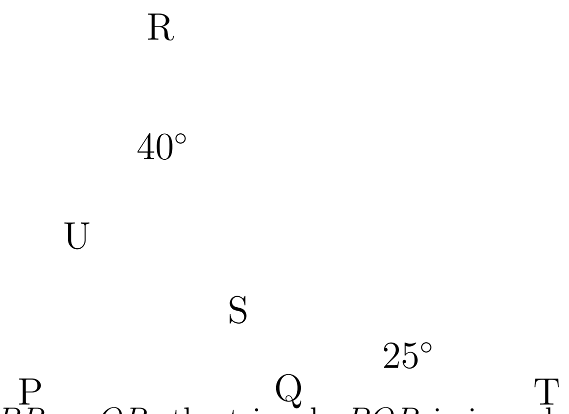
2001 MUMS Schools Maths Olympics
Questions and Solutions

Question 1. What is the sum of the first five square numbers ending in a 1?

Solution 1. Note that for a square number to end in a 1, its square root must end in a 1 or a 9. So the sum of the first five square numbers ending in a 1 is:

$$1^2 + 9^2 + 11^2 + 19^2 + 21^2 = 1 + 81 + 121 + 361 + 441 = 1005$$

Question 2. In the diagram, $PR = QR$, $\angle PRQ = 40°$, and $\angle PTU = 25°$. What is $\angle RST$, in degrees?



Solution 2. Since $PR = QR$, the triangle PQR is isosceles and $\angle RPQ = \angle RQP$. Considering the angles in triangle PQR gives the equation $\angle RPQ + \angle RQP = 180° - \angle PRQ = 180° - 40° = 140°$. Therefore, $\angle RPQ = \angle RQP = 70°$ and $\angle SQT = \angle RQT = 180° - \angle PQR = 180° - 70° = 110°$. Considering the angles in triangle SQT gives us $\angle QST = 180° - \angle SQT - \angle STQ = 180° - 110° - 25° = 45°$. Hence, $\angle RST = 180° - \angle QST = 180° - 45° = 135°$.

Question 3. Simplify

$$\frac{16\sqrt{7}}{\sqrt{7} + \frac{1}{\sqrt{7}}}$$

Solution 3.

$$\frac{16\sqrt{7}}{\sqrt{7} + \frac{1}{\sqrt{7}}} = \frac{16\sqrt{7}}{\sqrt{7} + \frac{1}{\sqrt{7}}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{16 \times 7}{7 + 1} = \frac{102}{8} = 14$$

Question 4. Mary has sat for 10 tests this term and has an average score of 68. What mark must she gain in the next one to raise her average to 70?

Solution 4. Let x denote the mark that Mary must receive in the next test. In the previous 10 tests, the total number of marks that Mary received was $68 \times 10 = 680$. To raise her average to 70 in the next test, the following equation must hold:

$$\begin{aligned} \frac{680 + x}{11} &= 70 \\ \Rightarrow \quad 680 + x &= 70 \times 11 = 770 \\ \Rightarrow \quad x &= 770 - 680 = 90 \end{aligned}$$

Question 5. The digits 3, 1, 4 and 1 can be arranged to form how many different 4-digit numbers?

Solution 5. If we treat the two 1's as two different numbers, then there would be $4! = 24$ ways of rearranging the four digits. (There would be one of four different numbers which we could choose for the thousands digit, one of three numbers for the hundreds digit, one of two numbers for the tens digit, and only one choice for the units digit.) However, since the two 1's can be interchanged, we have counted each number twice. Therefore, there are only $\frac{24}{2} = 12$ different 4-digit numbers that can be made from the digits 3, 1, 4 and 1.

Question 6. What is the surface area, in square centimetres, of a cube having volume 343 cm^3?

Solution 6. Since the cube has a volume of 343 cm^3, the side length of the cube must be $\sqrt[3]{343}$ cm = 7 cm. Thus, the area of one square face of the cube is $7 \text{ cm} \times 7 \text{ cm} = 49 \text{ cm}^2$. The surface area of the cube is composed of six of these square faces and is $6 \times 49 \text{ cm}^2 = 294 \text{ cm}^2$.

