.$$

Repeating this argument several times to break off each term, we get the desired result. □

For those who prefer a more geometric statement, we can restate Theorem 1 as the following:

Theorem 2. *Suppose we have an n-dimensional rectangular prism with sides of lengths $a_1, a_2, \ldots, a_n$ and diagonal of length c. Then:*

$$a_1^2 + a_2^2 + \cdots + a_n^2 = c^2.$$

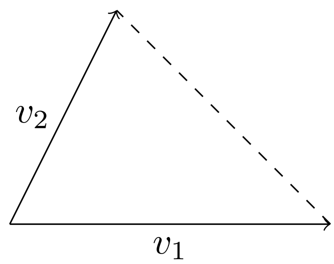
Proof. This immediately follows from Theorem 1: we can represent the sides of the rectangle by orthogonal vectors $v_1, v_2, \ldots, v_n$, with $\|v_i\| = a_i$ for each $i = 1, 2, \ldots, n$. The diagonal is then given by the vector $v_1 + v_2 + \cdots + v_n$, so we have:

$$c^2 = \|v_1 + v_2 + \cdots + v_n\|^2 = \|v_1\|^2 + \|v_2\|^2 + \cdots + \|v_n\|^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

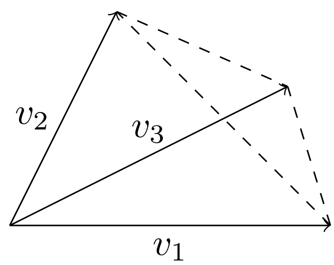
□

Second generalization

The second generalization of Pythagoras' theorem can be stated in terms of higher-dimensional solids called simplices. A 1-dimensional simplex is a line, a 2-dimensional simplex is a triangle, and a 3-dimensional simplex is a tetrahedron. If we have two vectors v_1 and v_2, we can consider the triangle that they span:



If we have three vectors v_1, v_2 and v_3, they span a tetrahedron in a similar way:



In general, n vectors $v_1, v_2, \ldots, v_n$ span an n-dimensional simplex. If the vectors $v_1, v_2, \ldots, v_n$ are orthogonal, let's call the simplex they span an *orthogonal simplex*. So, for example, a 2-dimensional orthogonal simplex is a right-angled triangle.

We can also generalize the notions of length, area, and volume to an n-dimensional simplex. I won't go into exactly how this is done, but it's basically the same as in dimensions 1, 2, and 3. In general, we'll talk about the *volume* of an n-dimensional simplex. (So, in particular, if $n = 1$, the "volume" is the length of the line segment, and if $n = 2$, the "volume" is the area of the triangle.)

Now, an n-dimensional simplex spanned by vectors $v_1, v_2, \ldots, v_n$ has $n + 1$ faces, each an $(n-1)$-dimensional simplex. n of these are adjacent to the vertex at the origin: these are spanned by some collection of $n - 1$ of the vectors $v_1, \ldots, v_n$. The remaining face is opposite the origin, and is spanned by the vectors $v_2 - v_1, v_3 - v_1, \ldots, v_n - v_1$. Just as we can talk about the lengths of the edges of a triangle, and the areas of the faces of a tetrahedron, we can talk about the volumes of the faces of an n-dimensional simplex. The second generalization of Pythagoras' theorem is the following:

Theorem 3. *If the faces of an n-dimensional orthogonal simplex adjacent to the origin have volumes $a_1, a_2, \ldots, a_n$, and the opposite face has volume c, then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 = c^2.$$

Proof of Theorem 3 in three dimensions

Note first that the statements of Theorems 2 and 3 appear to have a lot of similarities. This is no coincidence: we will see that these are "dual" statements in some sense. In fact, we will prove Theorem 3 by applying Theorem 1 to vectors representing the faces of the simplex, just as we proved Theorem 2 by applying Theorem 1 to vectors representing the edges of the prism. To get us warmed up for this proof for arbitrary dimension n, we'll first see how it works in the more familiar 3-dimensional case.

