Paradox

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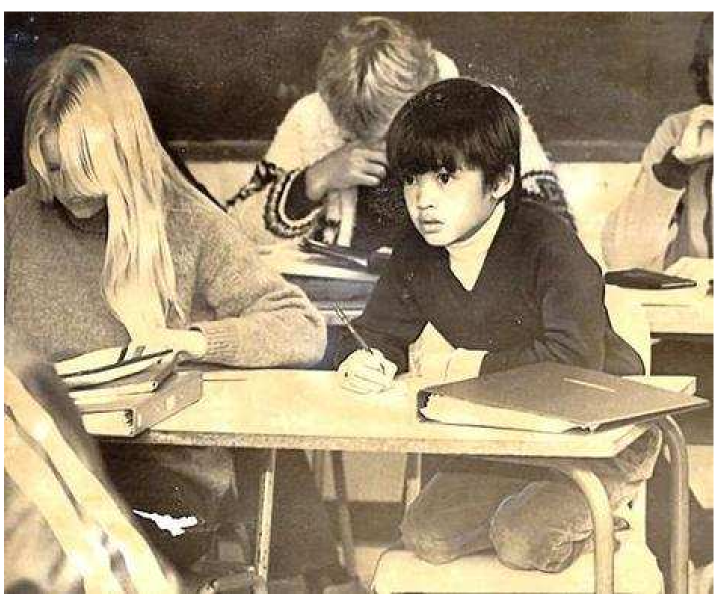

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Paradox

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In this edition of Paradox

Regulars

Special Features

Articles

On the Cover:
Top left: A 13 year-old Terry Tao receives his gold medal at the 1987 International Maths Olympiad held in Canberra from Prime Minister Bob Hawke.
Top right: Terry, during his brief stop in Melbourne in September, outside the MUMS room.
Bottom: A seven year-old Terry sits a maths test with students ten years his senior.

Words from the Editor

There is an advertisement for Intel doing the rounds at the moment that features Ajay Bhatt, co-inventor of the USB, strutting down a corridor at the Intel headquarters, leaving swooning women and awestruck men in his wake. The tag-line reads, 'Our rockstars aren't like your rockstars.'

If the mathematics community has a 'rockstar' it is surely Terry Tao, and he received a suitably rockstar welcome at Melbourne Uni in September when he whisked into town for the Melbourne leg of his Australia-wide lecturing tour. With a Copeland Theatre overflowing with eager maths students and celebrity spotters alike, devastated latecomers (including myself) had to be turned away at the door. If only the acoustics at the lecture had been worthy of a rock concert; the microphone was on so low that the softly spoken Terry could, by all accounts, barely be heard.

But my disappointment in missing the lecture was short-lived; on a whim Paradox had asked Terry if he would be willing to grant us brief interview, notwithstanding his incredibly busy lecturing schedule. To our delight he consented, and the next thing we knew Paradox was face to face with the great man himself. The resulting interview can be found on page 12 of this edition. I hope the article conveys the general thrust of the interview, but if anyone is interested in a full transcript then Paradox will happily oblige.

Given Terry Tao's status as a top mathematical researcher, it is easy to forgot that he also regularly teaches classes at UCLA, mostly to post-graduate students. Zhihong Chen, on exchange at UCLA, managed to enrol in one of Terry's classes, and his account of the experience can be found on page 19.

Elsewhere in this bumper edition you can attempt the Paradox Wallpaper Challenge, indulge your passion for absurdly large counterexamples, or explore the intersection between the two seemingly disjoint fields of maths and the law.

The next edition of Paradox will be published after the summer holidays, so if three long, dreary months without mathematical stimulation proves unbearable consider writing an article for Paradox. We are always looking for more contributors and contributions. Until next year!

— Stephen Muirhead

Words from the President

As the end of semester begins to looms ever closer, the spectre of exam-doom appears ever ...*grosser*. Ok, so there is a reason why I'm a maths major and not an arts student!

In other news, it is a good time to be a President. Apparently nowadays one simply needs to be President for a Nobel Prize to appear in the mailbox. I've been expecting to hear the announcement of my nomination for the last week or so (though probably not for literature).

On a (slightly) more serious note, MUMS plans to end the semester in the scintillating fashion that we're so well known for. Yes, we'll be having our trivia night on the last day of semester as tradition requires. Expect the usual hijinks to ensue; in the past we've had apple bobbing, epic bridge building competitions and smarties flying across the room into awaiting mouths of eating. To be honest, I don't actually have a clue what's going on this year and hope to be pleasantly surprised. So, barring any other more important commitments (as if there's anything more important anyway), I totally expect to see you, yes you the reader, at our trivia night. What else could you possibly want to do on the last day of semester?

— Han Liang Gan (awaiting his Nobel Prize nomination)

Maths in the News:

1) Osama bin Laden is a wizz at mathematical computation (*The Age*, 18/10/2009). Apparently, when not plotting ways to terrorise innocent people, he enjoys 'showing off his mathematical ability by challenging people to beat his arithmetic with a calculator.' We can only hope there is no connection between these two attributes.

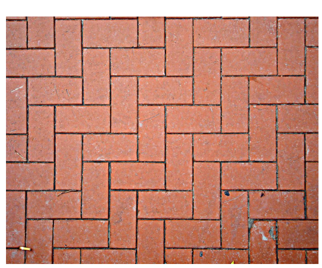
2) Sudoku has been solved (*mX*, 23/3/2009)! A professor at South Carolina University has developed a mathematical algoritmh for automatically solving Sudoku puzzles in just five easy steps. As with automatic crossword solvers, frustrated players can now turn to a program to crack even the most difficult of problems. What is unclear is why anyone would get any joy out of automatically solving a Sudoku.

The Paradox Wallpaper Challenge

The maths quad has always held a special place in the hearts of all maths students; with its majestic Jacaranda tree that flowers only during Swot Vac when there isn't anyone around to see it, its legendary resident cat (may he rest in peace), and the yearly invasion of architecture students peering through fingers held at arms length, testing who-knows-what principle of construction. Another reason why the quad is so interesting is its distinctive 'wallpaper' tiling; some swear it depicts cannibalistic pacmen, others favour interlocked lips. Yet whatever its interpretation, all 2nd Year pure maths students worth their salt have at some point asked the question: what wallpaper group is it?

The mathematics of wallpapers are familiar to anyone with a grounding in group theory. Any repeated wallpaper pattern has a set of symmetries, or operations on the plane which give you back an overlay of the original pattern. These symmetries may take the form of reflections, rotations or glide reflections, and considered together form a group. Interesting, despite the infinite number of wallpaper patterns, there are only a finite number of symmetry groups that they can have. In fact, there are only 17 of these groups.

The tiling in the maths quad has no reflections, contains a rotation of 180 degrees, and allows a glide reflection. This is an example of the *pgg* symmetry group. Another type of wallpaper with the same symmetries is the classic tiling of bricks.



Inspired by the tiling of the maths quad, Paradox sets the following challenge...

Find examples of all 17 wallpaper groups on the Melbourne Uni campus.

The person, or team, who are the first to send us a complete list of 17 locations wins the prize of $20! If no-one submits a complete list of all 17, or indeed if all 17 do not exist (*p6* may prove problematic!), the submission with the greatest number of distinct examples by the date of publication of the next edition (expected to be in April) will take the prize. Submitted locations must be explicit enough for Paradox to be able to verify them. Drop submissions into the MUMS room, or email them to Paradox.

To help you in this quest, an easy reference for distinguishing all 17 groups is included below.[1] Good luck!

	Reflection?	
Least Rotation	Yes	No
60 degrees	*p6m*	*p6*
90 degrees	**Mirrors at 45 degrees?** Yes: *p4m* No: *p4g*	*p4*
120 degrees	**Rotation centre off mirror?** Yes: *p31m* No: *p3m1*	*p3*
180 degrees	**Perpendicular reflections?** Yes No **Rotation centre off mirror?** *pmg* Yes: *cmm* No: *pm*	**Glide reflection?** Yes: *pgg* No: *p2*
none	**Glide axis off mirror?** Yes: *cm* No: *pm*	**Glide reflection?** Yes: *pg* No: *p1*

Puzzle

Can you carve out a portion of a unit cube in such a way that it will allow another unit cube to pass through it?

(The solution is on page 20)

[1]http://en.wikipedia.org/wiki/Wallpaper_group.

Wrap-up from the UMO

The 2009 University Maths Olympics was held on Friday 18th September in the JH Michell Theatre. Ok those are enough boring details. Let's move on to some of the more interesting aspects of the day:

- There was a record turn-out of 27 teams. By record, I mean the most that I can remember.
- Assoc. Prof. Barry Hughes, our esteemed MC, turned up in fancy dress.
- I was almost deafened by a blast from a whistle for someone red carded for running. Please stop running, for my sake.
- Barry Hughes produced what is arguably the worst mathematical joke ever. It was that horrendous one about how the integral of 1/cabin is a houseboat (log(cabin)+ c, get it?).
- I managed to lose my black pen that day. If anyone has seen it, please hand it in to the MUMS room.
- The MUMS mascot, a giant blue TI-83+ calculator was forcibly ejected by Barry Hughes under strict adherence to the 'No Calculators' rule.
- We got to use the reverse side of those *Quality of Teaching* survey forms for scrap paper. Finally, some use has come out of those sheets.
- I forgot to thank Barry Hughes publicly at the actual event, so I'm taking this space to do so now.
- Oh and some team called Mudkip Tangents only just won, ahead of Oresome Foursome and THYND. Full results, questions and solutions can be found at http://www.ms.unimelb.edu.au/˜mums/olympics/.

