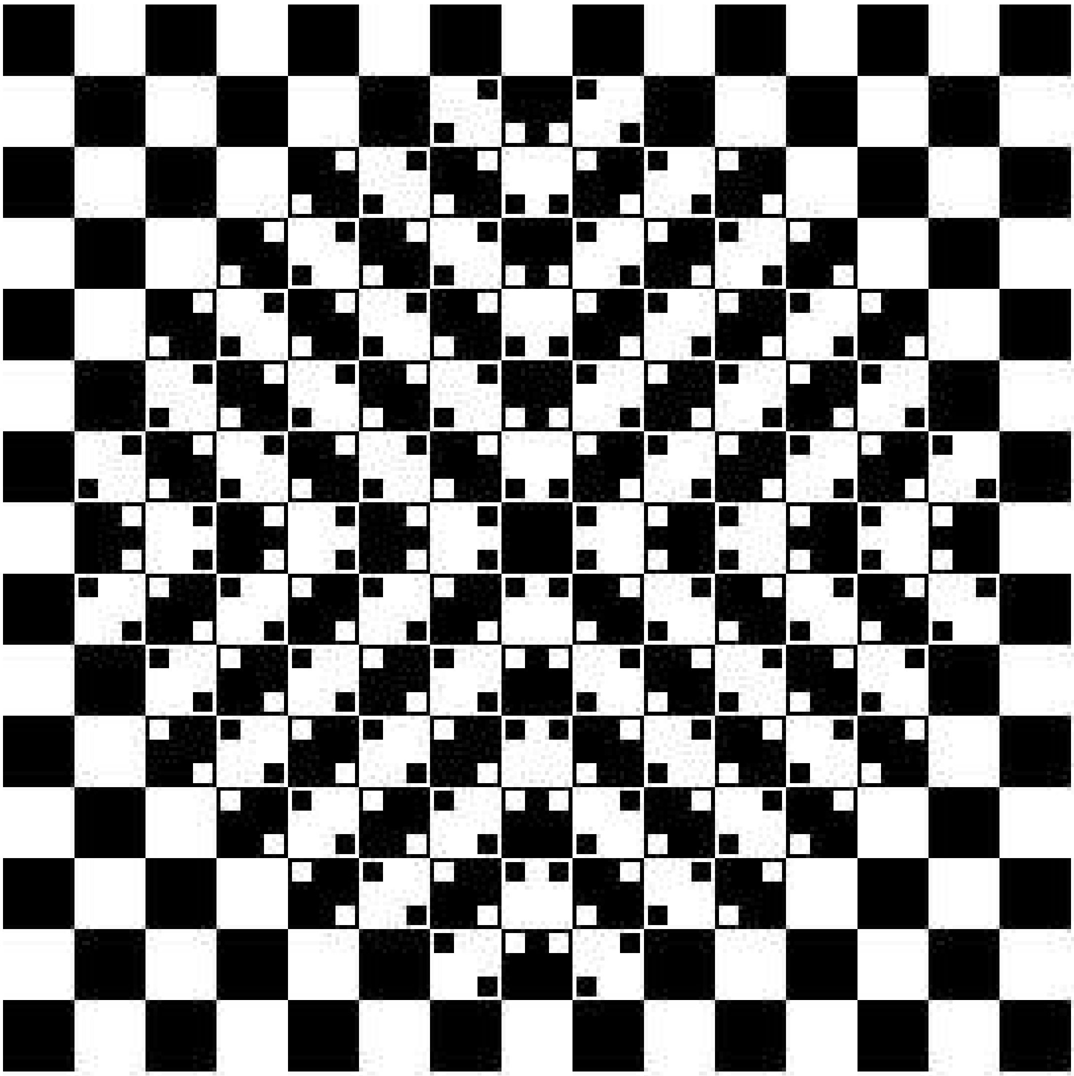
Issue 2, 2003

Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society



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Words from the Editor...

It is with tremendous pleasure that I bring you this year's second issue of Paradox, one of the most action-packed and hilarious issues ever to have been published. If this is your first encounter with Paradox, then you may not know that it is the publication of MUMS, the Melbourne University Mathematics and Statistics Society. (You may be wondering then, why the acronym is not MUMSS... to tell you the truth, I'm not sure myself!) If you have come across Paradox before, then it is highly likely that you have also come across *Knot Man*, our resident comic strip action hero. Unfortunately, *Knot Man* is on holiday at the moment, so is *knot* available to appear in this issue. However, there is still a myriad of mathematical jokes included for you to laugh and/or groan at.

