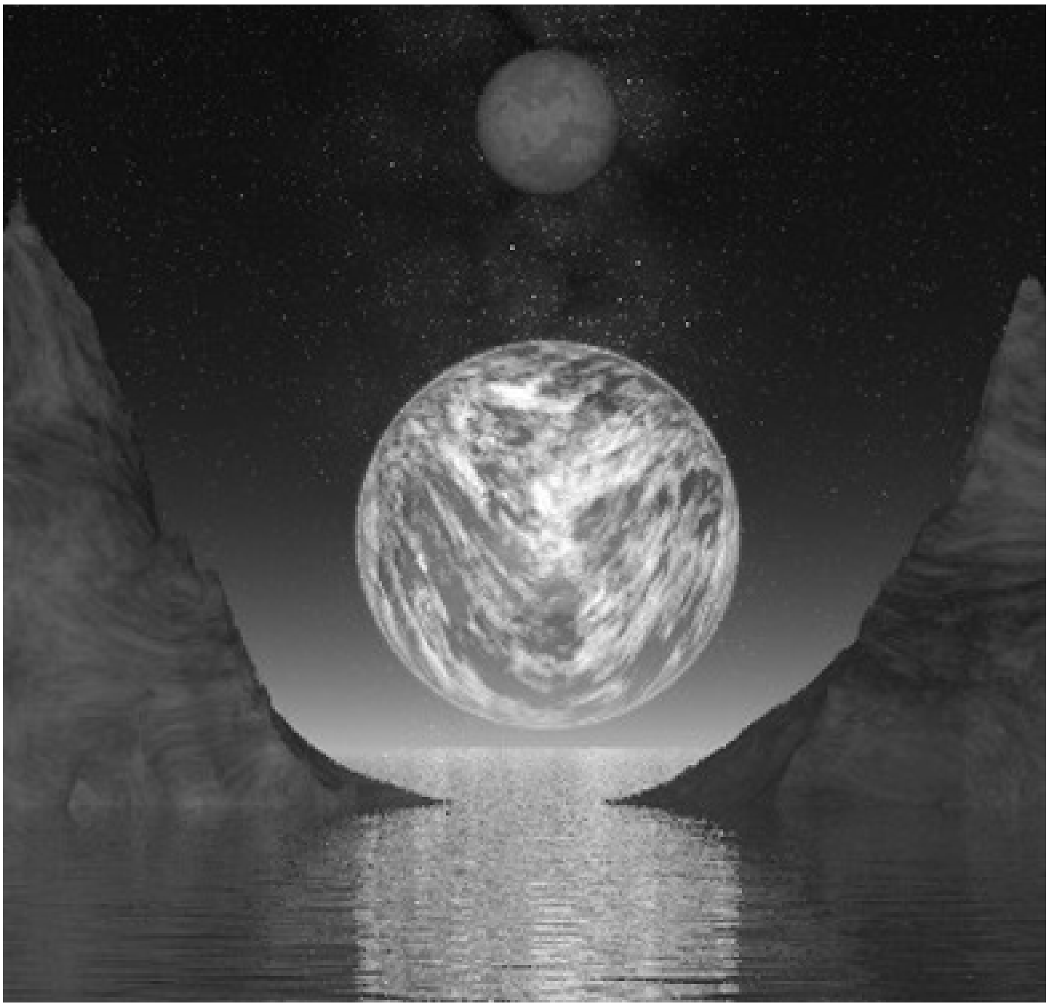
Paradox

Issue 1, 1999

The Magazine of the Melbourne University Mathematics and Statistics Society



Parabolic Curves in the Plane of the Ecliptic, Ken Musgrave, 1997.

Paradox

Editor:	Jian He
Layout:	George Doukas
Sub-Editors:	Jeremy Glick, Desmond Lun, Kuhn Ip
Artist:	Sally Miller
Web Page:	http://baasil.humbug.org.au/Mums/Paradox
E-mail:	paradox@ms.unimelb.edu.au
Printed:	February, 1999

Greetings from the Editor

Welcome to the special Orientation Edition (very limited! very collectible!) of *Paradox*. As you may or may not know, *Paradox* is the magazine of the Melbourne University Mathematics and Statistic Society (MUMS). Unlike most other magazines we do not use pictures of women intently staring at you to catch your attention, where the simple joy of mathematics would more than suffice (OK so this is a shameless lie but at least maths doesn't stare). We try to make each issue as entertaining as possible while at the same time thought-provoking. Hence there will be comics, jokes, puzzles, problems and not least of all paradoxes, some of which could earn you up to $20 for being the first to submit a correct solution. We would also like to have more contributors. Anyone interested can contact us at `paradox@ms.unimelb.edu.au`. Suggestions, submissions and complaints can also be forwarded to the above address or the MUMS pigeon-hole next to the Maths and Stats Office in the Richard Berry building.

— Jian He, *Paradox* Editor

A Word from the Head of Department

Welcome!

A few more words for the newcomers:

It's all a bit different from what you have experienced at school: huge class sizes, rapid pace, and a mix of the old and familiar with totally new things that may seem quite difficult at first. If you put in enough time and effort, you will find our subjects accessible and fulfilling, but there are some tricks you need to know.

1. Your lecturers are actually quite nice people who are delighted to help you. They should announce office hours during which they are "at home" for students to drop in and talk about their mathematics problems, and not enough students take advantage of this.