All in all, it was a great event! Bar a very trivial (notice how lecturers always claim something is trivial when it isn't?) error that will be duly forgotten, the event went exactly to plan and everyone had fun. I mean, they had to, because it was written in the rules that they must. Thanks again for everyone who turned up (and thus had fun), and I hope to see you all again next year!

— Han Liang Gan

The Integrating Machine

A mechanical integrating machine!? In the Maths department!

When I'd first heard about this machine my mind had conjured up images of a giant, room-size contraption, graphs fed in on one side and definite integrals spat out the other, complete with a hand-crank to make it work. Lying in an wooden box measuring about 100cm by 30cm, and looking a bit like a giant compass, reality was somewhat less impressive. John Groves, the machine's custodian for the last several decades, told me a bit about its history. Around 80 years old, the Intergraph had its heyday in the pre-computer age, when it allowed quick, if not entirely accurate, approximations of definite integrals. In the 60s, once affordable computers started appearing, it fell into disuse, and eventually John rescued it from under piles of junk to give it a more honourable resting place in his office, alongside other mathematical curiosities such as a parabola sketcher and an ellipse sketcher. He maintains an interest in such machines because of their intricate construction — he compares it to the fascination some people have for old clocks — and the 'Made in Geneva' tag sported by the Integraph certainly speaks of old-world precision.

More a curve sketcher than a numerical machine, the Integraph allows you to produce an integral curve for any original curve you care to put to paper, allowing you to read off the definite integral via the Fundamental Theorem of Algebra.[1] It works off a simple basic principle; the gradient of the integral curve is in proportion to the value of the height of the original curve, and so, as the user traces one end of the machine over the original curve, the other end

[1] $\int_a^b f(x) = F(b) - F(a)$.

draws a curve in a direction corresponding to this gradient. Simple enough in theory, in practice the contraption is a complex combination of hinges and sliding parts, and a closer look gave me a real appreciation for the workmanship that had gone into it. But to get a true feel for it, I needed to try it out.

'Look after it', John warned me as I took the thing down to the MUMS room to experiment, reminding me that it was worth somewhere in the five figures. 'They don't make things like that anymore.'

We'd penciled in around half an hour to do some basic definite integrals; a parabola, a half-circle and the density of the normal distribution, but confronted with a complex machine, and an instruction booklet weighing in at an ominous 50 pages, things took a little longer than expected. The first task was getting the Integraph to draw anything at all. Everything that could go wrong did; pencils screwed in too tight or too loose, clasps left undone when they should have been clamped, curves veering off the paper or off the table. Our first success at drawing — our machine producing a cubic as the integral of a standard parabola — was duly celebrated, until someone noticed the cubic curve was not flat at zero, meaning the calibration was wrong. And it was this second aspect that proved the more troublesome. Everything on the machine had to be aligned properly; the wheels parallel to the x-axis, the tracing arm aligned to give a y-value of zero, the drawing arm parallel to the x-axis of the integral curve. It all had to be physically perfect. And by the time we had mastered all of this, and were ready to produce some results, two hours had already elapsed and we were fast running out of time.

First up was the integral of an odd function, between -1 and 1. The integral curve ended up pretty much where it had started, equating to a definite integral of 0, so our hours-long attempt at calibration was finally paying off. Now to get a proper definite integral. We attacked the Normal distribution density, famous for not having an indefinite integral of closed form, thus whose definite integrals are mere approximations. Perhaps we could approximate them better with our newly calibrated Integraph! Tracing a standard normal between -2 and 2, we ended up with an integral curve which had a displacement of 167.0mm. Multiplying by the initial calibration factors (the gradient multiplier, the scales on the axes) we ended up with a definite integral of 0.9355 (to 4 dp) compared to the z-value in the tables of 0.9545 (to 4 dp). Perhaps time to update those tables?

A quick flick through the instructions demonstrates the versatility of the Integraph, beyond just solving definite integrals. For instance, by switching the position of tracer and pencil, it can differentiate any given curve. More surprisingly, given any closed shape, the Integraph, through a process of successive integrals, can find the centre of gravity of this shape. Finally, using repeated integration, it can sketch polynomials of any order, and thus by inspection solve the roots of these polynomials. To this end we used the Integraph to sketch a cubic, by starting with a linear graph and integrating twice, factoring constants in at each step.

The Integraph is a fascinating tool, but its accuracy depends much on the fastidiousness of its user. Requiring more than just careful calibration, since the actual tracing of the curve to integral has to be done manually a steady-hand can make the difference between success and failure. And since you are tracing the curve while rolling the 5kg-odd machine along behind you this is no easy task.

Nevertheless, the Intergraph is a window into a world which no longer exists, and using it you can't help experience a certain wonder at the achievements of this past age. The romantics would have it that the pre-computer era was more disciplined, more careful, more technically sound. And perhaps this is true. But, after spending three hours to set up a machine to tell me that the integral of x^2 on $x \in [-1, 1]$ is 0.6523, I was experiencing another sensation entirely; a resounding pity for those for whom the Integraph was not a curiosity but a necessity. Like the slide rule, there are those who lament the loss of the Integraph from the desks of mathematicians. But such people are in all likelihood not the ones who were ever forced to use them!

For those eager to check out this fascinating machine, there is one on permanent display in the Wilson lab in the Richard Berry building. Paradox would like to thank John for generously allowing us to test out the machine.

— Stephen Muirhead

Paradox **Speaks to Terry Tao**

We know everything, we just can't prove it.

—- Terry Tao

Terry Tao is a man who needs no introduction. Called the 'Mozart of Maths', the youngest ever winner of a gold medal at the International Maths Olympiad has well and truly lived up to his childhood potential, and a few years ago became Australia's first Fields medalist. He is nothing short of a legend amongst Australian mathematicians and maths students alike.

Understandably then, when Terry Tao made a whirlwind stop in Melbourne recently, he was a man in high-demand. Delivering three lectures in three days, his time was definitely not to be wasted. That's why, when I heard we'd be interviewing him for Paradox, I had to be damn sure I had my questions down. . .[1]

First impressions are often deceptive, and with Terry this was no different. As we started off the discussion, under the slightly intimidating gaze of a professional journalist from *The Australian* newspaper, he seemed more nervous than I was, with fast head movements and minimal eye contact. But being the battle-hardened interviewee that he is, Terry warms quickly to the task, and after a few hesitant one-word answers, he relaxes and begins to discuss openly. Our first topic is something that he is well known for commenting on: the state of mathematics in Australia.

Maths in Australia

Even though he is unable to visit Australia often, Terry retains a strong interest in the well-being of maths education in Australia, especially at the tertiary level. Through his blog,[2] he has drawn attention to the retrenching of staff in various maths departments around Australia over the last couple of years. Upon his return to the country, the first thing Terry did was assess how things are going on the ground. 'It doesn't seem to be getting much worse', he remarked carefully, 'but it doesn't seem to be getting better either.'

[1] Though, being so excited, I first had to re-learn how to hold a pen!
[2] http://terrytao.wordpress.com/

Terry is worried by the fact that maths departments are being broken down and refocused to serve the needs of more vocational disciplines, and that Australia runs the risk of failing to provide the same research opportunities for maths graduates available in other countries, such as the US. 'The top five universities are still world class and always have been, with people who are just as good, if not better [than those in the US], so it's still looking good in that respect. However there is a lot of pressure on the departments in the tiers below. For example, Victoria University has a well regarded inequalities unit, which is in danger of being cut as part of restructuring. In the worst case scenario, Australia may be left with only the top five universities' [with real maths departments].[3] Whilst the other universities will still be able to churn out bachelor degrees, already people in industry have started complaining that the graduates at these universities weren't being given the skills they needed for the workforce.'

Terry contrasts the bean-counting approach taken by some universities in Australia with the mindset prevalent in the United States, where many more universities, even those outside the top tier, compete to maintain their research quality. 'The funding situation is more stable and the universities are not trying to squeeze every last dollar out of student enrollment. They take a more long-term approach and invest in their research program. In the US, many of the administrators come from academic backgrounds, so they sort of 'get' research, and don't try to cut corners.' Nevertheless, though critical of the situation in Australia, Terry is careful to point out that not everyone should be tarred with the same brush. 'There are many good administrators in Australia as well.'

Maths in the wider society

Terry believes one of the reasons behind the slow erosion of maths faculties around Australia, and the surprising lack of public outcry about it, is the poor perception of maths in the wider society. 'Whilst it is the most basic of basic sciences, people don't see the relevance of maths, when it is far more relevant than they can imagine. Maths is used in everything and any major technological advance has its foundations in mathematics.'

One reason for this poor image is that mathematics is not well portrayed in

[3] Terry is (obviously) including Melbourne University as one of these five.

the public media. 'You get the slightly crazy genius stereotype and none of the cool role models that you might get in the other disciplines.' The abstractness of maths is another factor that prevents people relating well to maths, concedes Terry. 'With astronomy you can look into space, with physics you can watch a pendulum and so forth, but with maths it is more intangible. . . you're looking for the hidden factors.'

However, Terry notes that 'often it is precisely because the underlying idea is an abstract one that it is widely applicable. There are mathematical tricks which can be used all over the place, but people don't think of these tricks as mathematical, so mathematicians don't get the credit.'

He gives the example of the classic problem involving 12 gold coins where one of them is counterfeit and only distinguishable by its weight. The challenge is to work out which coin is fake using a scale no more than three times, and the trick is to weigh the coins in groups. While this appeared to be an exclusively abstract problem, during the first world war, when doctors needed an efficient way to test soldiers for TB and syphilis, someone realised the analogy; in any given population the prevalence of these diseases was very low, so it was more efficient to mix the blood of many soldiers together and test this sample instead. This 'group testing' is now ubiquitous. Terry's point is that 'instead of using biology to solve a specific problem, we can use maths to solve a more generalised version of the problem that can then be applied to many different fields.'