This issue includes articles on Poncelet's Porism (a gem of a theorem from geometry), the infamous 'P vs NP' problem (which is now worth a million bucks) and an interview with the renowned mathematician Raymond Lickorish. Of course, there are also a few maths problems for you to try your hand at for cash prizes, as well as the solutions to last issue's problems. I hope that you all enjoy reading this issue of Paradox and have a super-duper second semester. If you have any queries, or wish to submit mathematical jokes and articles, please feel free to e-mail us at `paradox@ms.unimelb.edu.au`.

— Norman Do, Paradox Editor

...and some from the President

Welcome to the second Paradox for 2003. Paradox is the magazine of the Melbourne University Mathematics and Statistics Society (MUMS). Each year MUMS puts on a wide variety of events aiming to provide entertainment, free food and educational activities for students studying maths and statistics.

Everyone studying maths and statistics is automatically a member of MUMS, so get involved in the activities and meet people trying to solve the same problems as yourself.

This semester we have a number of activities planned. For the socialites amongst you, we have a BBQ and a trivia night. For the sportsmen and women, we have the Olypmics, and for the studious types, we have seminars and an honours information session, giving you a chance to find out what it is really like doing postgraduate study here at the University of Melbourne.

This issue I would also like to welcome the competitors in the Schools Maths Olympics. This event, which this year has attracted 28 schools from across Victoria is always a highlight of the calendar providing an opportunity for school students to demonstrate their sporting and academic prowess.

The top five teams are invited to compete in the University Maths Olympics. Perhaps unsurprisingly, these teams often do very well, beating many University students into submission.

Hope you enjoy this Paradox and the activities that MUMS has in store for the rest of the semester.

— Joe Healy, President of MUMS

MUMS Dates Semester 2, 2003

Date	Event
17/8/2003	Schools Maths Olympics
12/9/2003	University Maths Olympics
4/10/2003	Welcome Back BBQ
15/10/2003	Honours Information Session
24/10/2003	Trivia Night

A Million Dollar Maths Problem P vs NP

The 'P vs NP' problem sits right on the boundary of mathematics and computer science; it is regarded as one of the most important unsolved problems in either field. One million bucks will be made available to the person that solves it.[1] In this article I will explain roughly what 'P vs NP' is about, without going into too many technical details.

To motivate the discussion, let us consider a classic example of a computational problem, called the 'Travelling Salesperson Problem' (TSP).[2] Suppose you are a travelling salesperson. You are given a list of the locations of n cities, say (x_1, y_1), ..., (x_n, y_n), all distances measured in kilometres. You start out at your home town, (x_1, y_1). Your car only has enough petrol to travel d kilometres, and there are no petrol stations. You need to determine whether it is possible to visit every city on the map, returning home at the end.

Here is an example of an *algorithm* (a precise sequence of instructions) that solves the TSP. Go through every possible ordering of the cities listed in the input description. For each ordering, add up the total distance travelled by visiting the cities in that order. If, for any particular ordering, the distance is $\leq d$, answer 'yes'. If, after testing every possible ordering, we haven't found any where the distance is $\leq d$, answer 'no'. Any programmer worth their salt would not have any difficulty converting this informal algorithm description into a program that could run on a real computer.

How long could this algorithm take to run? There are $(n-1)!$ different possible orderings to try.[3] Let us make the reasonable assumption that it takes the computer at least T seconds to perform a 'basic operation' such as adding two numbers together, or testing which of two numbers is larger. For each ordering, the computer has to add n numbers together, which takes $n-1$ operations, and test whether the answer is $\leq d$, giving n operations. Therefore the total time taken to solve a given instance of the TSP is potentially $Tn(n-1)! = Tn!$ seconds.

[1] Visit `http://www.claymath.org/Millennium_Prize_Problems/P_vs_NP/` for more information and details on how to enter!

[2] The TSP is more commonly known in the literature as the Travelling Sales*man* Problem.

[3] Recall that $N! = 1 \times 2 \times 3 \times \cdots \times N$.

For example, if $T = 10^{-9}$ seconds, which is optimistic for today's desktop computers, then the case with twenty cities would take the computer 77 years. This sounds pretty slow! However, from the theoretical computer scientist's view, it is missing the point entirely. The real issue is that $Tn!$ *grows very rapidly* as a function of n. In fact, it grows more rapidly than any fixed polynomial $p(n)$; that is, $Tn!/p(n) \to \infty$ as $n \to \infty$. Computer scientists express this by saying that the algorithm *runs in superpolynomial time*. Notice that this has nothing to do with the speed of your computer (that is, the value of T).