In 3 dimensions I'll construct vectors representing the sides of a tetrahedron using the *cross product*, or *vector product*. For those who aren't familiar with it, the cross product of vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ is given by:

$$v \times w = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1).$$

This has the following properties:

1. $(au+bv) \times w = a(u \times w) + b(v \times w)$ and $u \times (av + bw) = a(u \times v) + b(u \times w)$ for all $a, b \in \mathbb{R}$ and all $u, v, w \in \mathbb{R}^3$.
2. $v \times w = -w \times v$ for all $v, w \in \mathbb{R}^3$.
3. $v \times v = 0$ for all $v \in \mathbb{R}^3$.
4. If v_1, v_2 and v_3 are orthogonal unit vectors, then so are $v_1 \times v_2$, $v_1 \times v_3$ and $v_2 \times v_3$.

From the properties above, it follows[1] that the area of the triangle spanned by vectors v_1 and v_2 is given by:

$$A = \frac{1}{2}\|v_1 \times v_2\|.$$

Now suppose we have orthogonal vectors v_1, v_2, v_3 spanning a tetrahedron with faces adjacent to the origin of areas a_1, a_2, and a_3, and opposite the face of area c. Now, the faces adjacent to the origin are the triangles spanned by any two of the three vectors v_1, v_2 and v_3, so using the cross product we can write:

$$a_1 = \frac{1}{2}\|v_2 \times v_3\|, \quad a_2 = \frac{1}{2}\|v_1 \times v_3\|, \quad a_3 = \frac{1}{2}\|v_1 \times v_2\|.$$

[1] Showing this is an interesting exercise, which I'll leave to the reader.

On the other hand, the face opposite the origin is spanned by the vectors $v_2 - v_1$ and $v_3 - v_1$, so we have:

$$\begin{aligned} c &= \frac{1}{2}\|(v_2 - v_1) \times (v_3 - v_1)\| \\ &= \frac{1}{2}\|v_2 \times v_3 - v_1 \times v_3 - v_2 \times v_1 + v_1 \times v_2\| \\ &= \frac{1}{2}\|v_2 \times v_3 - v_1 \times v_3 + v_1 \times v_2\|. \end{aligned}$$

Now, since v_1, v_2 and v_3 are orthogonal vectors, it follows from properties 1 and 4 that $v_2 \times v_3$, $-v_1 \times v_3$ and $v_1 \times v_2$ are also orthogonal. Hence, by Theorem 1, we have:

$$\begin{aligned} c^2 &= \frac{1}{4}\|v_2 \times v_3 - v_1 \times v_3 + v_1 \times v_2\|^2 \\ &= \frac{1}{4}\|v_2 \times v_3\|^2 + \frac{1}{4}\|v_1 \times v_3\|^2 + \frac{1}{4}\|v_1 \times v_2\|^2 \\ &= a_1^2 + a_2^2 + a_3^2 \end{aligned}$$

which proves Theorem 3 in the three-dimensional case.

Wedge products

The key step in the proof of Theorem 3 in three dimensions was to use the cross product to write the areas of the faces of a tetrahedron in terms of the lengths of vectors, to which we could apply a more familiar version of Pythagoras' theorem. To make this method work in higher dimensions, we need something analogous to the cross product. Since we would like to use this to measure volumes of simplices, this "product" should take in $n-1$ vectors $v_1, \ldots, v_{n-1}$ in $\mathbb{R}^n$ and output a vector whose length is related to the volume of the $(n-1)$-dimensional simplex spanned by $v_1, \ldots, v_{n-1}$.