2. There's lots of individual and group help available through the First-year Learning Centre. If you haven't found it yet, just look for the "Purple Peril" ceiling decoration on the ground floor at the main corridor intersection.

3. You don't "learn" much in lectures, but then again, that's not what lectures are for. Lectures are to guide you towards your own learning: it is working through the lecture material in your own time, puzzling over the problem sheets or assignments, chatting with fellow students outside class, attending and participating in small group tutorials, and so on, that really turns seeing and hearing the material into knowing and understanding the material. But don't skip lectures, or think that lectures are just about copying stuff down from the board or the screen. Listen to the words and try to soak in the basic ideas as you go. Then get busy and do your bit!

A few more words for the veterans:

You won't be seeing too many changes, except for the relocation of our computer laboratories from the Alice Hoy Laboratory to the Richard Berry Building, near the student photocopy machines, and the relocation of the Note Room from G53 (opposite the vending machines) to G65 (around the corner from the vending machines). But there have been a few changes that you won't notice, including big changes in first year subjects.

A few last words:

There'll be plenty of advice around later in the year about career prospects, subject choice, and a whole lot more. There'll even be fun, games and the ritual public humiliation of senior staff (it's called the Maths Olympics, in semester two). And there are lots of other good things associated with the Melbourne University Mathematics and Statistics Society (like this magazine and the Friday afternoon seminars about interesting things), so do join in.

I'm looking forward to a great year, and hope you enjoy it as much as I plan to.

— Barry Hughes, Associate Professor and Head of Department

Chaitanya says ... a message from the president

Greetings, and a special welcome to those reading *Paradox* for the first time. For those of you who haven't heard of MUMS before, our job is to encourage your interests in maths and stats. Membership is automatic once you are a student in this department, and anyone else interested is welcome to join. The committee works hard to organise a range of activities, which we hope you find entertaining or thought-provoking.

Paradox is one such fine example. *Paradox* has advanced over recent times to become a must-read for anyone fascinated by the beauty (and jokes!) that the mathematical world has to offer. I encourage you to have a go at the problems and puzzles given, and even to contribute to future issues!

So what else does MUMS have in store for this year? We will continue to run our regular seminars on Friday afternoons, followed by free refreshments. Given by students, staff and visitors, we aim to cover a wide range of topics so that at least some should appeal to most undergraduate and postgraduate students. They also give you ideas of some of the research interests in the maths and stats department.

But as usual, our major activity for the year will be the popular Maths Olympics. Held in second semester, teams of school and university students and staff will compete for 45 frantic minutes solving maths questions and running around Theatre A. Some sick people in the past began training for it up to six months in advance! More details next semester.

We also plan to stage another trivia competition and social functions for undergraduate and postgraduate students. For news on upcoming events, visit our web page

`http://baasil.humbug.org.au/Mums`

or our notice board in the First Year Learning Centre. If you would like to be involved in our committee, come along to our Annual General Meeting in late March. Please feel free to contact us during the year for suggestions and feedback on what we do!

— Chaitanya Rao, MUMS President `kumar@ms.unimelb.edu.au`

Maths and stats chess team

This year, MUMS would like to enter a team into the Winter Interclub Chess tournament. While winter sounds like it's a long way off, it's never too early to start training, so anyone who would be interested in playing should contact Jeremy at `jglick@ecr.mu.oz.au`, or come and find him slouched over a chess board in G08.

Send donations now!

No big effort is required, just 10c will do, but first, read the following proof ...

$$\begin{aligned} \$1 &= 100c \\ &= (10c)^2 \\ &= (\$.1)^2 \\ &= \$0.01 \\ &= 1c \end{aligned}$$

The least interesting proof

Proof: All positive integers are interesting.

Assume the opposite. That means there is some lowest non-interesting integer. This makes it interesting. Hence we arrive at a contradiction. Therefore, all positive integers are interesting.

Response: All positive integers are boring.

Assume the opposite. That means there is some lowest non-boring integer. Who cares?

One last song

$\aleph_0$ bottles of beer on the wall,
$\aleph_0$ bottles of beer,
if one of those bottles should accidentally fall,
There'd be $\aleph_0$ bottles of beer on the wall!

$2^n - 1$: Discursions

The sequence $\{M_1 = 1; M_n = 2^n - 1, n > 1\}$, which begins $1, 3, 7, 15, 31, 63, \ldots$, has a long history in tales of mathematics. One example is the legend about the inventor of chess. The story goes that the ruler of Persia so liked the game that he asked the inventor to name a reward. The reply was to ask to have one grain of wheat for the first square of the board, two grains for the second square, four for the third, eight for the fourth and so on, doubling at each square, until the inventor had been rewarded for all sixty-four squares. The ruler's beancounters, however, would have been less than impressed with this greedy mathematician when they tallied up the total amount of wheat required: the total after p squares is precisely M_p, and M_{64} grains is enough to give every person alive today more than a billion grains of wheat!