Terry lists other examples where maths has been incorporated into everyday life, such as the use of matrices in the Google page-rank algorithm, compressed sensing to make MRIs more efficient, and the use of disentangling algorithms on mobile phones to prevent interference when multiple phones are used in the same room. 'The world makes more sense when you look at it through maths. [Technology] is not just magic boxes that do things,' he jokes.

However, the rise of scientifically-themed television shows offers some hope in improving the image of mathematics. He believes the 'best way to make maths accessible is not to make it the focus of the show. . . which means not trying to ram it down people's throats.'[4] He mentions shows such as N3mb3rs and Mythbusters, who 'have mathematicians on staff and try to incorporate mathematics, but [who] don't try to make it too obvious.'

[4]On that note, we hope you find Paradox entertaining and accessible, and it's much better than having copies rammed down your throat. Trust me, I'm a doctor.

Though many students look back on their days of maths class with horror, Terry doesn't believe that people can be split into those who 'get' maths and those who don't. 'A lot of people say they wouldn't touch maths with a 10 foot pole[5] and yet they love Sudoku puzzles. Sudokus are really mathematical in the way you have to think about things and reason. So I think a lot of people are capable of doing this sort of thing, they are just not exposed to it.'

Terry believes that having a good teacher can make a 'huge difference' and feels that this was true in his case. 'You can have a bright student with a lousy teacher and he can be turned off the subject for life. But the converse holds true as well. I know quite a few research mathematicians who started out rather late. It's not like only 1% of 1% of the population can do this stuff. Some of the people I work with started off in completely different fields.'[6]

It is perhaps his own positive experiences which drive Terry to be as good a lecturer as he is a research mathematician. 'It makes you feel useful,' he candidly admits, because 'one of the problems with doing all this abstract stuff is that sometimes it is hard to feel like you are doing anything productive. But if you are talking to a student who doesn't get it and then suddenly they 'click' and you can see it in their eyes, then that is a really nice feeling. You've explained something and they're not going to lose that.'

Life as a professional mathematician

Terry freely concedes that an academic's life is 'different' from other jobs. 'Most jobs you're working nine-to-five and then you go home and you're done. If I have free time, then I'll always be thinking about maths. [A mathematician] is never really working full on and never really quits their job: you're never totally away from it.'

When asked for his approach to solving difficult problems, and perhaps even the secret to why he is so successful where others fail, Terry is typically modest. 'Often it's about keeping yourself in the right mindset. If you want to solve a problem there are lots of things you need to keep at your fingertips, and you have to have everything in your head at once. Working, in this sense is a subconscious thing. You're probably going to laugh at me but I don't really know how I'm doing it!'

[5] American idiom of unknown origin. Better known in Australia as the good old 3.048m pole.
[6] So there is still hope for yours truly!

While many undergraduates, in their young and foolish days, have dreamt of solving the famous unsolved problems in mathematics,[7] Terry warns that the life of a research mathematician is rarely as glamorous. 'A lot of the technological advances don't come as revolutions. You just try to make this year's product a little bit better than last year's, and after 10 years you have a really good product. That's how maths works these days. Mathematicians are pushing the envelope of their abilities and taking these small steps in understanding. It's more like an evolution than a revolution.'

Indeed, Terry actively shies away from working on these 'impossible goals' in mathematics, one which he likens to 'treasures on top of cliffs that no one can climb'. Instead he tackles what he calls 'basic research', the preparatory work for other people. 'For applied topics, I might get interested if someone can abstract out a problem for me to work on!'

Nevertheless Terry freely admits that mathematicians often fail to see the bigger picture. 'A lot of the maths that we do is not used for maybe 10 or 20 years. Then someone will come to us with a problem and we're like 'oh, we solved that a long time ago, we just never thought of an application!"

However, there are some areas where Terry believes true breakthroughs in mathematics can still be made. 'A lot of biomaths is just beginning to get off the ground. We want faster gene sequencing and to work out which genes are causing which diseases. Protein-folding involves a lot of complicated maths and we don't yet know how to do it properly. Another area is in financial maths. A misunderstanding of the mathematics underpinning risk analysis was one of the causes of the current financial crisis.'

Collaboration is also an integral part of the way mathematicians work these days. As part of his lecture at Melbourne Uni,[8] Terry showed how he and other research mathematicians used various tools such as forums, blogs and wikis to share their ideas and progress with each other, especially where solutions may span many diverse subfields of mathematics. As an example, Terry recounts how he posted Question 6 of this year's International Maths Olympiad paper[9] on his blog. In a powerful demonstration of collaborative mathematics he had over 30 partial solutions submitted. With people working together, they eventually got a complete solution in 2 days. This was a bit of fun, he jokes, 'but most of the problems I do now can't be solved in two days!'

[7] Everyone on the MUMS Committee, for example!

[8] 'Mathematics and the Internet', part of the Clay-Mahler series of lectures.

[9] A question only solved by three competitors.

Some advice from Terry

Many undergraduate readers of this magazine may be considering going overseas in order to do post-graduate study, and for their benefit I asked Terry to reflect on his personal experiences, and what advice he would give to people tossing up between staying put in Australia or heading overseas.

'I think it's a great idea, I definitely benefited a lot from going overseas. Certainly I think you should do your post-graduate study at a different university from where you were an undergrad. If you don't, you can get into the mentality where you're just doing some sort of fancy version of your undergraduate study. Post-graduate study is really quite different, you have to be much more independent and you have to go and seek out things on your own. If you want to learn something you can't just wait until your classes or exams, you have to go to the library or the internet or wherever. There are a lot of good places in Australia, but you should not be afraid of going overseas.'

Fun facts about Terry, gleaned from the interview

- In his first class as a lecturer, he was only one year older than his students. 'It was kinda cool to be the oldest person in the room for a change,' he jokes.
- At the 1987 International Maths Olympiad in Cuba, Terry was thrown into the pool by one of the non-medalists on the team, obviously jealous!
- Terry enjoys reading fantasy novels, and still remembers the characters and places from Eddings and Pratchett novels.
- He says he can solve a Rubik's cube 'in theory'.[10]
- He has an Erdös number of three, a Bacon number of ∞,[11] and a Colbert number of zero![12]
- Riemann Hypothesis: true or false? 'True! But not solvable in the next 50 years.'

[10] Note this probably means he has a non-constructive proof!

[11] If you don't know what these mean see http://en.wikipedia.org/wiki/Erdos-Bacon_number

[12] 'The Colbert Report', Sept 26, 2006: 'Terence Tao, mathematician...I'll give you a number: ZERO!'

- What about the Twin Prime Conjecture and Goldbach's Conjecture? 'True, true, all true. In number theory, we know everything, we just can't prove it.'

Epilogue

With our allotted time rapidly running out, we felt we had to use this opportunity get Terry's autograph, and what better than to get him to sign a MUMS t-shirt.[13] Though, this being a semi-official interview, and with the journalist from *The Australian* still in the room, we weren't at all sure how he would respond. To our relief, not only did he graciously accept, he even looked a little bit chuffed to be asked. With the journalist frantically scribbling in his notepad, Terry recounts how he was similarly ambushed by some high-school kids that had recognised him on the street in Sydney.

Despite his obvious modesty, I get the feeling even Terry can't help but feel a little satisfied by such rare events, not so much for the sake of his own ego, but more the fact that a mathematician, any mathematician, is getting the same sort of media and public attention that we have so long heaped on other members of our society.[14]

Yet these thoughts come crashing back to earth when we offer to help Terry find his way back to the Richard Berry building from the AMSI offices on the other side of campus. As we gratefully seize this opportunity to ask him questions we hadn't had time for in the interview, I am struck by how inconspicuous the three of us are, looking just like any other group of students on campus, getting barely a glance from passers-by. He may be the 21st century's only celebrity mathematician, but the people on South Lawn clearly could not care less!

— Charles Li

Charles and Terry outside the MUMS room.

[13] Which now sits proudly in the MUMS room.
[14] Sportsmen being the obvious example, but let's not go there.

A Quarter with Professor Tao

A 760 for the SAT math section at eight; IMO gold medalist at 13; Ph.D. from Princeton at 20; full professor at 24; Fields Medalist at 31. Mathematicians in Australia would know that these are not the achievements of John Nash of 'A Beautiful Mind' fame, or 'that person from 'Good Will Hunting" but rather a former child prodigy from humble Adelaide. In 2006, Terence Tao became the first Australian and one of the youngest ever to win the coveted Fields Medal, garnering a host of acclaim from renowned mathematicians around the world and becoming an instant celebrity amongst students of mathematics in Australia.

At the start of 2009, I was one of these students who had the fortune of meeting Terence Tao in person. As an exchange student at UCLA for the Fall of 2008, I was delighted to discover that Professor Tao was teaching a graduate course in real analysis in the following quarter. Upon gathering the relevant paperwork to bypass the bureaucracy, I successfully enrolled in the class as an undergraduate student.

Terry Tao has always been known as a young achiever, yet it is still easy to forget that he is only in his early thirties. When he walked into the classroom for the first lecture, there was a noticeable moment of realisation for some; the groundbreaking mathematician did not look much older than the class he was teaching. Casually dressed and donning a small backpack, Tao could well be mistaken for a student, while a passerby could be forgiven for thinking that his teaching assistant was the professor.

The prevailing wisdom amongst students is that great researchers do not necessarily make great teachers; however, Terry Tao certainly does not fall in this category. Meticulous and organised, Professor Tao prefers a multi-pronged approach to learning. He is a traditionalist, a user of the chalk and blackboard during class, yet he also realises the value of the internet and the blog, often posting detailed notes on his personal webpage.[1] He also did not hesitate to assign problems from a textbook authored by another mathematician. He often enjoys discussing the intuition and motivations behind a mathematical result, preferring to focus on conditions that would cause a theorem to fail rather than those that would allow it to hold.