Question 7. What is the area of the smallest circle in which you can fit six equilateral triangles of area 3 without overlap? (Express your answer *exactly*)

Solution 7. If we inscribe a regular hexagon in a circle and draw its diagonals through the centre, we can see that in a circle of radius r, we can *exactly* fit six equilateral triangles of side length r. For an equilateral triangle to have area 3, we require the following equation to hold:

$$\begin{aligned} \text{Area} = \frac{1}{2}r^2 \sin 60^\circ &= 3 \\ \frac{1}{2}r^2 \times \frac{\sqrt{3}}{2} &= 3 \\ r^2 &= \frac{12}{\sqrt{3}} \\ r^2 &= 4\sqrt{3} \end{aligned}$$

Thus the area of the required circle is $\pi r^2 = 4\sqrt{3}\pi$.

Question 8. Luke is a fugitive from justice. He steals a car in Melbourne at 8:00 am and aims to drive it to his freedom. Unfortunately the Magna he stole is a real bomb and he can only travel at a constant speed of 80 km/h. The police are notified of the theft and commence a chase at 8:30 am from the location of the theft. They follow the trail of oil which has been dripping out of the Magna's engine and so follow Luke's route exactly, at a constant speed of 100 km/h. At what time will Luke be arrested? Remember to specify in your answer whether the time is am or pm.

Solution 8. Suppose that Luke is arrested t hours after 8:00 am. Then in that time, Luke has travelled $80t$ kilometres and the police have travelled $100(t-0.5)$ kilometres. Since these two distances have to be equal, we have the equation:

$$\begin{aligned} 80t &= 100(t-0.5) \\ 80t &= 100t - 50 \\ 20t &= 50 \\ t &= \frac{50}{20} = 2.5 \end{aligned}$$

Therefore, Luke is arrested 2.5 hours after 8:00am – that is, at 10:30 am.

Question 9. Norm and Geordie each roll a die. What is the probability that the product of the two numbers rolled is less than 6? (Express your answer as a fraction in simplest form.)

Solution 9. Let the number that Norm rolls be N and the number that Geordie rolls G. Since there are six possibilities for N and for G, all equally likely, there are $6 \times 6 = 36$ possibilities for the pair (N, G). The only possible pairs whose product is less than 6 are $(1,1)$, $(1,2)$, $(2,1)$, $(1,3)$, $(3,1)$, $(1,4)$, $(4,1)$, $(1,5)$, $(5,1)$ and $(2,2)$.

Since there are 10 of these pairs, the required probability is $\frac{10}{36} = \frac{5}{18}$.

Question 10. Three jolly professors, Tim, Swarup and Kris, are gambling by a billabong. They start with sums of money in the ratio 7 : 6 : 5 and finish with sums of money in the ratio 6 : 5 : 4, in the same order of wealth. One of the professors won \$12. How many dollars did he start with?

Solution 10. Suppose that the total amount of money of all the professors in dollars is M. Since they started with sums of money in the ratio 7 : 6 : 5, they each started with $\frac{7M}{7+6+5}$, $\frac{6M}{7+6+5}$ and $\frac{5M}{7+6+5}$, respectively. Since they ended with sums of money in the ratio 6 : 5 : 4, they each ended with $\frac{6M}{6+5+4}$, $\frac{5M}{6+5+4}$ and $\frac{4M}{6+5+4}$, respectively. Thus, the first professor won $\frac{6M}{15} - \frac{7M}{18} = \frac{M}{90}$, the second professor won $\frac{5M}{15} - \frac{6M}{18} = 0$ and the third professor lost $\frac{5M}{18} - \frac{4M}{15} = \frac{M}{90}$. We are told that one of the professors won \$12, so $M = 90 \times \$12 = \$1080$. Hence, he started with $\frac{7M}{18} = \frac{7}{18} \times \$1080 = \$420$.

Question 11. Many years ago Melbourne University had under 5000 students enrolled. A third of the students were in first year, two-sevenths were in second year, one-fifth were in third year and the rest were postgraduate students. The mathematics department offered a popular course in which were registered a fortieth of all the first-year students, a sixteenth of all the second-year students, and a ninth of all the third-year students, while the remaining third of the maths class were all postgraduates. How many students were there in the maths class?