The terms of the sequence $\{M_n\}$ are now called the *Mersenne numbers* in honour of the seventeenth century French friar, great proponent of science and disseminator of mathematics, Marin Mersenne. He came to them via another set of numbers, the perfect numbers. A number is perfect if the sum of its divisors is exactly twice the number itself. For example, 6 is perfect since $1+2+3+6 = 12$, as is 28 since $1+2+4+7+14+28 = 56$. In other words a perfect number is equal to the sum of all its divisors not including itself. Over two thousand years ago Euclid had proved that if $M_n = 2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect. This result actually gives all the even perfect numbers. It is suspected but still not proven that there are no odd ones.

By the time Mersenne was writing about perfect numbers, it was known that M_n was not prime unless n itself was. The first four primes n=2, 3, 5, and 7 give the first four perfect numbers 6, 28, 16×31 and 64×127. Note, however, that the next prime $n = 11$, gives $M_n = 2^{11} - 1 = 2047 = 23 \times 89$, so prime n does not guarantee prime M_n. It was also known that $n = 13, 17$ and 19 gave prime M_n. Mersenne declared that for $n \leq 257$, the perfect numbers (i.e. those 2^{n-1} times a prime M_n) were given by

$$n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$$

There was no proof attached to this declaration, but recently a rule Mersenne may have used has been reconstructed from his later writings[1]. Mersenne's list is in error since, for n=67 and 257, M_n is composite and so $2^{n-1}M_n$ not perfect, whilst for n=61, 89 and 107 M_n *is* prime.

Nonetheless, correct prime M_n (called Mersenne primes) are still much sought after in the hunt for large prime numbers. By 1771 Euler had confirmed that indeed the eighth number on Mersenne's list by showing that M_{31} was prime. This remained the largest number known to be prime until 1851. Mersenne primes regained the ascendancy in the 1870's with the work of the French mathematician Édouard Lucas on M_{127}. Apart from two brief periods in the computer age, the largest known prime has been a Mersenne prime ever since (currently $2^{3021377} - 1$, verified in 1998).

In 1876, Lucas published a note containing results from which he thought had proven $M_{127} = 2^{127} - 1$ was prime[2]. The stated method involved the sequence of Fibonacci numbers $\{F_1 = 1, F_2 = 1; F_n = F_{n-1} + F_{n-2}, n > 2\}$ that begins $1, 1, 2, 3, 5, 8, 13, \ldots$, and showing, amongst other things, that M_{127} divides $F_{2^{127}}$ Incidentally, that Leonardo of Pisa (Fibonacci) is known around the globe today is due largely to Lucas, who affixed the name to the sequence of numbers studied by Fibonacci many centuries earlier in the 'rabbit multiplication' problem. In a later paper, Lucas gave a different method that was based upon showing that M_{127} divided $L_{2^{126}}$, where L_n is a sequence now called the Lucas numbers $\{L_1 = 1, L_2 = 3; L_n = L_{n-1} + L_{n-2}, n > 2\}$. Lucas found a recurrence for these numbers,

$$L_2 = 3; L_{2^{k+1}} = L_{2^k}^2 - 2, k \geq 1$$

from which he could find $L_{2^{126}}$. Instead of that, however, Lucas described an ingenious method for doing both steps at once, i.e. finding not just $L_{2^{126}}$ but its remainder when divided by M_{127}: he constructed a 127×127 chessboard, used modular arithmetic (mod M_{127}) via binary notation with pawns as ones and empty squares as zeros and iterated the recurrence $k = 125$. Without computers, this would have taken some days, and Lucas said he only did it once. In his later writings, he then wasn't always sure that he had proved M_{127} was prime. His method works, however, so he is usually credited with the discovery.

Over the following few years Lucas generalised his prime number tests. His ideas, particularly concerning Mersenne primes, have influenced much prime testing since. In particular, Lucas established a successful rule that could determine whether certain numbers were prime. This approach could sometimes replace the tedious method of dividing a number by all primes less than its square root (which in addition required a list of primes).

So François Édouard Anatole Lucas (1842-1891) was more than just the man who discovered the largest number known to be prime before the advent of the computer. Born in Amiens, he graduated in mathematics from the prestigious École normale in 1864 having passed the competitive exam for admission to the teaching staff of the *lycées* (high schools). Instead, though, he went to work at the Paris Observatory as an assistant. He left there to become an artillery officer in the Franco-Prussian war in 1870-1. After that he taught higher mathematics at various *lycées*, notably at Paris St. Louis (1879-1890). His unfortunate demise came after an incident in Marseilles during a meeting of the French Association for the Advancement of Science. After a dinner, a pile of earthenware plates fell, a fragment of one gashing Lucas in the cheek. He died in Paris a few days later of the blood disease erysipelas[3].

Lucas wrote papers in many subjects and was an active member of learned societies.

His main interest was in number theory. Lucas is perhaps more widely known, however, for his work on games and recreational mathematics, of which the four-volume collection *Récréations Mathématiques* is still relatively easy to find[4]. In this area, Lucas is most of all associated with the Tower of Hanoi puzzle. This provides the last tales about $2^n - 1$ here. The Tower of Hanoi puzzle comprises a tower of eight disks stacked on one of three pegs (see illustration). The largest disk is at the bottom, with others strictly decreasing in size as you go up the tower. The aim of the game is to transfer the entire tower to another peg without ever placing a larger disk on top of a smaller one. Usually a secondary aim is to transfer the tower in the minimum number of moves possible.