The *complexity class P* is the class of problems which can be solved in *polynomial time*. That is, a problem is in P if there is an algorithm for solving it that always takes at most $p(n)$ steps, where $p(n)$ is some polynomial, and n is the size of the input data. For example, determining whether a sequence of n numbers is listed in ascending order is in P. To solve this problem, we simply compare each number with the next in the sequence; this takes $n - 1$ steps, which is a polynomial in n.

There are good theoretical reasons for thinking of polynomial time algorithms as 'fast' algorithms. They will always beat superpolynomial time algorithms for sufficiently large n. (However, in practical situations, the relevant values of n may be small enough that some superpolynomial time algorithm beats the best polynomial time algorithm.)

It is widely suspected that the Travelling Salesperson Problem is *not* in P, largely because so far no-one has found a polynomial time algorithm that solves it. On the other hand, *nobody has yet been able to prove that such an algorithm does not exist.* Consequently we simply don't know whether the TSP is in P or not.

The *complexity class NP* is the class of problems *whose solutions can be checked* in polynomial time.[4] This is a somewhat technical idea, but in the case of the TSP, it simply means the following. If I present you with a route through the cities on the map, it is very easy to check (in polynomial time) whether this route is at most d kilometres long. Therefore the TSP is in NP.

The 'P vs NP' problem is to determine whether P = NP. In other words, is it true that every problem whose solutions can be verified in polynomial time, is itself solvable in polynomial time? The suspected answer is of course *no*. To prove that P $\neq$ NP, you need to find an NP problem that has the following property: for *any* algorithm which solves the problem, and for *any* polynomial $p(n)$, there is some instance of the problem (presumably with very large n) for which the algorithm takes more than $p(n)$ steps to run. What makes this so difficult to prove is that it is a statement about *all* possible algorithms that solve the problem.

If you are interested in learning more about computational complexity, a great place to start is *Introduction to the Theory of Computation* by Michael Sipser, PWS Publishing Company, 1997.

— David Harvey

[4]NP actually stands for 'non-deterministic polynomial' time.

An Interview with Professor Raymond Lickorish

"So what do we do in an interview? I have never been interviewed before."

These were the very first words Professor Lickorish said to me as I entered his office.

"Well, tell me about your life," I said.

The interview then flowed fluently without me saying much more.

···oooOooo···

In April this year, I was fortunate enough to have had the opportunity to interview Professor Raymond Lickorish, who was at the University of Melbourne as a visiting scholar. Professor Lickorish is the current head of the Department of Pure Mathematics and Mathematical Statistics at Cambridge, and a ubiquitous name in Knot Theory.

Professor Lickorish has been a Cantabrigian since he was an undergraduate. He graduated with First Class Honours in 1960 and began a Ph.D. in 1961 at Cambridge under Chris Zeeman in Piecewise Linear Topology. While undertaking his Ph.D., Professor Lickorish went to the Institut des Hautes Etudes Scientifiques in France for a year on exchange. In 1963, Professor Lickorish began lecturing at the University of Sussex while proceeding with his Ph.D., where he taught mathematics to Virginia Wade, the 1977 Wimbledon Ladies Singles Champion. He returned to Cambridge the next year to finish his Ph.D., and became a member of the academic staff in the Department of Pure Mathematics and Mathematical Statistics, where he has stayed ever since.

I asked Professor Lickorish on how he see the differences between universities in Britain and Australia, to which he replied:

> *Britain has three times the number of people as Australia and a much older history. Consequently, there are many more universities in Britain than in Australia. However, the universities in Britain are dominated by Oxford and Cambridge and every student wants to be in one of those two. Consequently, Oxford and Cambridge have become centres of elite students and staff, and the academic output is at a very high level. The downside is that as a result a lot of the other universities become unheard of, especially to people overseas, even though they are still very good universities.*

After a pause, Professor Lickorish commented on the current situation in Britain:

> *Recently, there has been a lot of student protests. The British government wants to squeeze more and more students into the universities. As a result there has been much debate over the role of the university in today's society, funding issues and the level of standards the universities should set.*

At this point, coincidentally, we could hear a student rally going on outside about the education system.

"I suppose the same problems are faced in Australia as well." Professor Lickorish chuckled.

Then the interview proceeded and the topic of Internet was brought up. From the conversation, I realised that unlike most people who were brought up without Internet in their teenage years, Professor Lickorish is a strong advocate for the Internet.

> *The advent of the internet has changed how mathematical research is done. Now whenever I want to look up a paper, I try MathSciNet or Google first, and usually I'd get what I want after a few minutes.*

Deep in thought, Professor Lickorish continued.