Solution 11. Let N denote the total number of students enrolled and let C denote the number of students registered in the maths class. Then $\frac{1}{3} \times \frac{1}{40} \times N = \frac{N}{120}$ first year students are in the maths class, $\frac{2}{7} \times \frac{1}{16} \times N = \frac{N}{56}$ second year students are in the maths class, and $\frac{1}{5} \times \frac{1}{9} \times N = \frac{N}{45}$. N must be divisible by 120, 56 and 45 and hence, N is divisible by the lowest common multiple of 120, 56 and 45 – that is, 2520. But the only multiple of 2520 which is less than 5000 is 2520 itself so $N = 2520$. Since first, second and third year students make up two-thirds of the maths class, we have the equation:

$$\begin{aligned} & \frac{N}{120} + \frac{N}{56} + \frac{N}{45} = \frac{2C}{3} \\ \Rightarrow \quad & \frac{2C}{3} = \frac{2520}{120} + \frac{2520}{56} + \frac{2520}{45} = 21 + 45 + 56 = 122 \\ \Rightarrow \quad & C = \frac{3}{2} \times 122 = 183 \end{aligned}$$

Question 12. In the diagram, the lengths of the sides of the triangle are 8, 9, 13 centimetres . The centres of the circles are at the vertices of the triangle, and the circles just touch. What are the radii of the three circles, in increasing order?

13

8 9

Solution 12. Let the radii of the three circles be a, b and c. Then we have the equations:

$$\begin{aligned} a + b &= 8 \\ a + c &= 9 \\ b + c &= 13 \end{aligned}$$

If we add the first two equations and subtract the third equation, we have:

$$\begin{aligned}(a+b)+(a+c)-(b+c) &= 8+9-13 \\ \Rightarrow 2a &= 4 \\ \Rightarrow a &= 2\end{aligned}$$

We can easily calculate the other radii: $b = 8 - a = 6$ and $c = 9 - a = 7$. So the radii of the three circles, in increasing order, are 2, 6 and 7.

Question 13. Lines from the vertices of a square to the midoints of the sides are drawn, as shown. If the area of the large square is 1, what is the area of the smaller square in the middle?

Solution 13. The original square can be rotated and a cross constructed as shown. Clearly, the area of the cross is the same as that of the large square. Hence the area of the small square is 1/5 of that of the large square.

Question 14. The diagram shows a 5 by 5 table. The top row contains the symbols A,S,I,D,E. The fourth row contains the symbols A,S,I at the centre. The remaining squares can be filled with A's, S's, I's, D's and E's such that no row, column or diagonal contains the same symbol more than once. What are the symbols in the bottom row, from left to right?

A	S	I	D	E
	A	S	I	

Solution 14. The table can be constructed in only one way to satisfy the given conditions. Each row of the table is the result of *cycling* the row above it two squares to the right. This will result in the second row reading "DEASI", the third row "SIDEA", the fourth row "EASID" and finally the bottom row reading "IDEAS".

Question 15. Let $f(n)$ be the number of letters used when writing out the digits of the base ten representation of n. For example, $f(27) = 8$ since "two seven" has eight letters. What is the value of $f(f(f(\cdots f(2^{16}))))$ where f is applied 10 times?

Solution 15.

$$\begin{aligned} f(2^{16}) = f(65536) = 3+4+4+5+3 = 19 \\ f(f(2^{16})) = f(19) = 3+4 = 7 \\ f(f(f(2^{16}))) = f(7) = 5 \\ f(f(f(f(2^{16})))) = f(5) = 4 \\ f(f(f(f(f(2^{16}))))) = f(4) = 4 \end{aligned}$$

Since $f(4) = 4$, we can see now that however many times we apply the function f to the value 4, it will remain 4. So $f(f(f(\cdots f(2^{16})\cdots))) = 4$ if the function f is applied five or more times.

Question 16. How many points with positive integer coordinates are there strictly inside the area bounded by the lines $x = 0$, $y = 0$ and the graph $y = 10/x$?

Solution 16. A point (x, y) will lie strictly inside the area bounded by the lines $x = 0$, $y = 0$ and the graph $y = 10/x$ if $xy < 10$, where x and y are positive. The only pairs (x, y) of integers which satisfy this are the pairs:

- $(1, k)$ with $1 \leq k \leq 9$
- $(k, 1)$ with $1 \leq k \leq 9$

- $(2,2)$, $(2,3)$, $(3,2)$, $(2,4)$, $(4,2)$ and $(3,3)$

There are nine points of the first kind, nine of the second kind and six of the third kind. However, we have counted the point $(1,1)$ twice, so there are $9+9+6-1=23$ points which lies strictly inside the given area.