One way to solve this problem is to forget about eight disks and look first at smaller numbers. The puzzle is trivial with one disk, since one move will do. Three moves are required with two disks, since the larger disk must end up at the bottom. A little work shows that seven moves are required for three disks. This is achieved by moving the two smaller disks onto one peg, moving the largest disk to the destination (either non-starting peg can be the destination), then transferring the smaller two on to it. Let M_n (surprise!) denote the minimum number of moves required to move a tower of n disks. Then $M_1 = 1$, $M_2 = 3$, $M_3 = 7$.

The method used for three disks gives a general solution to the problem of n disks: first move all but the largest disk on to one peg (requiring M_{n-1} moves), then move the largest disk to the destination (one move) before finally transferring the $(n-1)$-disk tower back on to the largest disk (again requiring M_{n-1} moves). Thus $M_n \leq 2M_{n-1} + 1$ since it is *possible* to transfer an n-disk tower in $2M_{n-1} + 1$ moves. But because the largest disk *has* to be moved at least once, and when it is moved, the other $(n-1)$ must form a tower on one peg, transferring an n-disk tower requires *at least* $2M_{n-1} + 1$ moves. So the solution for the minimum number of moves possible is

$$M_1 = 1; M_n = 2M_{n-1} + 1, n > 1$$

and, yes, they are the Mersenne numbers again. To see this, add one to both sides of the recurrence equations and relabel by $N_n = M_n + 1$, giving $\{N_1 = 2; N_n = 2N_{n-1}, n > 1\}$ from which $N_n = 2^n$ so $M_n = 2^n - 1$. Solving the original puzzle, 255 moves are required to move a tower of eight disks when there are three pegs.

It is perhaps surprising that the origin of the Tower of Hanoi puzzle is still debated. Lucas marketed it from 1883 as 'The authentic brain-teaser of the Annamites, a game brought back from Tonkin by Prof. N. Claus of Siam, Mandarin of the College of Li-Sou-Tsian'[5]. There are two anagrams in the French: Lucas was born in Amiens, so "N. Claus d'Amiens" is Lucas; meanwhile "Li-Sou-Tsian" is an anagram for Saint Louis. The exoticisms and in-jokes continue in the instructions, which start 'This game was found, for the first time, in the writings of the illustrious Mandarin FER-FER-TAM-TAM, which are to be published sooner or later, by order of the government of China.'[6] Lucas regarded Fermat as 'one of the greatest geniuses of mankind' and was on the commission for the publication of Fermat's works[7]. From these, it seems unlikely that anyone other than Lucas came up with the puzzle. There is one persistent legend concerning the origin of the Tower of Hanoi, which is detailed below. In *Récréations Mathématiques*, however, Lucas wrote that the legend and the puzzle itself had both been thought up recently in Paris, and at the same time that (his nephew) Raoul Olive was the nephew of the inventor[8].

The legend? One version of the story is that under the dome of the great temple at Benares (in India), there are three needles fixed in a brass plate. The Creator placed a sixty-four disk tower on one of these needles at the Creation. Priests transfer disks according to the sacred rules, and when all sixty-four disks have been transferred to another needle, the Tower and Brahmins will fall and the world come to an end[9]. Thus the puzzle is sometimes known as the Tower of Brahma or the End of the World puzzle. Now a sixty-four disc tower requires $M_{64} = 2^{64} - 1 > 10^{19}$ moves before it is reassembled on another peg. One move each second (and no mistakes) would see the task completed in more than 500 billion years, whatever that means. Returning to the grains of wheat (also M_{64}), our greedy inventor of chess will be waiting a very long time for their reward to be counted!

NOTES

1. Drake, S. The rule behind Mersenne's numbers. *Physics-Riv. Internaz. Storia Sci.* 13:421-424 (1971) quoted in Williams, H.C., *Edouard Lucas and primality testing*, Canadian Mathematical Society series of monographs and advanced texts vol. 22, Wiley-Interscience, New York (1998).

2. Information from this section was taken from Williams *op. cit.* rather than Lucas' original papers. Satisfactory proofs of some results in this 1876 paper did not come for some decades, but the results did in the end turn out to be correct. Why Lucas chose to investigate M_{127} is uncertain; it seems he was unware of Mersenne's list at that time.

3. One-page biographies of Lucas abound. One is Campell, P. J., Lucas' solution to the non-attacking rooks problem, *J. Rec. Math.*, 9(3):195-200 (1976-77), another by Gridgeman, N.T., on p. 1624 of *Biographical dictionary of Mathematicians: reference biographies from the Dictionary of Scientific biography*, Scriber, New York (1991). Obituary: *Nature* 44:574 (1891).

4. Lucas, E. *Récréations Mathématiques*, Gauthier-Villars, Paris Vol. 1 (1891, 2^{nd} Ed.) Vol. 2 (1896, 2^{nd} Ed.) Vol. 3 (1893) Vol. 4 (1894). Lucas wrote the first two volumes: the latter two were compiled from the papers left at his death by his editors Delannoy, Laisant and Lemoine. Andrew would be interested if anyone finds the university copy, particularly the first volume, which is catalogued as a first edition (1882).

5. From the box cover of the Tower of Hanoi puzzle (1883), shown at `http://www.cs.wm.edu/~pkstoc/toh.html`. This translation is from there also.