> *I remember that in the past there were a lot of cases of people publishing identical results in different journals due to a lack of communication in the world wide academic community. However thanks to the Web this is no longer a problem. I'm beginning to regret having archived filing cabinets full of technical papers, since now I just have to do a search on the web and I'll find the paper faster than looking through the filing cabinets.*

While the conversation was focusing on research and Mathematics, I asked Professor Lickorish on how or when he started to take Mathematics as a life-long career, and what does he thought about Mathematics itself.

> *Well I liked Mathematics since I was a teenager, but I can't pinpoint a time in my life where I decided that I was going to be a Mathematician. I suppose that if you've done a lot of Maths and have enjoyed it while doing it, gradually you develop an attachment to it. Also as you become more successful as a Mathematician, your confidence grows and you stay in the field.*

I smiled. This topic always comes down to the charms of Mathematics...

> *To me the single most enjoyable aspect about mathematics is, after you get the answer, to reason your way through and show that your answer is correct. Mathematics is one of the few fields where it is more important to understand and argue correctly, than the result being useful. At a student level, students should learn how to reason, and how to use reasoning correctly. Undergraduate Maths courses should teach ideas as well as methods.*

This is when I asked Professor Lickorish on personal anecdotes in academia. To which he made the remarks:

> *When I was at the University of Sussex, I learnt about the value of tradition. At the time the department of mathematics at Sussex was still very young, and we spent long lengths of time discussing the syllabus of various courses. I discovered then how much more tempting it is to repeat what was done in the previous year, than to restructure the course.*

I have been the head of DPMMS for the last five years. My first major task as head was to supervise the completion of the new Centre for Mathematics Sciences at Cambridge, a project costing 60 million pounds. The job is frustrating at times, since you are acting as the bridge between the university administration and the department. My advice to the young people out there is that "don't assume everybody is pleasant", because after becoming head I have met plenty of people which you would otherwise try to avoid.

"What about your life outside academia?" I asked, while digesting the Professor's wise words.

I enjoy walking. I used to hike a lot in my younger days. I also like spending time in the garden, watching the plants grow and blossom. Swimming in the river Cam in the summer is also one of my favourite pastimes.

I also travel frequently. At Cambridge a lecturer is granted a year's leave for every 7 years of service, which provides an excellent opportunity to travel. As visiting scholar I have been to universities in the US and France, and now I'm at the University of Melbourne.

"Having been to so many countries, where's your favourite place in the world?"

I love France. The weather is pleasant and there is very nice wine. A place I visited many times is Ardeche, which is just south of Lyon. England is a very cold and crowded country you know.

But mind you, Melbourne is lovely too. The climate is mild, and the food on Lygon Street very palatable.

To conclude the interview, I asked Professor Lickorish about his opinion of Knot Man, the comic in Paradox, in which Professor Lickorish made a guest appearance in one of the episodes.

Yes I have read Knot Man. The authors gave me an especially enlarged copy to have a look. It is a very clever comic, and I think the authors will have a very successful career in this. I would definitely like to see future episodes of Knot Man.

— Geordie Zhang

Pentagonal Paradox Challenge

1 Introduction

If you love origami and reckon you have an eye for symmetry, then this challenge is for you. Alternatively, if you have some scrap paper, a few fingers, and love procrastination, then this is *definitely* for you!

Below you will find instructions on how to fold a piece of paper into two different pentagons. Then, the challenge is to work out whether they are regular or not. Remember: a pentagon is regular if and only if all of its sides are equal in length *and* all of its internal angles are equal.

2 Folding Instructions

Pentagon A

To make this pentagon you need a strip of paper, no more than about two centimetres wide. Essentially, all you need to do is to (carefully!) tie a knot, and a pentagon will magically appear. For more detailed instructions, refer to Figure 1 and follow these directions.

1. Lay the strip of paper down flat. Pull one end up and towards the middle.
2. Take this end down and behind the rest of the strip, to form a loop.
3. Pull this end up again and thread it through the loop you just formed.
4. You should now have the beginnings of a nice knot. Carefully pull the ends out until the knot is tightened. Make sure the paper is as flat as possible.
5. If you've done it right, the knot forms a pentagon!

Pentagon B

This pentagon is easier to construct than Pentagon A. There are no fancy knot tying tricks here, just very simple origami, and all you need is an A4 piece of paper. If have you never seen origami diagrams before, then here's a very quick tutorial.

- Folds are indicated by dashed lines.
- Existing creases are indicated by dotted lines.
- Arrows show you which way to fold the paper.