[1] Terry Tao maintains his blog at http://terrytao.wordpress.com/ with notes to his courses as well as his research.

The course itself, while challenging, was not prohibitively hard; at the start of the course, the assignments were long and tricky, however, as the quarter continued, the assigned problems became more accessible. Many of us felt the professor was deliberately recalibrating the difficulty to suit the abilities of students. In between all the serious discussions and assessments on mathematics, there was still some space for humour. The professor once claimed that he had a "three-line proof" to a not-so-trivial theorem, and after some hand-waving, the crux of the proof involved drawing three lines. Another time, shortly before the final exam, he quipped: 'Once you've done all the exams and quals,[2] that's when the real learning begins.'

Tao's celebrity status is of course not limited to our shores; throughout the quarter, the class size would fluctuate, with various students suddenly appearing in class on some days while the more enthusiastic did not even pretend to be legitimately enrolled. When the last class for the quarter ended, in what can only be described as a calculated ambush, two students stepped into the room and in full view of our class, requested a photo with Professor Tao, who obliged with a wry smile.

Those who were in his class had subtler ways of having an audience with the professor; his office hours would be utilised by some students to discuss all things mathematical, both related and unrelated to the course. At the end of the course, I also used this as a means for a general chat with Professor Tao about various topics including 'jobs in industry versus academia'. The end of a quarter with Professor Tao was capped off by him signing my textbook.

— Zhihong Chen

Solution to the Puzzle on Page 7

Yes! Consider a unit cube viewed from directly above one of its vertices. The cross-section of the cube from this angle is a hexagon of side-length $\frac{\sqrt{3}}{\sqrt{2}}$, and the largest square that fits inside this has side-length $\sqrt{6} - \sqrt{2} \approx 1.035$. Thus a cube of side-length less than 1.035 can pass through it. See *YouTube* for a demonstration! Interestingly, the largest cube which can pass through, the Prince Robert's Cube, has side-length $\frac{3\sqrt{2}}{4} \approx 1.061$.

[2]Most PhD programs in Mathematics in the USA will require students to complete coursework and pass qualification exams in major areas of mathematics, including analysis, algebra and geometry.

Maths and the Law

When I tell people that I study both Maths and Law, the typical response is surprise: 'That's a strange combination! How do they relate to each other?' Most of the time, the answer is that they don't. This article gives an overview of some areas of overlap between the two.

Famous Lawyer-Mathematicians

There are not many Law/Maths students out there,[1] but to console those of us who feel outnumbered by the masses of Law/Arts, Law/Commerce and Commerce/Science students, here are some examples of Law/Maths students who have made it to the pinnacle of their profession.

Lord Denning (1899-1999)

> Quite apart from anything else, the elementary mathematics of judges are prone to error, except in the case of Lord Denning who was a wrangler, and his maths are too good for anyone else to understand.[2]

Described as 'the most celebrated English judge of the 20th century',[3] Lord Denning is a figure familiar to any law student who reads their prescribed course materials. He was a judge in the House of Lords from 1957-62 and Master of the Rolls from 1962-82. Known for his easy-to-read and sometimes radical judgments, his achievements in law included inventing the Mareva injunction and establishing 'deserted wife's equity'.

What is less well-known is the fact that Lord Denning studied pure mathematics at Oxford and graduated with first class honours in 1920.[4] He subsequently taught maths at Winchester College for a short period, before deciding to study law and join the bar.

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[1] As an interesting side note, MUMS seems to be a magnet for Law/Maths students. On our current committee (see http://www.ms.unimelb.edu.au/~mums/committee/), four of the 12 members are Law/Maths students.

[2] Anthony Nicholson, *Esprit de Law* (1973) 236.

[3] Clare Dyer, 'Lord Denning, Controversial "Peoples Judge", Dies Aged 100', *The Guardian*, 6 March 1999 http://www.guardian.co.uk/uk/1999/mar/06/claredyer1.

[4] Edmund Heward, *Lord Denning: A Biography* (1990) 11; Iris Freeman, *Lord Denning – A Life* (1993) 65.

Harry Blackmun (1908-99)
Harry Blackmun was a justice of the US Supreme Court from 1970-94. He is best known as the author of the Court's opinion in *Roe v Wade*,[5] which enshrined a constitutional right to abortion. *Roe v Wade* was fiercely debated and Blackmun reportedly received death threats over the case.

Like Lord Denning, prior to achieving fame in the legal arena, Blackmun had studied mathematics at university, graduating from Harvard College *summa cum laude*[6] in 1929.[7]

Pierre de Fermat (d. 1665)
Given that this is a maths magazine, Fermat needs no introduction. These days, Fermat is remembered solely for his mathematical achievements. But Fermat's primary occupation was actually as a lawyer and jurist; maths was a part-time preoccupation, albeit one at which he excelled. Fermat received the title of councillor at the High Court of Judicature in Toulouse in 1631, a title he held until his death.[8]

Mathematics in Law

> Mathematics, a veritable sorcerer in our computerized society, while assisting the trier of fact in the search for truth, must not cast a spell over him.[9]

This section looks at some of the areas in which courts may rely on mathematics (usually in the form of probability or statistics) in reaching their decisions. Each area includes references to secondary sources if you are interested in a more comprehensive treatment of the maths in question.

Standards of Proof

In civil cases, the standard of proof is 'on the balance of probabilities'.[10] In other words, for the plaintiff to succeed, the court must be satisfied 'on the

[5] 410 US 113 (1973).
[6] 'With the highest praise'.
[7] Wikipedia, *Harry Blackmun*, http://en.wikipedia.org/wiki/Harry_Blackmun.
[8] Wikipedia, *Pierre de Fermat*, http://en.wikipedia.org/wiki/Pierre_de_Fermat.
[9] *People v Collins*, 438 P 2d 33 (1968).
[10] In contrast, the standard of proof in criminal cases is 'beyond reasonable doubt'.

balance of probabilities' that they have made out their case.

The extent to which maths plays a role in this standard of proof is unclear.[11] Lord Simon in *Davies v Taylor* stated that:

> [b]eneath the legal concept of probability lies the mathematical theory of probability. Only occasionally does this break surface – apart from the concept of proof on a balance of probabilities, which can be restated as the burden of showing odds of at least 51 to 49 that such-and-such has taken place or will do so.[12]

But Dixon J in *Briginshaw v Briginshaw* expressed his disapproval of such 'mechanical comparison[s] of probabilities'[13] and adopted a more common-sense approach requiring the plaintiffs allegation to be made out to the 'reasonable satisfaction' of the court.

Given that in many cases it is impossible to assign precise probabilities to events in question, it can be seen why the common-sense approach is more widespread. Nonetheless, some judges have seized the opportunity to use the probability they learnt in high school. An example is the NSW Court of Appeal's judgment in *Rose v Abbey Orchard Property Investments Pty Ltd*.[14] The defendant owned a car park, and the plaintiff had slipped on an oil patch on the floor in the car park. The plaintiff sued the defendant for injuries she had suffered in the fall, arguing that the defendant had breached its duty of care to maintain a proper system of inspection to avoid such accidents. The plaintiff had been injured at 3:10pm. It was unknown how long the oil patch had been on the floor, but the last inspection prior to the accident was at 2:10pm. The Court held that a proper system required inspections at intervals of not more than 20 minutes; the defendant would thus be liable if, on the balance of probabilities, inspections at 2:30pm and 2:50pm could have prevented the accident. The Court reasoned:

> But... the probabilities are twice as great that the oil was spilt in the 40 minutes period between the last inspection and 2:50pm rather than the 20 minute period after 2:50pm... Accordingly, we think that as a matter of probability the oil was spilt before 2:50pm and not after that time. To so find is not to engage in speculation but

[11] See Justice D H Hodgson, 'The Scales of Justice: Probability and Proof in Legal Fact-Finding' (1995) 69 *Australian Law Journal* 731.

[12] [1974] AC 207, 219.

[13] (1938) 60 CLR 336, 361.

[14] (1987) *Aust Torts Reports* 80121.

> to make a finding in accordance with probability theory. In a civil case the plaintiff has only to prove her case on the balance of probabilities. It follows, therefore, that on the probabilities the oil was spilt at a time which would have permitted a proper system of inspection to remove it prior to the time of the accident.[15]

Mens Rea of Murder

In Australia, a person who kills someone can be convicted of murder if they satisfy one of the two types of 'mental states' (or *mens rea*) of murder: intent or recklessness. A person is reckless in the requisite sense when he/she knew that death or grievous bodily harm would probably result from his/her conduct but nonetheless chose to act that way.[16]

The question is: what does knowing that death would *probably* result actually mean? Here, the courts have been keen to divorce the definition of recklessness from any mathematical notion of probability; the High Court has pointed out that the average person does not think about the consequences of his/her actions in terms of mathematical probabilities.[17] This has lead to a rather messy state of affairs where 'probably' is given a different meaning to possibly, but includes situations where the mathematical probability of death is less than 0.5.

A good illustration of this is the interesting case *R v Faure*.[18] The defendant shot his girlfriend in the head and was charged with murder. He claimed that he hadn't intended to kill his girlfriend; rather, they had been playing a game of Russian roulette, using a six-chamber revolver with a single cartridge in one of the chambers. The defendant and victim took turns pulling the trigger while pointing the revolver at the others head, agreeing to stop after each had had two turns. On the defendant's final turn, the revolver discharged, killing the victim. The issue was whether the defendant could be convicted of reckless murder: presuming his story to be true, did the defendant know that death or grievous bodily harm would probably result from this game of Russian roulette?