Continued page 10


ISSUE #1
The Adventures Of
Paradox Kid
by Jeremy Glick & Sally Miller
MATHEMATICS 101
Assessment :
- Exam 60%
- 4× Assignments 10% each
THE UNIVERSE IS A LONELY AND EMPTY 3-MANIFOLD AND FROM THE BACK ROW OF THEATRE A, P.K. WAS FINDING IT PARTICULARLY BORING.
SUDDENLY...
OH NO! IT'S PERTURBED BAYSHUN!
THE INSANE PSYCHOTIC PHYSICS STUDENT!
THEY SAY HE WAS EXPOSED TO TOO MANY GAMMA RAYS AS A CHILD...
OK. GIMME ALL YOUR EQUIPMENT - I WANT CALCULATORS, MATH-O-MATS PROTRACTORS - EVERYTHING!
... AND YOU THERE - HAND OVER THAT ABACUS!
WHO KNOWS WHY HE RAN - MAYBE FEAR, MAYBE COURAGE, MAYBE LOVE FOR HIS ABACUS... WE'LL NEVER KNOW, FOR BEFORE HE COULD REACH THE DOOR HE WAS FRIED BY PERTURBED...
ZAP!
EXIT
LOOK, PERTURBED - I WON'T STAND BY AND WATCH WHILE YOU ELECTROCUTE INNOCENT MATHS STUDENTS.

P.K. DONS HIS SUPER VISOR AND IS TRANSFORMED INTO PARADOX KID, WITH
MORE POWER THAN A QUINTIC (5),
MORE BRAINS THAN THE BASKET BEHIND A GUILLOTINE, AND
MORE SEXUAL ALLURE THAN A HYPERBOLIC ATTRACTOR
AAAAAHHH
KAZAM!
POW!
ZAP!
SIMPLEX!
AFTER A FIERCE BATTLE, PARADOX KID DELIVERS ONE FINAL CRUSHING BLOW, LEAVING PERTURBED BAYSHUN'S HEAD OSCILLATING UNCONTROLLABLY...
DOING!
AND REMEMBER EVERYONE - THE EQUATION OF MOTION OF PERTURBED'S HEAD IS $\frac{d^2\theta}{dt^2}+\alpha^2\sin\theta+\beta\frac{d\theta}{dt}=0$
...AND WITH THAT, PARADOX KID RETURNS TO HIS SEAT, TAKES OFF HIS SUPER VISOR, WIPES HIS BROW, AND RESUMES HIS LIFE AS P.K. THE EVERYDAY MATHS STUDENT.
TO BE CONTINUED...

From page 7

6. *ibid.*

7. Lucas, *op. cit.* Vol. 1 p. v. Fermat's *Oeurves* were published in four volumes, the first time in 1891, 1894 and 1896, the last in 1912.

8. From de Parville's account as translated in Ball, W.W.R., *Mathematical recreations and problems*, Macmillan, London (1892) p. 79, also appearing in Northrop, E.P., *Riddles in mathematics*, Penguin; Harmondsworth (1960), which has several sections on $2^n - 1$ problems. Similar versions can be found on the original Tower of Hanoi instructions; in Lucas, *op. cit.*, Vol. 3 p. 58, and also in many computer science texts and web-pages about algorithms.

PROBLEMS

1. Looked at one way, the sequence of Lucas numbers is the first different sequence given by a Fibonacci-type recurrence (look at $W_1 = 1, W_2 = 2, W_n = W_{n-1} + W_{n-2}, n > 2$ and see what happens).

   Show that for all positive integers n, F_n divides F_{2n}, where F_n is the n^{th} term of the Fibonacci sequence $\{F_1 = 1, F_2 = 1, F_n = F_{n-1} + f_{n-2}, n > 2\}$. From this give an alternative representation of the Lucas numbers.

2. There are many variations of the Tower of Hanoi puzzle. One of these is to increase the number of pegs. Find an upper bound for the number of moves necessary to transfer a nine-disk tower when there are four pegs. Lowest upper bound submitted wins the money. (Hint: it is much less than 511, the number of moves required with three pegs). Extra prize for a non-exhaustive proof that yours is the minimum number of moves required.

— Andrew Oppenheim

Coming up . . .

MUMS is continuing its program of events in 1999, with the first major gathering being the upcoming AGM. Many positions will be up for grabs, so interested maths and stats students of all levels will be able to get involved in the group. Events of a more social nature will begin soon afterwards, so keep looking at noticeboards around Richard Berry for updates.

It disappears in a puff of logic ...

Here is a trite logic puzzle that you have no doubt heard before, with an extra twist.

On a small island in the Pacific Ocean, the inhabitants see each other at least once a day, however, the island is mirrorless, and they cannot see themselves. Each inhabitant has either blue or brown eyes. If someone knows that they have *blue* eyes, then they will leave the island during the night following their discovery. Everyone is living happily, (no-one knows the colour of their own eyes), until a stranger comes to the island and announces: "At least one person has blue eyes".

1. Show that if there are n people with blue eyes, then they will all leave on the n^{th} night.
2. If there were, say 10 people with blue eyes, then everyone on the island knew that at least one person had blue eyes, and further they knew that everyone else knew that at least one person had blue eyes, so what new information was offered by the stranger that catalysed the exodus?

Before you complain — remember last edition's jokes!

Q: Have you heard the one about the statistician?
A: Probably ...