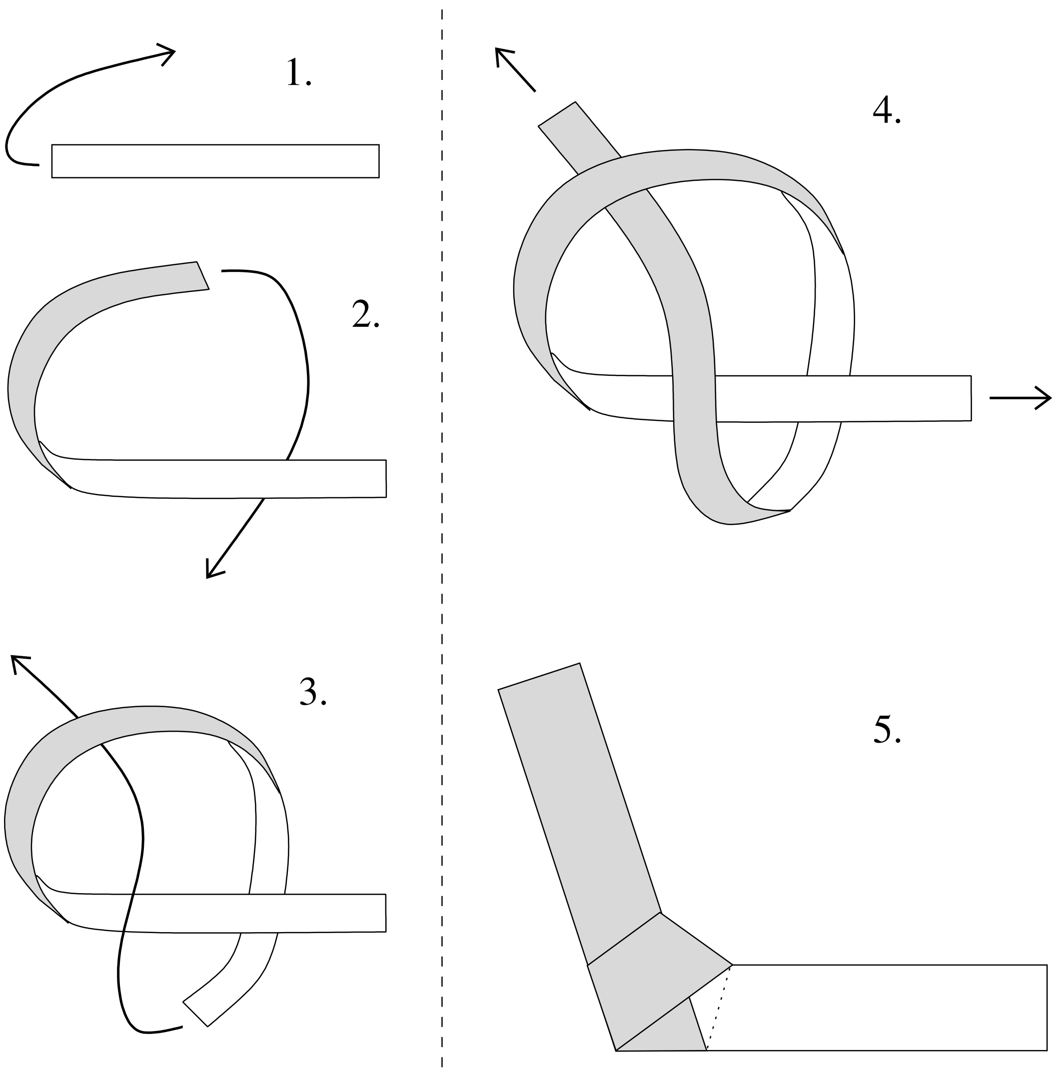


Figure 1: Pentagon A

- A big, unfilled arrow means to *un*fold the paper.

Now refer to Figure 2 and follow these directions.

1. Start with an A4 piece of paper, in "portrait" orientation. Fold the paper so that the bottom-left corner meets the top-right corner.
2. Turn the paper anticlockwise slightly, so that the previous fold line is horizontal. Fold the paper in half, right to left.
3. Unfold the paper, so you get back to the previous position.
4. There should now be a creaseline in the middle. Fold the "wings" into the centre, so that the edges line up with the centre creaseline.
5. Thus is formed Pentagon B!

3 The Challenge

Now that you have constructed both pentagons, can you see whether they are regular or not? If you can, and can *prove* it, then you could be in for fame and fortune!

Solutions may be emailed to `paradox@ms.unimelb.edu.au` or you can drop a hard copy of your solution into the MUMS pigeonhole near the Maths and Stats Office in the Richard Berry Building. The clearest and most elegant solutions may be worthy of a prize and may be published in a future edition of Paradox.