The product we will use is called the *wedge product*. For the purposes of this article, we can think of this as a map:

$$\begin{aligned} \overbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{n-1 \text{ times}} &\to \mathbb{R}^n \\ (v_1, v_2, \ldots, v_{n-1}) &\mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_{n-1}. \end{aligned}$$

It is possible to write down fairly simple formulas for the wedge product, such as:

$$v_1 \wedge \cdots \wedge v_{n-1} = \det \begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ v_{1,1} & v_{1,2} & \cdots & v_{1,n} \\ \vdots & \vdots & & \vdots \\ v_{n-1,1} & v_{n-1,2} & \cdots & v_{n-1,n} \end{pmatrix}$$

where $v_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,n})$ and e_i is the unit vector pointing along the i^{th} coordinate axis in $\mathbb{R}^n$. Here we interpret the determinant by taking a cofactor expansion across the top row to get a linear combination of basis vectors. The wedge product satisfies the following properties analogous to those of the cross product:

1. If $v_1, v_2, \ldots, v_{n-1}, w \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, then:

$$\begin{aligned} v_1 \wedge \cdots \wedge v_{i-1} \wedge (av_i + bw) \wedge v_{i+1} \wedge \cdots \wedge v_{n-1} = \\ a(v_1 \wedge \cdots \wedge v_{i-1} \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_{n-1}) \\ + b(v_1 \wedge \cdots \wedge v_{i-1} \wedge w \wedge v_{i+1} \wedge \cdots \wedge v_{n-1}). \end{aligned}$$

2. $v_1 \wedge \cdots \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_{n-1} = -v_1 \wedge \cdots \wedge v_{i+1} \wedge v_i \wedge \cdots \wedge v_{n-1}$ for all $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$.
3. $v_1 \wedge \cdots \wedge v_{n-1} = 0$ if we have $v_i = v_j$ for some $i \neq j$.
4. If $v_1, v_2, \cdots, v_n \in \mathbb{R}^n$ are orthogonal unit vectors, then so are the vectors

$$\hat{v}_i = v_1 \wedge v_2 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_n$$

for $i = 1, 2, \ldots, n$.

Just as for the cross product, these properties imply that the volume of the simplex spanned by vectors: $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ is given by:

$$V = \frac{1}{n-1} \|v_1 \wedge \cdots \wedge v_n\|.$$

Proof of Theorem 3: general case

Suppose we have an n-dimensional simplex spanned by n orthogonal vectors $v_1, v_2, \ldots, v_n$, with faces adjacent to the origin with volumes $a_1, a_2, \ldots, a_n$, and face opposite the origin with volume c as in the statement of Theorem 3. Then we have:

$$a_i = \frac{1}{n-1} \|v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_n\|$$

and

$$c = \frac{1}{n-1} \|(v_2 - v_1) \wedge (v_3 - v_1) \wedge \cdots \wedge (v_n - v_1)\|.$$

So, using properties 1, 2, and 3, we have:

$$\begin{aligned} c^2 &= \left(\frac{1}{n-1}\right)^2 \|(v_2 - v_1) \wedge (v_3 - v_1) \wedge \cdots \wedge (v_n - v_1)\|^2 \\ &= \left(\frac{1}{n-1}\right)^2 \left\| v_2 \wedge v_3 \wedge \cdots \wedge v_n - \sum_{i=2}^{n} v_2 \wedge \cdots \wedge v_{i-1} \wedge v_1 \wedge v_{i+1} \wedge \cdots \wedge v_n \right\|^2 \\ &= \left(\frac{1}{n-1}\right)^2 \left\| \sum_{i=1}^{n} (-1)^{i-1} v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_n \right\|^2 . \end{aligned}$$

Since $v_1, v_2, \ldots, v_n$ are orthogonal, by properties 1 and 4, so are the vectors:

$$v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_n$$

so by Theorem 1, we have:

$$\begin{aligned} c^2 &= \left(\frac{1}{n-1}\right)^2 \sum_{i=1}^{n} \|(-1)^{i-1} v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_n\|^2 \\ &= \sum_{i=1}^{n} \left(\frac{1}{n-1}\right)^2 \|v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_n\|^2 \\ &= a_1^2 + a_2^2 + \cdots + a_n^2. \end{aligned}$$

Duality

The two generalizations of Pythagoras' theorem exhibit a kind of duality. We can move from one, which concerns "volumes" (or lengths) of 1-dimensional things, to the other, which concerns volumes of $(n-1)$-dimensional (or *codimension 1*) things, by replacing the set of vectors:

$$\{v_i \,|\, i = 1, 2, \ldots, n\}$$

by the set of vectors:

$$\{v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_n \,|\, i = 1, 2, \ldots, n\}.$$

Duality of this kind, where objects of dimension 1 are interchanged with objects of codimension 1, is widespread in mathematics. As we have seen, it underlies the relationship between two simple generalizations of Pythagoras' theorem. Other examples include the notion of dual polyhedra and duality in projective geometry,[2] as well as many more rich and subtle theories such as Poincaré duality in cohomology theory.