The judgment in *Faure* noted that the probability of the revolver discharging

[15] Ibid [68].
[16] *R v Crabbe* (1985) 156 CLR 464.
[17] *Boughey v The Queen* (1986) 161 CLR 10, 1920 (Mason, Wilson and Deane JJ).
[18] [1999] 2 VR 537.

at some point in the Russian roulette game was 671/1296, but went on to say that 'the case is not to be approached as Pascal would have approached it.'[19] A defendant could be regarded as knowing that death would probably result even where the mathematical probability of death is relatively low: eg. if the trigger was only pulled once (probability of discharge = 1/6).[20]

Prosecutor's Fallacy

In a criminal case, a prosecutor presented evidence that the defendant's blood type matched that found at the crime scene. This blood type is only found in 10% of the population. The prosecutor thus reasoned that there was a 10% chance that the defendant would have this blood type if he were innocent, and concluded that there was a 90% chance that the defendant was guilty as charged.[21]

The flawed logic in the above scenario is an example of the prosecutor's fallacy, a term coined by Thompson and Schumann in 1987.[22] Intuitively, the prosecutor's conclusion regarding the defendant's likelihood of guilt cannot be right, as it could be applied to *any* person with that blood type, all of whom bar one would be innocent. To draw any conclusions about the defendant's guilt from the blood type match, it is necessary to take into account the *a priori* likelihood of guilt. If there is little other evidence pointing towards the defendant's guilt, a conviction based on the match is unsound.

For a more mathematical analysis of the fallacy, we can apply Bayes' Theorem. Suppose E = observed evidence (eg. the blood type match in the above scenario), I = defendant is innocent, $\bar{I}$ = defendant is guilty, and $P(E|I)$[23] is small. The prosecutor's fallacy assumes that if $P(E|I)$ is small, then $P(I|E)$[24] is correspondingly small. But, in fact, Bayes' Theorem tells us:

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[19] Ibid 547 (Brooking JA).

[20] Ibid 551.

[21] This scenario is taken from William C Thompson and Edward L Schumann, 'Interpretation of Statistical Evidence in Criminal Trials: The Prosecutor's Fallacy and the Defense Attorneys Fallacy' (1987) 11 *Law and Human Behaviour* 167.

[22] Ibid.

[23] The probability that the evidence would be observed if the defendant were innocent.

[24] The probability that the defendant is innocent given the observed evidence.

$$P(I|E) = \frac{P(E|I) \cdot P(I)}{P(E)} = \frac{P(E|I) \cdot P(I)}{P(E|I) \cdot P(I) + P(E|\bar{I})P(\bar{I})}$$

This means that if P(I)[25] is much larger than $P(E)$, then $P(I|E)$ can also be quite large. Returning to our blood type match example, if the other evidence against the defendant is weak so that $P(I) = 0.80$, then:

$$P(I|E) = \frac{0.1 \times 0.8}{0.1 \times 0.8 + 1 \times 0.2} = \frac{2}{7} \approx 0.29$$

As can be seen, there is quite a high probability that the defendant could be innocent — a conviction in such circumstances would be worrying. Thompson and Schumann's experiments point to a real risk that a significant minority of jury members could reason in accordance with the prosecutors fallacy, especially when only given $P(E|I)$.[26]

An infamous and tragic example of the prosecutor's fallacy at work is the Sally Clark case.[27] Clark was a British solicitor whose first son died suddenly at 11 weeks old, and whose second son also died suddenly at 8 weeks old. She was charged with two counts of murder. At her trial, the paediatrician Sir Roy Meadow gave evidence that the probability of two children from an affluent family dying of SIDS was 1 in 73 million; the figure was obtained by squaring the probability of a single death from SIDS (1 in 8543).[28] Clark was subsequently convicted by a 10-2 majority verdict and sentenced to life imprisonment.

Following her conviction, the Royal Statistical Society issued a press release criticising Meadow's statistical evidence.[29] It noted that in simply squaring the probability of a single SIDS death, Meadow had made an unfounded assumption that SIDS deaths were independent events, when in fact genetic or environmental factors can predispose families to SIDS. Further, the manner in

[25] The prior probability of innocence. This is an estimate of the defendant's innocence on the basis of the other evidence in the case (i.e. not including E).

[26] Thompson and Schumann, above n 21.

[27] Wikipedia, Sally Clark, http://en.wikipedia.org/wiki/Sally_Clark.

[28] *R v Clark* [2003] EWCA Crim 1020, [96] http://www.bailii.org/ew/cases/EWCA/Crim/2003/1020.html.

[29] Royal Statistical Society, 'Royal Statistical Society Concerned by Issues Raised by Sally Clark Case' (Press Release, 23 October 2001).

which Meadow presented his evidence may have led the jury into the prosecutor's fallacy. The RSS noted that some press reports had misinterpreted the figure of 1 in 73 million to represent the probability that Sally Clark was innocent; given the sensational nature of the figure,[30] it is possible that jury members also make this mistake. The RSS stated:

> The jury needs to weigh up two competing explanations for the babies' deaths: SIDS or murder. Two deaths by SIDS or two murders are each quite unlikely, but one has apparently happened in this case. What matters is the relative likelihood of the deaths under each explanation, not just how unlikely they are under one explanation (in this case SIDS, according to the evidence as presented.[31]

Clarks conviction was overturned in 2003, with the Court of Appeal stating that Meadow's statistical evidence 'should never have been before the jury in the way that it was when they considered their verdicts.'[32] Clark was released after three years in jail, but she did not recover from her ordeal and died of accidental acute alcohol intoxication in March 2007.[33]

Another example of misuse of statistics in criminal trials to achieve conviction is *People v Collins*.[34] Here, witnesses to a robbery had observed certain characteristics of the perpetrators (a Caucasian female and an African-American male), each of which were possessed by the defendants. The prosecutor presented the following probabilities to the jury:

Yellow automobile	1/10
Man with moustache	1/4
Girl with ponytail	1/10
Girl with blond hair	1/3
African-American man with beard	1/10
Interracial couple in car	1/1000

The prosecutor proceeded to multiply all the probabilities together and con-

[30]Sir Roy Meadow likened the figure of 1 in 73 million to the chances of backing an 80-1 outsider in the Grand National (an English horse race, like the Melbourne Cup) four years running, an winning each time.

[31]Royal Statistical Society, above n 29.

[32]*R v Clark* [2003] EWCA Crim 1020, [177] http://www.bailii.org/ew/cases/EWCA/Crim/2003/1020.html.

[33]'Obituaries: Sally Clark', *The Times*, 19 March 2007 http://www.timesonline.co.uk/tol/comment/obituaries/article1533755.ece; 'Alcohol Killed Mother Sally Clark', *BBC News*, 7 November 2007 http://news.bbc.co.uk/2/hi/uk_news/england/essex/7082411.stm.

[34]438 P 2d 33 (1968).

clude that the chance of an innocent couple possessing all these incriminating characteristics was one in 12 million; the jury convicted. It is not surprising that the Supreme Court of California quashed the convictions on appeal; the figures for the individual probabilities had little evidence to support them, the prosecutor had erroneously assumed that each of the characteristics were independent, and the prosecutor's fallacy is also likely to have reared its ugly head.[35]

Defence Fallacy

At the other end of the spectrum to the prosecutor's fallacy is the defence fallacy. This is where members of the jury are persuaded that evidence of characteristic matches (eg, the blood type match considered above) is irrelevant, because the evidence merely shows that the defendant and the actual criminal are members of the same group. For example, if (as above) we take the prevalence of a particular blood type found at the crime scene to be 10%, in a city of 1 million people there would be 100,000 people who could be incriminated by this evidence. Victims of the fallacy would not see any probative value in the fact that the defendant and actual perpetrator both belong to such a large group.

This conclusion is reasonable when there is no other evidence against the defendant, but it becomes a fallacy when there are other indicators of the defendant's guilt. In the latter situation, the defence fallacy fails to recognise that the blood type match drastically narrows the group of people who could be suspects, while failing to exclude the defendant.

Thus, for example, if the other evidence in the case is sufficiently strong so that the prior probability of innocence, $P(I)$, is around 0.2,[36] the additional evidence of the blood type match can significantly increase the likelihood that the defendant is guilty. Again applying Bayes' Theorem, we have:

$$P(I|E) = \frac{0.1 \times 0.2}{0.1 \times 0.2 + 1 \times 0.8} = \frac{1}{41} \approx 0.024$$

[35] For a more detailed discussion, see Michael O Finkelstein and William B Fairley, 'A Bayesian Approach to Identification Evidence' (1970) 83 *Harvard Law Review* 489.

[36] While this is low, it would probably be insufficiently low to satisfy the 'beyond reasonable doubt' standard of proof in criminal cases.

Thompson and Schumann's experiments suggest that jury members' susceptibility to the defence fallacy is even greater than in the case of the prosecutor's fallacy. In their second experiment, which was similar to the blood type match scenario given here, 66% of the participants made one or more judgments consistent with the defence fallacy.

Conclusion

> [I]magine a law student approaching the study of the Constitution for the first time ... He buys his law books, opens his notebook and begins with a historical survey [of s 92]. ... The student [understandably] closes his notebook, sells his law books, and resolves to take up some easy study, like nuclear physics or higher mathematics.[37]

Hopefully this article has shown that maths and law do intersect in some interesting ways. The prosecutor's fallacy and the defence fallacy suggest that having lawyers who are comfortable with mathematics, especially statistics, might assist the administration of justice. Sally Clark's case serves as a potent reminder that mixing inaccurate statistical reasoning with the law can result in terrible consequences.