Question 17. Solve the following equation for the integer n:

$$\sqrt[3]{n+\sqrt{n^2+8}}+\sqrt[3]{n-\sqrt{n^2+8}}=8$$

Solution 17. Let $a=\sqrt[3]{n+\sqrt{n^2+8}}$ and $b=\sqrt[3]{n-\sqrt{n^2+8}}$. Then the equation given to us is $a+b=8$. We can also find the product of a and b.

$$\begin{aligned} ab &= \sqrt[3]{n+\sqrt{n^2+8}}\sqrt[3]{n-\sqrt{n^2+8}} \\ &= \sqrt[3]{(n+\sqrt{n^2+8})(\sqrt[3]{n-\sqrt{n^2+8}})} \\ &= \sqrt[3]{n^2-(n^2+8)} \\ &= \sqrt[3]{-8} \\ &= -2 \end{aligned}$$

Now we note that:

$$\begin{aligned} 2n &= (a^3+b^3) \\ &= (a+b)^3-3ab(a+b) \\ &= 8^3-3\times(-2)\times 8 \\ &= 512-(-48) \\ &= 560 \end{aligned}$$

Therefore, $n=280$.

Question 18. On the planet Dankuhn lives a being which can be one of three sex types: male, female and emale. Any two different sexes may breed and the offspring from such a union is of the third sex. How many of an emale's great great great great grandparents were emales?

Solution 18. The following table shows the number of emales, females and males for each generation in the past six generations of an emale. So we can see that every emale has 22 great great great great grandparents.

	1	2	3	4	5	6
emales	0	2	2	6	10	22
females	1	1	3	5	11	21
males	1	1	3	5	11	21

Question 19. Find the number of three digit numbers whose digit sum is ten.

Solution 19. First, we will determine all integer triples (a, b, c) with $0 \leq a \leq b \leq c \leq 9$ where $a + b + c = 10$. Note that $a = 0, 1, 2$ or 3.

- If $a = 0$, then $b + c = 10$, so the only possible triples are $(0, 1, 9)$, $(0, 2, 8)$, $(0, 3, 7)$, $(0, 4, 6)$ and $(0, 5, 5)$.
- If $a = 1$, then $b + c = 9$, so the only possible triples are $(1, 1, 8)$, $(1, 2, 7)$, $(1, 3, 6)$ and $(1, 4, 5)$.
- If $a = 2$, then $b+c = 8$, so the only possible triples are $(2, 2, 6)$, $(2, 3, 5)$ and $(2, 4, 4)$.
- If $a = 3$, then $b + c = 7$, so the only possible triple is $(3, 3, 4)$.

Thus, we can see that there are 13 possible triples which satisfy the conditions. We can divide these into the following:

- All digits are non-zero and distinct – each of these triples corresponds to six distinct three-digit numbers.
- All digits are non-zero and two coincide – each of these triples corresponds to three distinct three-digit numbers.
- All digits are distinct and one of them is zero – each of these triples corresponds to four distinct three-digit numbers.
- Two digits coincide and the other is zero – each of these triples corresponds to two distinct three-digit numbers.

There are four, four, four and one, respectively, of each of these types of triples. Thus, the total number of three-digit numbers whose digit sum is ten is $4 \times 6 + 4 \times 3 + 4 \times 4 + 1 \times 2 = 54$.

Question 20. "Baker's Dozen" doughnuts are sold only in boxes of 7, 13 or 25. To buy 14 doughnuts you must order two boxes of 7, but you cannot buy exactly 15 since no combination of boxes contains 15 doughnuts. What is the largest number of doughnuts that cannot be ordered using combinations of these boxes?

Solution 20. Note that we can buy any number of doughnuts from 45 through to 51 as follows:

$$
\begin{aligned}
45 &= 1 \times 7 + 1 \times 13 + 1 \times 25 \\
46 &= 3 \times 7 + 0 \times 13 + 1 \times 25 \\
47 &= 3 \times 7 + 2 \times 13 + 0 \times 25 \\
48 &= 5 \times 7 + 1 \times 13 + 0 \times 25 \\
49 &= 7 \times 7 + 0 \times 13 + 0 \times 25 \\
50 &= 0 \times 7 + 0 \times 13 + 2 \times 25 \\
51 &= 0 \times 7 + 2 \times 13 + 1 \times 25
\end{aligned}
$$

Since these are seven consecutive numbers, we can buy any number of doughnuts which is 45 or greater, by buying extra boxes of 7. However, it is not possible to buy 44 doughnuts.