A mathematician is showing a new proof he came up with to a large group of peers. After he's gone through most of it, one of the mathematicians says, "Wait! That's not true. I have a counter-example!" He replies, "That's okay. I have two proofs."

Desmond Lun, former layout editor, before he left for Berkeley, was overheard in the corridor ...

"Yeah, I used to think it was just recreational ... then I started doin' it during the week ... you know, simple stuff: differentiation, kinematics. Then I got into integration by parts ... I started doin' it every night: path integrals, holomorphic functions. Now I'm on diophantine equations and sinking deeper into transfinite analysis. Don't let them tell you it's just recreational. Just say $\{\}$. Fortunately, I can quit any time I want."

Easy money

The following are some problems for prize-money. The person who submits the best (i.e. clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number, and will have their solution published in the next edition of *Paradox*. Solutions may be e-mailed to `paradox@ms.unimelb.edu.au`. (LaTeX format would be appreciated though not demanded.) If you do not have access to e-mail then drop in a hard copy of your solution to the MUMS pigeon-hole near the Maths and Stats Office in the Richard Berry building.

1. (\$10) Let p be a prime greater than 3. Prove that

$$p^5|(2p)! - 2(p!p!)$$

where $x|y$ means that x divides y.

2. (\$15) Let $f(x)$ and $g(x)$ be the non-zero polynomials:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

$$g(x) = c_{n+1}x^{n+1} + c_n x^n + \cdots + c_0$$

and let $g(x) = (x + r)f(x)$ for some real r.

If $A = \max\{|a_0|, |a_1|, \ldots, |a_n|\}$ and $C = \max\{|c_0|, |c_1|, \ldots, |c_{n+1}|\}$, prove that $A/C \leq n + 1$.

3. (\$10) Let P be a point inside a triangle ABC. Let x, y, z be the distances from P to the sides BC, CA, AB of the triangle, and let the lengths of those sides be a, b, c respectively. let R be the circumradius of ABC. Show that:

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \sqrt{\frac{a^2 + b^2 + c^2}{2R}}.$$

4. (\$10) Find a (closed-form) formula for the value of:

$$\sum_{i=0}^{n}(-4)^i \binom{n+i}{2i}$$

in terms of n.

5. (\$5) Using Euler's constant e, and the symbols for addition, subtraction, multiplication, division, square root, parentheses and "round down", express the integers 1 through 15. The round down function, indicated by $\lfloor\rfloor$ returns the largest integer less than the amount enclosed between the brackets. For example, $\lfloor 5.42 \rfloor = 5$. 1 can be written as e/e, or in a number of other ways. Preference will go to the answer which uses the fewest 'e's.

Paradox **from the past**

Years ago, long before many of us had integrated our first polynomial, the Melbourne University Mathematical Society published a magazine called *Matrix*, a predecessor to *Paradox*. In those days, before computers were as commonplace as they are now, articles were laboriously typeset, and editions were released once a year. In the next few editions of *Paradox*, we will be reprinting articles from *Matrix*, preferably those written by current

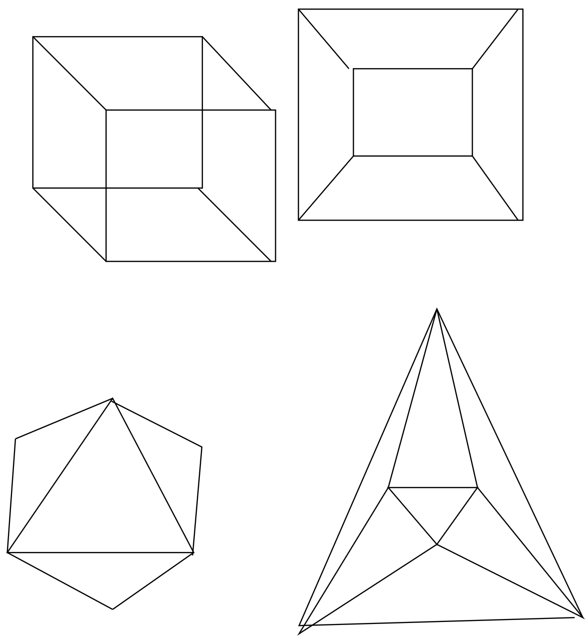
Melbourne University staff members. We hope that by including these articles we can help recreate the enthusiasm and optimism that was prevalent then (so much so that in the 1964 edition of *Matrix*, J. W. Craggs wrote an article commenting that there were *too many* jobs for mathematicians, and not enough graduates!).

The following article was written by Dr. Frank Barrington, as a nineteen year old, and appeared in the 1963 edition of *Matrix*. Aside from a few typesetting errors, the article is enjoyable to read, and shows some straightforward but interesting results.

For those of you interested in reading more articles of this nature, Dr. Barrington has generously handed over his old editions of *Matrix* to the maths library, and they can be found there. If any readers have any other old copies of *Matrix*, we would love to get our hands on them, as at the moment we have only four editions available.

Platonic solids

In most modern geometry courses, the study of solids and figures is neglected, although the principles behind such a study are relatively simple. Perhaps it is too old to merit much enthusiasm in the face of the modern need for powerful mathematical techniques yet, in itself, the investigation of solid bodies is an interesting and stimulating byway.