I'd like to conclude on a frivolous note by referring to the hilarious antics of Theodore Rout, who in 2003 exhorted the High Court of Australia to exercise its jurisdiction 'to apply the law of mathematics and physics... to add, subtract, divide and multiply.'[38]

Rout has a website[39] on which he claims, amongst other things, to have shown that $\frac{0}{1} = \frac{1}{0}$. Aggrieved at the lack of recognition given to him by the academic community, Rout aired his grievances before two members of the High Court in the form of an electoral petition against MP Bob McMullan. A theme running throughout Rout's submission is a conspiracy regarding dividing and multiplying by zero:

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[37] Sir Robert Garran, *Prosper the Commonwealth* (1958) 413-15, cited by the High Court in *Cole v Whitfield* (1988) 165 CLR 360.

[38] Transcript of Proceedings, Rout, *An Application by C4/2002* (High Court of Australia, Kirby and Heydon JJ, 14 March 2003) http://www.austlii.edu.au/au/other/hca/transcripts/2002/C4/1.html.

[39] http://home.pacific.net.au/~t_rout/

> And the law is their set of dividing and multiplying by zero. As long as they maintain their incorrect dividing and multiplying by zero, then they enable me to cause things to cease to exist, and that is why I have the power to do so...
>
> Now, I have proven everything is on nothing so if everything is on nothing and you multiply it by zero, then the entire universe and the world does not exist. I have proven it conclusively.
>
> [T]he High Court of Australia does have jurisdiction to apply adding, subtracting, dividing and multiplying and I am asking the Court to do precisely that, to subject them to their own dividing and multiplying by zero.[40]

Rout goes on to argue that, since $\frac{0}{1} = \frac{1}{0}$ can be shown to imply that all numbers are equal, MP Bob McMullan's election was fraudulous as he received no more votes than any other candidate. No prizes for guessing whether Rout succeeded.

— Julia Wang

Some Mathematical Anecdotes

1) Once Lord Kelvin used the word 'mathematician' while lecturing and then, interrupting himself, asked his class: 'Do you know what a mathematician is?' Stepping to his blackboard he wrote upon it: $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$, and then turned to his class and said, 'a mathematician is one to whom this is as obvious as that twice two makes four is to you.'

2) In 1915 Emma Noether was invited by Hilbert and Klein to join the maths department at the University of Göttingen. On an objection by the Philosophy department she was denied a formal place on the faculty, the argument being that a woman's place was not in the University Senate. Hilbert's reaction was: 'Gentlemen! There is nothing wrong with having a woman in the Senate. The Senate is not a bath.'

3)Apparently at Harvard there was once a graduate Maths course whose final consisted of just one line: 'Make up an appropriate final exam for this course and answer it. You will be graded on both parts.'

[40] Ibid.

Counter Revolutions

The Riemann hypothesis[1] is often called the greatest and most important unsolved problem in maths. And interestingly, despite resisting proof for so long, the mathematical community holds near unanimous belief in its truth; to the extent that there is a mountain of mathematical results which commence; 'assuming that the Riemann hypothesis is true, then…'. Over time, then, the utter desperation to prove the result has both increased and diminished; increased because a significant portion of modern mathematics is founded on its truth, and diminished because such conviction in its truth renders a rigorous proof almost superfluous.

But why this certainty? Well for one thing, as Terry Tao remarked when Paradox posed this question to him,[2] there is simply no reason for it to be false; in fact we would be astounded if it was. Perhaps more persuasively, the result has been checked for the first 10^{13} Riemann-Zeta function zeros. In other words, if there was a counterexample, it would have to be enormous!

Yet mathematical history is littered with famous counterexamples to conjectures previously believed to be true. Even some of the greats, most notably Fermat, have ended up with (posthumous) egg on their face, embarrassed by counterexamples to their conjectures. When Fermat assumed that $2^{2^n}+1$ was prime for all integers n, little did he suspect that a hundred years later Euler would demonstrate that $2^{2^5}+1 = 641 \times 6,700,417$!

If a counterexample at $n = 5$ looks like carelessness, ones which emerge at values much, much larger start to be really surprising. And when the mathematical community is convinced a result is true until one day a large counterexample rudely interjects, you get true upheavals in mathematics.

The Prime Race

All primes aside from two take the form $4k+1$ or $4k+3$ for some integer k. A natural question then is to ask which of these types are more common. As there are infinitely many primes, a succinct way to state this is: for each n,

[1] In a nutshell, it is the statement that the complex roots of the Riemann-Zeta function all have real part $\frac{1}{2}$. See http://en.wikipedia.org/wiki/Riemann_hypothesis for more details.

[2] See the interview with Terry on page 12 of this edition.

what is the relationship between $\pi_1(n)$, the number of primes less than n of form $4k+1$, and $\pi_3(n)$, the number of primes less than n of form $4k+3$? This question, the so-called Prime Race, was first described by Chebyshev in 1853, who conjectured that $\pi_3(n) \geq \pi_1(n)$ for all n. Indeed, given the data below,[3] it is easy to see why.

n	# of $4k+1$	# of $4k+3$
100	11	13
200	21	24
300	29	32
400	37	40
500	44	50
1000	80	87
2000	147	155
3000	211	218
5000	329	339
10000	609	619
100000	4783	4808

While Chebyshev may have been right almost all of the time,[4] an exhaustive check of the first 30,000 values of n shows that the race leader does in fact swap, and so $\pi_1(n) > \pi_3(n)$ for certain values of n. In fact, Littlewood in 1914 showed that the race leader swaps infinitely often! The reason that this was not established sooner is that the first example of such a swap is at $n = 26,861$, a value far, far beyond what would be verified to get an intuitive belief in the conjecture. Thus Chebyshev's intuition was tripped up by a very large counterexample; embarrassing, but he wouldn't be the last.

Pólya's Conjecture

Yet the counterexample in the Prime Race is mere child's-play next to some of the others. Sure, verifying up to 27,000 may be tedious if done by hand, but in the age of the computer such a counterexample would be found in an instant. Much, much worse was to come…

[3]http://www.dms.umontreal.ca/~andrew/PDF/PrimeRace.pdf.

[4]Indeed it is still an open conjecture as to whether the proportion of time that we have $\pi_3(n) \geq \pi_1(n)$ tends to 1 as $n \to \infty$.

In 1919, Pólya was considering whether numbers had an odd or an even number of prime factors when written out in their full prime factorisation. For example, $6 = 2 \times 3$ has an even number of factors, whereas $18 = 2 \times 3 \times 3$ has an odd number. By defining $O(n)$, the number of positive integers less than n with an odd number of prime factors, and $E(n)$, the number of positive integers less than n with an even number of prime factors, Pólya initiated his own version of a 'race', and conjectured that $O(n) \geq E(n)$ for all $n > 2$.

Knowing the history of the Prime Race, it is a fair bet that Pólya took the time to check his conjecture for many early values of n, perhaps even as far as several hundred thousand or more. Unfortunately for Pólya and his computation, the lowest is at a whopping $n = 906, 150, 257$. No wonder it took mathematicians 61 years to find it!

But it gets worse...

The mother of all large counterexamples has an even more turbulent history. For, although we know a counterexample must exist, and although we know it must exist within a certain bound, this bound is so unfathomably enormous that no-one has yet found an explicit counterexample! It is proof by counterexample without an explicit construction!

This counterexample relates to one of the most important results on the primes of all, the Prime Number Theorem, which can be stated: The number of primes less than n, denoted $\pi(n)$, is asymptotically approximated by $Li(n) = \int_2^n \frac{1}{log(t)} dt$. While this theorem was proved in 1896, what was not certain was whether $Li(n) > \pi(n)$ for all n, as appeared to be the case. Though this result held for all values of n physically checked, and though it had been proved that $Li(n) - \pi(n)$ should be roughly growing as n got larger, in 1914 Littlewood managed to prove that $Li(n)$ couldn't possibly be always greater than $\pi(n)$. Yet he was unable to demonstrate what this counterexample was, or how large we should expect it to be.

Enter Skewes, a student of Littlewood, who set about trying to discover an upper bound for this counterexample. After several false attempts, including one that assumed the truth of the Riemann Hypothesis, he finally came up in 1955 with the infamous Skewes' number; $10^{10^{10^{963}}}$, a number so utterly enormous that a comparison with anything in observable life becomes meaningless. Sufficient to note that, whereas a $googleplex = 10^{10^{100}}$, is already larger than the number of elementary particles in the universe, even $10^{googleplex}$ is insignificant when compared to Skewes' number. Aside from the ridiculously enormous Graham's number, it remains the largest number ever used in a mainstream mathematical proof.

Fortunately for those wanting to construct an explicit counterexample, the upper bound has been greater reduced. The most recent result puts it at less than 1.4×10^{316}, still larger than the number of particles in the universe, but at least within the same ballpark! Perhaps more surprisingly, it has been suggested that the first counterexample lies at around 1.397×10^{316}, though this is unproven. The search for the elusive counterexample goes on, though if the value is indeed this large then we should not be holding our breath.

So is the Riemann hypothesis on shaky ground?

What becomes obvious from the examples above is that there are many false statements for which the first counterexample is utterly enormous to the point of being undiscoverable. Thus the nominal evidence for the Riemann conjecture, that the first 10^{13} zeros satisfy the conjecture, is really no evidence at all.

Moreover, these examples all related to the behaviour of the primes, and are indicative of their unruly and unpredictable behaviour. Similarly, the Riemann hypothesis is inextricably linked to the distribution of the primes, and so these examples are particularly pertinent when asserting the truth of this famous hypothesis.

But perhaps the most damning counterexample of them all was the disproof of the Merten's conjecture in 1985 after almost a hundred years of speculation that it was true. This conjecture is similar to Pólya's conjecture in considering whether numbers have an odd or even number of prime factors. The key difference is that, whereas n still ranges over all the natural numbers, for the purposes of our running sums we only count numbers that are square-free, and discard numbers that are not. Using the notation from above, Merten's conjecture states that $|O(n) - E(n)| \geq \sqrt{n}$ for all n.