— Damjan Vukcevic

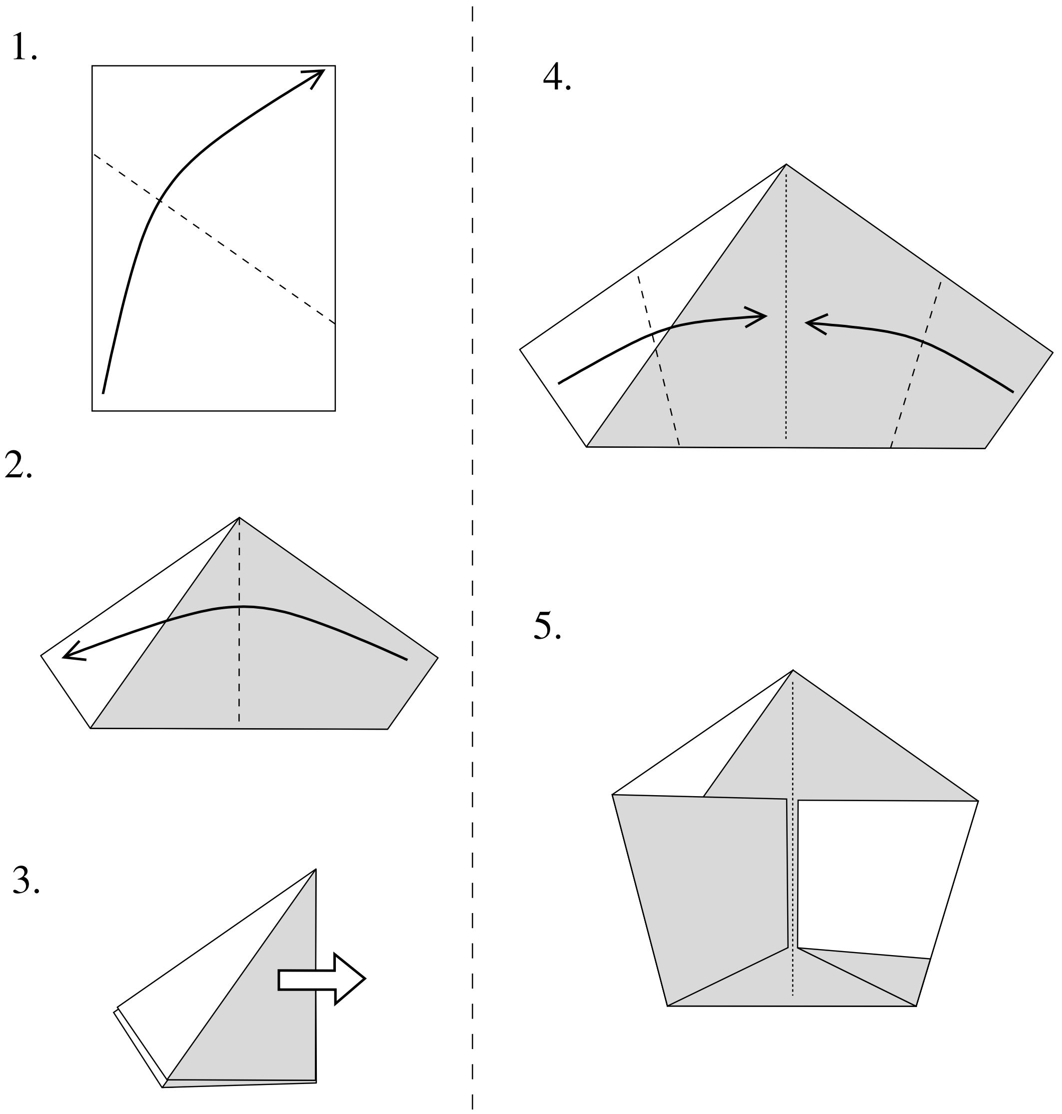


Figure 2: Pentagon B

Poncelet's Porism and the Problem with Plane Geometry

Plane geometry is easily the coolest area of elementary mathematics. What do the theories of linear algebra, probability, or analysis have that could possibly compare with the beauty of the nine-point circle theorem?[5]

The theorems that we are talking about in this article are called porisms. The word porism is obscure in origin — it is variously defined as "an odd or surprising theorem", "a surprising construction theorem", or a "construction theorem in which a construction is performed which, if it works for one starting point, works for infinitely many starting points, and if it doesn't work for one starting point, doesn't work for any other starting point either". This last definition seems stupid at first, but when you see a few examples, you get a vibe for it.