The following discussion is concerned with regular convex polyhedra, more commonly called Platonic solids. A polyhedron may be defined as a solid figure bounded by portions of planes, called faces. (We restrict the number of planes to be finite.) A polyhedron is convex if the whole figure lies on one side of each of these planes. Further, it is regular if each of face is a regular polygon, a regular n-gon, say, the same number of faces coming together at each vertex. We shall now look at the existence of these regular convex polyhedra, and determine their number.

One useful and convenient way of studying polyhedra is by means of a perspective drawing, made by viewing a polyhedron from just outside just outside the centre of one of its faces. (This is sometimes referred to as a Schlegel diagram.) The polyhedron itself must be imagined, of course, as a transparent ”match-stick” model, the matches corresponding to the edges, which are the intersections of the faces of the polyhedron. Examples of such drawings are shown in figure 1, for the cases of the cube and regular octahedron.

For any polyhedron, denote the number of vertices by v, the number of edges by e, and the number of faces by f. Owing to the fact that convex polyhedra may be represented by plane diagrams, we may deduce the simple relation

$$v + f - e = 2$$

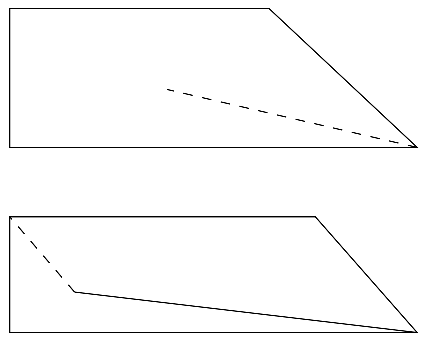
attributed to Euler. This holds not only for perspective diagrams like the above, but for any simple plane "map", consisting of points, with lines between these points, dividing the plane into regions which do not overlap. (Here, "regions" and "faces" are equivalent. In the case of the Schlegel diagram, the part of the plane outside the boundary of the diagram corresponds to the face through which the figure is viewed.)

The validity of the formula may be seen by constructing such a map, starting with a single point ($v = 1$, $e = 0$, $f = 1$, the only face being the whole plane excluding the single point) and adding edges one at a time. Initially $v + f - e = 2$, and any step in the construction of the map consists of either

1. a new edge joining an old vertex to a new one, or

2. a new edge joining two old vertices. (See figure 2.)

In the first case, the number of edges is increased by one, as is the number of vertices, and the number of faces remains the same; so $v + f - e$ is unchanged. In the second case, e and f are increased by one but v remains the same; so again, $v + f - e$ is unchanged.



We now return to the regular polyhedron. Suppose each face has s edges, and suppose that t of these faces meet at each vertex. A polygon has at least three sides, so $s \geq 3$. Also at least three faces must meet at each vertex for the vertex to exist, so $t \geq 3$. The numbers s and t characterise the polyhedron; we shall see what these restrictions, together with Euler's formula, imply.

Each face has s edges, and therefore the f faces have fs edges in all. But each edge is common to two faces, so each edge has been counted twice. Thus

$$fs = 2e$$

By a similar argument for the vertices,

$$vt = 2e$$

Multiplying Euler's formula (1) by $2t$, $2vt + 2ft - 2et = 4t$, and thus $f(2s + 2t - st) = 4t$ from (2) and (3).

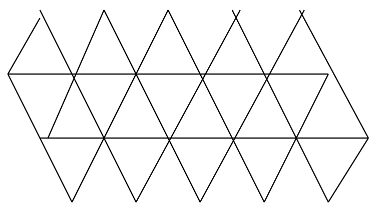
Now $f > 0$, and $t > 0$, so $2s + 2t - st > 0$. Hence $st - 2s - 2t < 0$. So $(s-2)(t-2) = st - 2t - 2s + 4 < 4$.

Now $s \geq 3$ and $t \geq 3$, so that $(s-2) \geq 1$ and $(t-2) \geq 1$. Whence $(s-2)$ and $(t-2)$ are positive whole numbers whose products are less than 4. The only possible products are $1 \times 1, 1 \times 2, 2 \times 1, 1 \times 3, 3 \times 1$, giving five pairs of values for s and t. The results of these manipulations are shown in the following table:

s	t	f	e	v	
3	3	4	6	4	Tetrahedron
4	3	6	12	8	Cube
3	4	8	12	6	Octahedron
5	3	12	30	20	Dodecahedron
3	5	20	30	12	Icosahedron

Many of the properties of Platonic solids are elucidated by the above table. Note that interchanging s and t always interchanges v and f. Geometrically, if the centres of each face of one of these solids are taken as the vertices of another polyhedron, the edges of the latter are the joins of the centres of neighbouring faces, so that each new vertex by construction lies on one old face and each new face cuts off one of the old vertices. However, e is unchanged. From the table this interchange can be made with the cube and the octahedron and with the dodecahedron and the icosahedron. Such a pair of polyhedra are said to be reciprocal. This property can also be seen from the symmetry of s and t in the above relations.

Thus there are at most five regular convex polyhedra. Nothing has yet been said about their existence but methods for their construction using nets as in figure 3, are generally well known and will not be discussed here.



Historically the Platonic solids were so named after Plato, who suggested that they symbolised the four "elements": earth, fire, air and water; the fifth, the dodecahedron, being a form characteristic of the whole universe. In more recent times our knowledge of polyhedral figures has been of benefit in areas such as quantum mechanics and group theory.