Despite this conjecture being true for at least the first 10^{14} values of n, it has nevertheless been shown that a counterexample must exist. Once more, this counterexample is so enormous that it has not yet been found explicitly. All we know is that lies somewhere less than 10^{41}.

The importance of this counterexample for the purposes of the Riemann hypothesis extends beyond a mere warning that large counterexamples can and do exist. More critically, Merten's conjecture logically implies the Riemann Hypothesis, in other words, had Merten's conjecture been true, the Riemann hypothesis would have been true as well. While patently the converse implication does not follow, this is nevertheless a clear warning that the Riemann hypothesis may too, one day, fall to a large counterexample. Whether that counterexample is the 10^{30}th zero, the $10^{googleplex}$th zero, or at a zero even larger, until the result is proven once and for all we simply cannot rule out that such a counterexample exists.

In conclusion, while it may be currently popular to assert that the Riemann hypothesis is definitely true, maybe it would be wise to take heed of Karl Marx: the revolution may well be coming.

— Stephen Muirhead

Postscript: Are you a revolutionary?

In light of the above, if you have ever felt the pressing need to prove one of the great unsolved conjectures, but are not sure your mathematics ability is quite up to it, finding a counterexample to an unsolved conjecture is a sure-fire way to get into the record books. And the task could be easier than you think. Consider;

Given any whole number n, perform the following iteration.

1. - If n is even, replace n with $\frac{n}{2}$.
   - If n is odd, replace n with $3n + 1$.
2. Repeat.

Does this iteration always eventually reach the value of 1?

This is the famous $3n + 1$ conjecture,[5] first proposed in 1937. Though this conjecture is widely believed to be true, and has been verified for $n < 10^{18}$, it would only take one value of n to start cycling for it to be proved false. And, best of all, anyone who can multiply by 3 and divide by 2 can join the fun. Posterity awaits…

[5] It is also known as the Collatz conjecture.

Uncovering a Track to Mt Solution

> When asked what it was like to set about proving something, the mathematician likened proving a theorem to seeing the peak of a mountain and trying to climb to the top. One establishes a base camp and begins scaling the mountain's sheer face, encountering obstacles at every turn, often retracing one's steps and struggling every foot of the journey. Finally when the top is reached, one stands examining the peak, taking in the view of the surrounding countryside and then noting the automobile road up the other side!
>
> — Robert J. Kleinhenz

> We have a habit in writing articles in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on. So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work.
>
> — Richard Feynman, American physicist, Nobel Lecture, 1966.

Background story

In the last issue of Paradox, I noticed the article entitled *Don't trust your instincts*, where the starting premise was the following question; what comes next in the sequence 1, 2, 4, 8, 16? Just arrived in Melbourne, I was surprised that the very same question was raised during a month spent in a middle of nowhere in Slovenia.

We were having a discussion on the merits of an IQ test, which involves asking similar questions. It was pointed out that the next number in that sequence can be anything, and the first example used to demonstrate that the answer is not necessary 32 was the following construction:

> Start with a circle drawn on a piece of paper with n vertices on it. Now, connect all pairs of vertices with a line. Let $A(n)$ be the maximal number of regions inside the circle formed by n vertices.

It should be noted that $A = 1, 2, 4, 8, 16, 31$ for $n = 1, 2, 3, 4, 5, 6$. The question is what comes next in the sequence? This prompted an unfriendly competition between me and another maths student, which I lost.

This article is about how I went about solving the problem.

My route to the solution

In a quest to find an explicit formulae, I did some calculations for $n = 7, 8, 9, 10$. Due to limitation on paper space, this wasn't very helpful. I simply couldn't draw as accurately as I would have liked. But the benefit of doing this was that it gave me some numbers to work with (even though some were just plain wrong!). I gained a little insight, and quickly concluded that induction will not work. I simply had to directly calculate it.

Having seen a few numbers and more importantly having seen a few diagrams, I began to think of the situation as a graph. This was after about 30 minutes of staring at diagrams. It then occurred to me that I can calculate $A(n)$ by using the well-known Euler characteristic

$$1 = V - E + F,$$

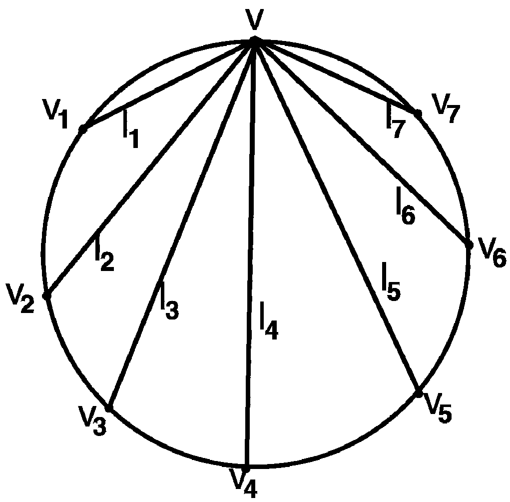
where V, E, and F are numbers of vertices, edges and faces respectively. Here, Terry Tao gives good advice on his blog;[1] subdivide your problem into manageable pieces. So, I formed a strategy in my head:

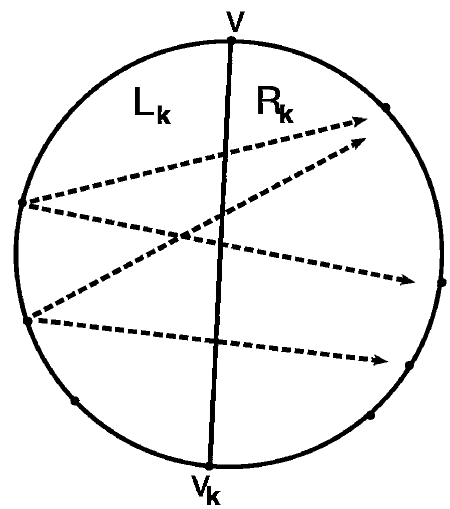
> *Step 1*: find a pattern for V;
> *Step 2*: calculate E;
> *Step 3*: use Euler's characteristic to find F.

Step 1

I knew from the start that this step would be the most difficult, since I had some idea for step 2 and a very clear idea for step 3. But this was where the process of drawing diagrams really helped.

[1] http://terrytao.wordpress.com/.





I started with a circle and a vertex. Obviously, there is no line drawn. Then I added another vertex on the circle and joined a line to the one already drawn. I repeated this process a few times (until $n = 10$).

This process made me realise that I should consider the 'local' viewpoint. This is what I mean; fixing n and a vertex v on a circle, I wanted to consider all lines that end at that vertex. The idea was to count the number of intersections with these lines. I then would use the fact that if I multiplied this number by n corresponding to a number of vertices on the circle, I would count each intersection four times. I would then only have to divide the number by 4 to get the total number of intersections, then add n to get V.

Each vertex on the circle has $n - 1$ lines coming out, so for convenient, I will name each $l_1, \ldots, l_{n-1}$, each joining v and v_i, as shown on the diagram. The number of intersections with line l_i will be denoted as I_i. So, I had the following data to work with:

n	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	$\sum$
3	0	0							0
4	0	1	0						1
5	0	2	2	0					4
6	0	3	4	3	0				10
7	0	4	6	6	4	0			20
8	0	5	8	9	8	5	0		35
9	0	6	10	12	12	10	6	0	56

Well, not exactly. I didn't draw very well, and therefore for $n \geq 8$ it was hard to be sure if I had the right numbers. Nevertheless, the first thing I noticed was, for all n;

- $I_1 = I_{n-1} = 0$;
- $I_2 = I_{n-2} = n - 3$;
- $I_k = I_{n-k}$ for all $1 \leq k \leq n$.

Now, for many including me, this reminds you of Pascal's triangle. But from a simple check, one can see that it is not exactly Pascal's triangle, as $I_k^n \neq I_{k-1}^{n-1} + I_k^{n-1} + const$. So, I went back to basics and thought about how these intersections were being formed. After about 20 minutes I had an another insight. I noticed that each line l_k divides the disc into two regions, say L_k and R_k for a region on the left and one on the right. Any line that intersects this line l_k must therefore connect two vertices, one from each region. There are $k - 1$ vertices in L_k, and $n - k - 1$ vertices in R_k. So, there must be $(k-1)(n-k-1)$ lines that intersect l_k. Assuming that no three lines intersect at one vertex, each line corresponds uniquely to an intersection on l_k. A simply check against the data verified that I was on the right track.

So now, the number of intersections with l_k is the sum of all the I_k's;

$$\begin{aligned}\sum_{k=1}^{n-1} I_k &= \sum_{k=1}^{n-1}(k-1)(n-k-1) = \sum_{k=1}^{n-1}(k-1)((n-2)-(k-1)) \\ &= \sum_{k=0}^{n-2} k((n-2)-k) = (n-2)\sum_{k=1}^{n-3} k - \sum_{k=1}^{n-3} k^2 \\ &= \frac{1}{6}(n-3)(n-2)(n-1).\end{aligned}$$

So the total number of intersections is

$$n \times \frac{1}{6}(n-3)(n-2)(n-1) \times \frac{1}{4} = \frac{1}{24}n(n-1)(n-2)(n-3).$$

Therefore, $V = n + \frac{1}{24}n(n-1)(n-2)(n-3)$.