— Dougal Davis

A father who is very much concerned about his son's bad grades in maths decides to register him at a Catholic school. After his first term there, the son brings home his report card: he's getting "A"s in maths now.
The father is, of course, pleased, but wants to know: "Why are your math grades suddenly so good?" "You know", the son explains, "When I walked into the classroom the first day, and I saw that guy on the wall nailed to a plus sign, I knew one thing: this place means business!"

Heap Big Indian chief had his three squaws sleeping on animal hides in his tepee. The first squaw slept on a cow hide, the second squaw slept on a horse hide, and the third squaw slept on a hippopotamus hide. By and by, they all became pregnant and presented the chief with his heirs. The squaw who slept on the cow hide had a son, the squaw who slept on the horse hide had a son, but the squaw who slept on the hippopotamus hide had twin sons...
Proof: The squaw of the hippopotamus is equal to the sons of the squaws of the other two hides.

[2] Google these if you want to learn more!

Blaise Pascal (1623–1662)

> *To speak frankly of geometry, I find it the highest exercise of the mind [. . .] but there is also in my case, that I am pursuing studies so distant from that frame of mind that I hardly remember what there is to it.*[1]

Pascal was an influential French mathematician perhaps most well known for his work on conics, probability and fluids, and also more popularly so for his eponymous Triangle. Apart from his work in mathematics, Pascal was an avid physicist, prose writer, and religious philosopher; the pursuits of the latter two caused some controversy towards the later portion of his life.

Conics and calculators

Some of Pascal's very early influences were the works of Girard Desargues, a fellow French mathematician who specialized in the field of projective geometry.[2] Specifically, Desargues' own work on conic sections spurred Pascal to work on what was known as the 'Mystic Hexagram'. He proved that "if a hexagon is inscribed within a conic, the three points of intersection of opposite pairs of sides [. . .] of that hexagon will always lie on a straight line."[3] He was sixteen when he submitted this result to Père Mersenne, and it is now known as Pascal's Theorem.

Before he turned twenty, Pascal also developed a mechanical calculator, with the intention that it would unburden his father from the repetitive sums he had to complete in his role as a tax commissioner. Pascal's calculator, or the Pascaline as it became known, was accurate and time-saving, however the machine failed to gain commercial success; an issue often attributed to the high manufacturing costs of its interior machinery.[4]

[1] Pascal in Nelson, J. *Pascal: Adversary and Advocate* (1981), 2.

[2] "The branch of mathematics that deals with the relationships between geometric figures and the images, or mappings, that result from projecting them onto another surface. Common examples of projections are the shadows cast by opaque objects and motion pictures displayed on a screen." *Encyclopaedia Britannica Online*.

[3] Quotation and image available from Adamson, D. *Blaise Pascal* (1995), 3.

[4] For a carefully detailed explanation of the internal workings of the Pascaline, see Nicole Ketelaars's paper *Pascal's Calculator*, AIMe Magazine 2001/2 [Online].

The famous triangle

In 1653, Pascal developed a visual representation of the binomial coefficients in his *Treatise on the Arithmetic Triangle*. In addition to the beauty inherent in its structure, Pascal's triangle holds a multitude of patterns and references to other areas of mathematics, such as having a relation to fractals, a particular Fourier transform, and the exponents of 11.