Steiner's Porism

Take two circles, T and U, one inside the other. Now construct a circle, C_1, which is tangent to both of them, as shown in the diagram below. Construct another circle, C_2, which is also tangent to T and U, and to C_1. We keep constructing this 'chain' of circles, as shown in the diagram, until we get to C_k. Suppose that C_k is tangent to C_1, as in the diagram. Steiner's Porism states that, if the ends of the chain "match up" like this for one position of C_1, then it doesn't matter where we put C_1 (as long as it's tangent to T and U), the chain always meets up with itself (as opposed to overlapping or falling short, as shown in Figure 5). So, because the chain matches up for the position of C_1 shown in Figure 3, it matches up for the position of C_1 shown in Figure 4, rather than not forming a perfect chain, like as shown in Figure 5.

The proof of Steiner's Porism makes use of a geometric transformation known as inversion, invented by Steiner himself. Using this technique, it can be shown that proving the problem for any arrangement of T and U is equivalent to proving it when T and U have the same centre. In this case, the circles C_i are all the same size, making the problem much simpler.

Soddy's Hexlet

Another amazing porism is known as Soddy's hexlet. In this case, imagine a sphere T with two smaller spheres, U and V, inside it, so that all the spheres are tangent to each other. Now construct a sphere S_1 which is tangent to T, U and V. Construct a sphere S_2 which is tangent to T, U, V and S_1, and then S_3 which is tangent to T, U, V and S_2. Continue this until you get S_6. Then S_6 is tangent to S_1!

[5] The nine-point circle of a triangle passes through the middle of each side, the feet of the perpendiculars from each vertex to the opposite side, and the midpoints of the lines connecting the vertices to the orthocentre (the point where the perpendiculars from the vertices to their opposite sides meet). The answer to the question is nothing.

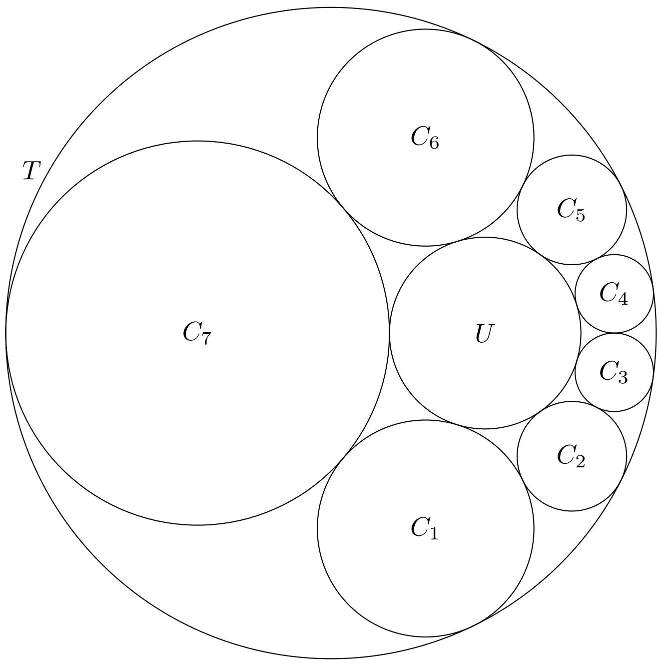


Figure 3: For this position of C_1, the circles match up.

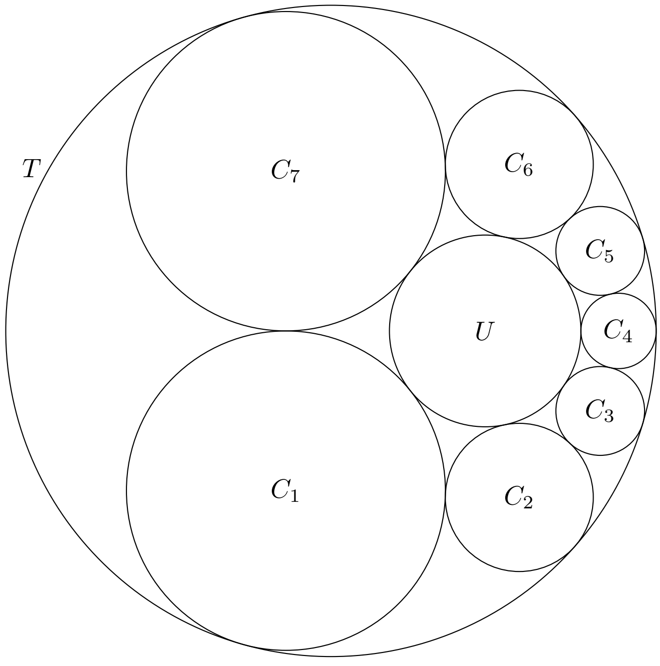


Figure 4: Now C_1 has moved, but the circles still match up.