— Frank Barrington, 1963

Solutions to last issue's problems

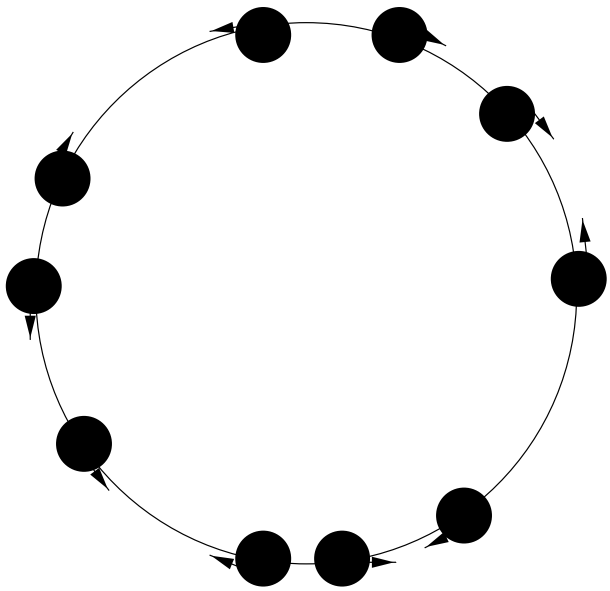
Last edition we put in as many problems as we could to keep you occupied during the summer. We don't have space to include solutions to all of them, but we've put solutions to a selection. If anyone is interested in the solution to a problem that is not included here, come and see one of the editorial team, and we'll give you a clue.

Problem 1. In the future, we expect that *Paradox* will be accepted as legal tender, so each copy will need to be individually numbered. As such, each copy will receive a six-digit number. However, legislation requires that each number differ from all others in at least two places (i.e. 123456 and 123457 could not both be used, but 123456 and 234567 could). What is the maximum number of *Paradox*es that could be printed, and why?

Solution:

We can split a six digit number into two parts: the first five digits, and the final digit. For two numbers to be valid, they must differ by at least one out of the first five digits (because they can differ by at most one out of the final digit!). Therefore, there is an upper limit of 10^5 different valid numbers. Further, it is possible to construct a set of 10^5 six digit numbers that differ in at least two places. To do this, we have the first five digits being the numbers from 00000 to 99999 (10^5 in all) and the sixth digit we make the sum of the previous five, modulo 10. To see that this is valid, we note that if the first five differ by two or more digits, then the numbers are valid, and if they differ by only one, then the sum modulo 10 of their digits must necessarily be a different number. Hence, the maximum number of copies we could print would be 10^5.

Problem 2. An arbitrary number of dodgem car drivers begin randomly placed around a circular rail. They all start with the same speed, however each driver randomly chooses to start going either clockwise or anticlockwise. After each collision each driver reverses direction without loss of speed (i.e. they bounce off each other). Prove that at some later time each of the cars will occupy their initial positions.



Solution:

We make the following observations:

1. The speeds of the cars don't change when they collide, so effectively it's as if they pass through each other but the drivers are swapped. Hence after a certain time t, the time it takes for a car to complete one circuit, there will be a car at each of the initial positions (but of course the drivers might be different).

2. The relative positions of the cars don't change as they bounce off each other. Therefore the cars can only be rotated around after time t, so after time nt, where n is the number of cars, all the cars will be back at their initial positions.

Thank you

We would like to thank the following people for helping us put together this issue: Kirsten Raynor, Andrew Oppenheim, Chaitanya Rao, Dr. Frank Barrington, Professor Derek Chan, Associate Professor Barry Hughes, Koula Courtot, Russell Sloan, Portia Neydorff, Anthony Wirth, John Dethridge, and a special thanks to Desmond Lun.

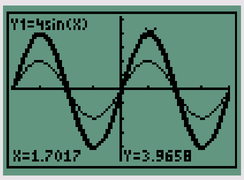
The world's shortest poem

The TI-83 Plus, an update to the TI-83, featuring Flash electronic upgradability and more memory, is the tool for everything from algebra through calculus, inferential statistics, and finance.

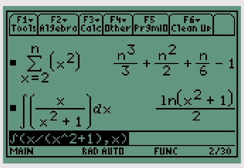
The TI-83 Plus and the TI-89, both featuring Flash electronic upgradability, are more useful for more classes than other calculators in their class. Whether you're covering algebra through calculus with the TI-83 Plus, or exploring advanced mathematics with the TI-89, they both have extraordinary feature sets designed to meet your needs. Now with Flash you can install unique applications

EXPLORATIONS™ Series Books. To make your TI calculators even more functional this series of workbooks was developed by educators for use within the context of your mathematics coursework.

The Only Calculators Your Classes Need Are in a Class by Themselves



Familiar TI-83 functionality on the TI-83 Plus



Symbolic manipulation on the TI-89

that provide significant functionality enhancements. Find the latest Flash applications at our web site: www.ti.com/calc/flash. To learn more about TI tools, Flash upgradability, and our popular educator support programs visit our web site, or call us at 1-800-TI-CARES.

The TI-89 featuring a robust Computer Algebra System (CAS), Flash electronic upgradability, and over 500k memory, is ideal for advanced calculus, differential equations, and beyond.

Texas Instruments

1-800-TI-CARES (Canada & U.S.) ti-cares@ti.com www.ti.com/calc