Step 2

To calculate a number of edges E, I used the fact that each intersection has four edges connecting to it, and each vertex on the circle has $n - 1 + 2$ edges connecting to it (including the circle). But then I would have counted each edge twice, so I need to divide the number by 2. So, we have

$$\begin{aligned} E &= \frac{1}{2}\left((n+1) \times \#\text{ vertices on the circles } + 4 \times \#\text{ intersections}\right) \\ &= \frac{1}{2}n(n+1) + \frac{1}{12}n(n-1)(n-2)(n-3). \end{aligned}$$

Step 3

Finally, combine all the data together, we get

$$\begin{aligned} A(n) &= F = 1 + E - V \\ &= 1 + \frac{1}{2}n(n-1) + \frac{1}{24}n(n-1)(n-2)(n-3). \end{aligned}$$

Final comment

In summary, the process of solving mathematical problem is often non-linear. In this case, I made several fruitless attempts. But a few lessons can be learnt from this:

1. It is a good start to play around with a problem. This allows you to get a feel for it, or at least some numbers to make a hypothesis.
2. You should form a solving strategy or break the problem into smaller pieces.
3. When stuck, always goes back to the basics. This is why problems should be understood at a fundamental level.

In this example, I had knowledge of the answer, but in research this is not always the case. So, the second lesson is particularly useful, when it is applied to research. Finally, there are always more questions to be solved:

> Consider a sphere with n vertices on it. For every three vertices, form the plane which divides the ball into two regions. Let $R(n)$ be the maximal number of regions formed by the above construction. If this problem is not hard enough, what is $R(n)$ in d-dimensions?

— Tharatorn Supasiti

Solutions to Problems from Last Edition

We had a number of correct solutions to the problems from last issue. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

Carol Badre solved problem 4 and may collect $3.

Brendan Duong solved problem 5 and may collect $3.

Andrew Conway solved problem 7 and may collect $5.

David Wakeham solved problems 1, 4, 5 and may collect $8.

Natalie Aisbett solved all the problems and may collect $15.

1. In a round robin tournament involving n teams, where every team plays each other exactly once, show that $\sum_k (w_k)^2 = \sum_k (l_k)^2$, where $w_k =$ the number of wins that team k collects and $l_k =$ the number of losses that team k collects.

   Solution: Firstly for each k we have $w_k + l_k = n - 1$. Also, each game contributes exactly once to $\sum w_k$ and to $\sum l_k$, so $\sum w_k = \sum l_k$, or equivalently $\sum(w_k - l_k) = 0$. Now, $\sum(w_k)^2 = \sum (l_k)^2 \Leftrightarrow \sum (w_k)^2 - \sum (l_k)^2 = 0 \Leftrightarrow \sum (w_k)^2 - (l_k)^2 = 0 \Leftrightarrow \sum (w_k + l_k) \cdot (w_k - l_k) = 0 \Leftrightarrow (n-1). \sum(w_k - l_k) = 0 \Leftrightarrow \sum(w_k - l_k) = 0$.

2. Draw n straight lines in a plane such that no three intersect. Show that the resulting regions can be 2-coloured, that is, coloured in one of two colours such that no two bordering regions share the same colour.

   Solution: We prove the result by induction. For $n = 0$ we can 2-colour the plane trivially. Consider a plane with n lines. Pick one line, remove it, and use the inductive assumption to 2-colour the plane that results. The nth line then divides the plane into two halves, both halves 2-coloured correctly but with regions bordering along the line having identical colours. To correct this simply invert the 2-colouring in one half of the plane.

3. n real numbers are written on the board. Each turn two numbers a and b are erased and replaced with $a + \frac{b}{2}$ and $b - \frac{a}{2}$. Can the set of original numbers every be regained?

Solution: Let s_i be the sum of the squares of all numbers written on the board after i turns. For any turn, $s_{i+1}-s_i = (a+\frac{b}{2})^2+(b-\frac{a}{2})^2-a^2-b^2 = \frac{ab}{2}+\frac{b^2}{4}-\frac{ab}{2}+\frac{a^2}{2} = \frac{a^2+b^2}{4} > 0$. Thus s_i is strictly increasing, and so the set of original numbers can never be recovered.

4. Four points A,B,C and D lie on a circle radius r such that $AB = CD = \sqrt{2}r$, $BC = 6$ and $AD = 8$. Find r.

   Solution: Let O be the centre of the circle. Notice that triangles AOB and COB are right-angled. Next, consider the quadrilateral as being composed of the four triangles ΔAOB, ΔBOC, ΔCOD and ΔDOA, and construct a new quadrilateral by switching the positions of ΔCOD and ΔBOC. This new quadrilateral has side lengths of $(\sqrt{2}r, \sqrt{2}r, 6, 8)$, and also has a diagonal passing through O. Using the 90 degree angle this diagonal subtends at the circumference, we know $6^2 + 8^2 = r^2$, thus $r = 10$.

5. On an nxn chess-board we infect $n-1$ of the squares. Each minute the infection will spread to a non-empty square if at least two of its four direct neighbours are already infected. Could the infection eventually spread to cover the whole board?

   Solution: Let P be the perimeter of the set of infected squares. Consider the four rotationally distinct ways in which a non-infected square can be infected. In each case at least two edges are removed from the perimeter of the set of infected squares, and at most two edges are added to it. Hence P cannot increase as a result of new infections. A completely infected board implies $P = 4n$, yet the largest initial P possible is $4(n-1)$. Hence the infection can never completely cover the board.

6. A triangle ABC has P on AB, Q on BC and R on AC such that ΔPQR is equilateral. Also, $AP = BQ = CR$. Prove that ΔABC is equilateral.

   Solution: Let the triangle be labelled using a, b, c, A, B, C notation, let $AP = BQ = CR = s, PQ = QR = RP = 1$, and let $\angle BPQ = x$, $\angle CQR = y$ and $\angle ARP = z$. Without loss of generality we have two cases; $a \geq b \geq c$ or $a \geq c \geq b$. In the first case we have $\angle A \geq \angle B \geq \angle C$, but we also have, looking in ΔCQR, $y + C = 60 + z$. Similarly $z + A = 60+x$ and $x+B = 60+y$ Together these imply that $x-z \geq y-x \geq z-y \Rightarrow 2x \geq y+z, 2y \geq x+z \Rightarrow z = \min\{x, y, z\}$. But by the sine rule in ΔCQR we have $\frac{s}{\sin y} = \frac{1}{\sin C}$. Similarly, $\frac{s}{\sin z} = \frac{1}{\sin A}$. Now if we assume that $A > C$ then also we have $\sin A > \sin C$, and so by the previous statement we have $\sin z > \sin y$, contradicting the fact that $z \leq y$. Therefore $A = C$.

Similarly, for case two we show that $x = max\{x, y, z\}$ which leads to the same contradiction when using the sine rule. Thus $A = B = C$ in both cases, and so the triangle is equilateral.

7. MUMS-land contains a thousand cities and possesses a dirt-road network such that a person at any city can get to any other city along them. The king of MUMS-land, Han, decides to pave some of the roads. Show that it's possible to pave some of these roads in such a way that every city is connected to an odd number of paved roads.

   Solution: Consider the original road network as a connected graph on 1000 vertices. Let the 'paved degree' of a vertex be the number of paved edges at this vertex. Define n to be the number of vertices with an even paved degree. Now, as each paved edge contributes one to the paved degree of precisely two vertices, we know $\sum_{vertices}$(paved degree) is even. Since there are 1000 vertices we deduce that n is even. Now, if $n = 0$ we have solved the problem. If $n \neq 0$, pick any two of the vertices with even paved degree and choose a path on the original graph between them. Invert every edge on this path. The paved degree of every vertex remains constant except the two end-points, whose paved degree reduces by one. Thus, n is reduced by exactly 2. Repeat this operation as many times as required until n is reduced to 0.

   Alternative solution (from Andrew Conway): Define $P(n)$ to be the proposition that any connected graph on n points S can be paved such that, if n is even, each point has odd paved degree, and if n is odd, precisely $n - 1$ points have an odd paved degree, with the single point being any of our choosing. We prove $P(1000)$ by induction. $P(1)$ is clearly true. For n even, choose a point R, and let $S = R + S_1 + S_2 + \ldots + S_k$ where each disjoint set S_i is connected internally, but not connected to any other S_i other than via R. Now, each of the S_i sets of even size can be coloured correctly by the inductive assumption. Moreover, as there are an odd number of S_i sets of odd size, if we colour an edge from R to each of these sets then each of the S_i with odd size can be coloured correctly also. Finally, R has odd paved degree by construction. For n odd we perform the same operation choosing R to be the vertex we wish to have even paved degree. In this case there will be an even number of odd size sets, and again we pave from R to each of them.

Paradox **Problems**

Below are some puzzles and problems for which cash prizes are awarded. Anyone who submits a clear and elegant solution may claim the indicated amount (up to a maximum of four cash prizes per person). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

1. ($1) What is the minimum number of *snaps* needed to break a nxm chocolate bar into individual pieces, assuming that a snap cannot act on two disconnected portions of the chocolate bar at the same time?

2. ($2) Let a, b, c be distinct integers, and let P be a polynomial having integer coefficients. Show that it is impossible to have $P(a) = b$, $P(b) = c$, and $P(c) = a$.

3. ($3) Consider an nxnxn cube as built from n^3 basic 1x1x1 cubes. Let a 2x2x2 cube with one basic cube missing be called a *block*. Prove that a 2^nx2^nx2^n cube with any basic cube removed (including in the interior) can be constructed entirely from blocks.

4. ($3) Prove that $x^2 + 3 = 4y(y - 1)$ has no solutions in the integers.

5. ($4) Each square of a 8x8 grid contains either a 1 or a 0. On this grid, you may choose any 3x3 or 4x4 subgrid to invert (swap all 0s to 1s, and all 1s to 0s). Using this operation repeatedly, can you always remove all the ones from the grid?

6. ($4) Find the smallest n such that given any n distinct integers one can always find 4 different integers a, b, c, d such that $a + b - c - d$ is divisible by 20.

7. ($5) Prove that $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}} < 2$.

Paradox would like to thank Zhihong Chen, Han Liang Gan, Charles Li, Lu Li, Tharatorn Supasiti, James Wan and Julia Wang for their contributions to this issue.