On chance

In 1654, Pascal collaborated with Fermat to analyse mathematical gambling problems presented to them by their friend, the Chevalier de Méré, and this formed the basis of the 'Problem of Points'[5] which he discussed in the lattermost third of his *Treatise*. His study on the probabilities associated with the game introduced the notion of the 'expected value' into mathematics, and thus Pascal and Fermat are often attributed as setting the probabilistic foundry for mathematicians such as Huygens, Bernoulli, de Moivre, and their successors.

Fluids and vacuums

Pascal's work with fluids was extensive enough that he developed a law of his own, which stated that where there is an increase in pressure at any point in a confined fluid, there is an equal increase at every other point in the container. This was indicated by his 1646 'barrel experiment', where he connected a vertical tube to a water-filled barrel, and then poured more water down this tube. The barrel then burst, causing Pascal to infer that the pressure increase at the junction where the barrel met the vertical pipe had indeed transmitted equally to all other points in the barrel.

Furthermore, Pascal disputed the Aristotelian concept of *horror vacui*: the belief that nature abhors a vacuum. By repeating the experiments of Galileo and Torricelli, and by conducting barometric research of his own,[6]Pascal saw no evidence for the purported 'invisible matter' that would supposedly take up what he saw as a vacuum.

[5]For a detailed explanation of the Problem, see `http://statprob.com/encyclopedia/blaisepascal.html`.

[6]Some simulated calculations of Pascal and Perier's barometer experiment can be found at `http://www.thermospokenhere.com/wp/02_tsh/B130___puydedome/puydedome.html`.

Writings, religion, and philosophy

Pascal identified with the Jansenist movement within Catholicism at around 1646, but due to the death of his beloved father, the ongoing disputes with his sister, and because of his own ill health, Pascal's faith began to waver. It was a particularly vivid religious experience[7] in 1654 which reignited his devotion to his religion, thus sparking the writing of the *Provincial Letters* and the *Thoughts*.

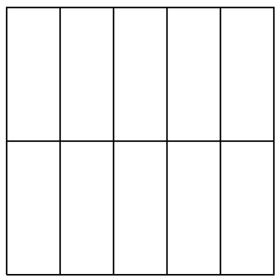
The *Letters* were a series of witty attacks on aspects of Catholic ethical reasoning; and as they were polemical pieces, they formed "part of a long running battle which did not end with the last of the eighteen completed letters".[8] *Thoughts*, on the other hand, took a much calmer approach to defend the Christian faith. By utilising philosophical arguments to logically assess religion, it is regarded by many to be his theological and literary magnum opus.

— Ruwan Devasurendra

Knot 2 (with apologies to Lewis Carroll)

Now this pie, as it happened, had been baked in a square dish, which presented the gnomes with something of a problem when dividing it fairly between them. Every gnome learns how to divide up a round pie into equal parts so that every piece had the same amount of crust almost as young as he learns to fish, but they were less sure how to divide a square pie. One gnome volunteered a measuring tape which he had been using to measure out his fishing lines that morning, in case it would be any help. All the gnomes agreed that of course it should, but didn't quite see how.

The first suggestion made was that they should cut the pie in half, then cut each half into fifths to get something like

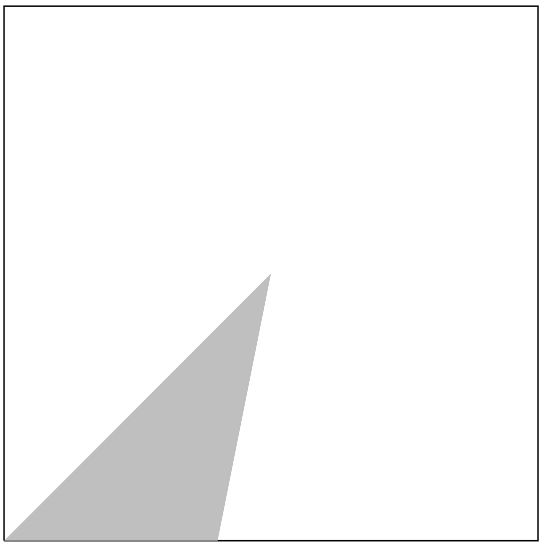


[7]Accounts vary, but it is generally attributed as a religious dream that resonated profoundly with the faith he had been out of touch with over the preceding years.

[8]Broome, J.H. *Pascal* (Scotland, 1965), 107.

This was, of course, unfair since some gnomes would get more crust than others, so they tried to think of something else. In order to get the same amount of crust on each piece, one gnome suggested that they should simply do as they would with a round pie: cut each slice to the centre and give each gnome the same length of crust around the outer edge. This suggestion was immediately disregarded as naïve by most of the ten gnomes, as the pieces would all be different shapes.

The most mathematically inclined of the gnomes was, however, less quick to dismiss this idea. After thinking for a while, he announced that this last proposal was indeed fair, as each piece would contain exactly the same amount of crust, and exactly the same amount of pie. When pressed as to why this was the case, he gave the following explanation: any piece of pie cut by this method which lay all on one side of the pie would look something like the shaded triangle below.



Since the altitude of the triangle would always be half the width of the pie, the area of any piece would simply be proportional to the length of crust around the edge. If they cut all the pieces with the same length of crust, all the pieces would therefore have the same area, and hence contain the same amount!

The gnomes were all quite satisfied with this solution, and were soon munching away on their evenly cut pieces of pie. Just as the last gnome was finishing on his last piece of crust, the cottage door was flung open with a terrible crash.

Through the open door strode a witch. She was tired and hungry after a long day spent casting spells on princesses and the like, and was in a terrible mood.

She was therefore very unhappy to see her dinner being gobbled by a bunch of nasty little gnomes who had broken into her house.

The terrified gnomes stared in horror at this fearsome apparition, and found themselves rooted to the spots where they had sat eating the pie. The witch gazed down at them with look of cold rage, contemplating how best to punish the gnomes for stealing her dinner. A wicked smile spread across her face as she was struck by an idea.

"Well, will you look at that," she said in a voice dripping with cool menace, "my pie seems to have vanished! Whatever will I do about dinner? I suppose I could make another pie, if only there were any meat left in the house... Perhaps I shall just have to go hungry," she said with a mock tear.

One of the gnomes, who had heard stories about witches' diets, didn't like where this was going.

"Though, I could always get the meat of some of these gnomes lying about!" cackled the witch. "Of course", she continued more quietly, "I shall not need to eat all of them. I'll have to choose which ones to eat and which ones to release somehow." She turned to the gnomes.

"How about we play a little guessing game?" said the witch sweetly. "I'll go and fetch some hats: some of them will be white and the rest of them will be black. Then you'll all line up one in front of the other, and I'll pop a hat on each of your heads. Starting from the back, I'll ask each of you what colour your hat is: when I ask you, you can answer 'black' or 'white', but otherwise no talking or it's not fair. Those of you who are right can leave, and the rest of you will help me with my dinner!"

With that the witch left the room and suddenly the gnomes could move again. Some of them ran to the door, and some to the window where they had come in, but all exits were sealed tight.

"Whatever will we do?" wailed the fat gnome. "I don't fancy being made into a pie!"

To be continued...

— Dougal Davis

Upcoming MUMS Events

Annual General Meeting
Friday 23 May 1pm–2pm: Old Geology Theatre 1

It's that AGM time of the year! Most of the current committee are graduating and stepping down so we really need some awesome new committee members to keep the exciting MUMS activities going! The following positions will be vacated for the AGM:

- President
- Vice President
- Treasurer
- Secretary
- Education Officer
- Publicity Officer
- Paradox Editor
- Up to 2 Postgraduate Representatives
  (must be enrolled in a postgraduate course)
- Up to 6 Undergraduate Representatives
  (must be enrolled in an undergraduate course)

Comprehensive explanations of roles, the meeting agenda, and constitution are available upon request. Next week (Friday 30 May) we have our last seminar with Dr Mark Fackrell, and then our usual end-of-semester Trivia Night!

Paradox would like to thank Dougal Davis, Ruwan Devasurendra, Jenny Fan, Ben Hague, Sam Lyons, Matthew Mack, and Terry Speed for their contributions.