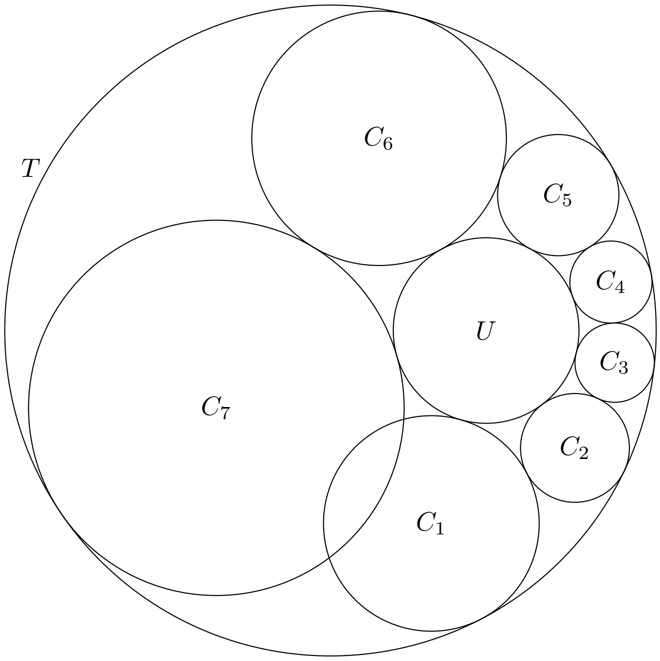


Figure 5: Now T and U have been changed, and the circles do NOT match up.

This incredible theorem can also be proven using inversion in three dimensions. In this case, the problem becomes equivalent to proving that we can place six circles of equal radius around another circle of the same radius, and they will all be tangent to their neighbours.

Poncelet's Porism

Now we come to a much harder problem. This was proposed by a French geometer named Jean-Victor Poncelet. Poncelet was an engineer in the Napoleonic Wars, and was left for dead on a Russian battleground. The Russians noticed his engineer's uniform, and decided he would be worth questioning. They marched him for miles through weather so cold that mercury froze. After five months on the march, he reached Saratova, where the unfortunate fellow was incarcerated in a Russian prison for two years. To pass the time, he tried to remember the geometry he had been taught at university. Writing at first with charcoal on the walls of his cell, Poncelet managed to reconstruct much of what he had learnt, and a whole lot of other stuff besides that no-one had heard of before, thus laying the foundations for much of the field of projective geometry.

Poncelet's porism, like Steiner's porism, starts with two circles, one inside the other. If we start with a point on the outer circle and construct successive sides of an n-sided polygon (or n-gon) which are all tangent to the inner circle, and the sides eventually match up, then it doesn't matter where we put the initial point, the sides will always match up to make an n-gon. So, because the sides match up in Figure 6, they match up in Figure 7, rather than falling short or overlapping, as in Figure 8.

The difference with Poncelet's Porism is that it doesn't have a simple proof. When we

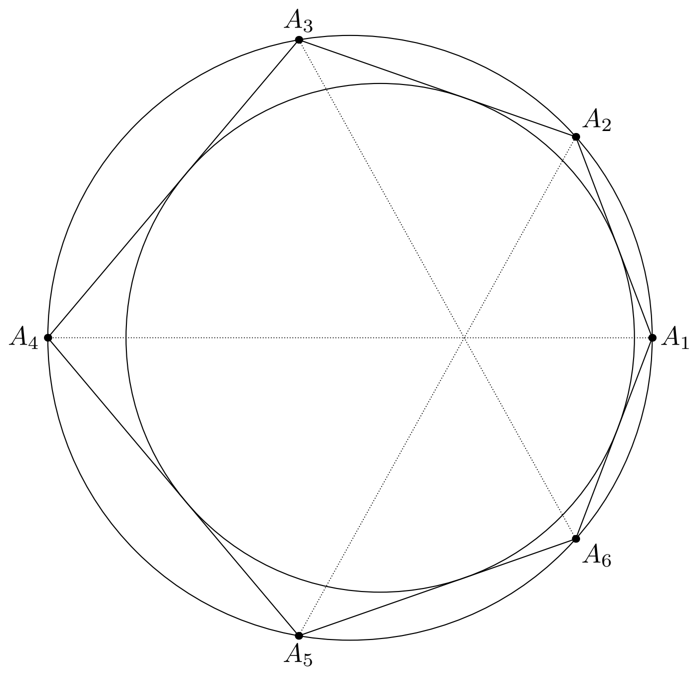


Figure 6: For this position of A_1, the lines match up.

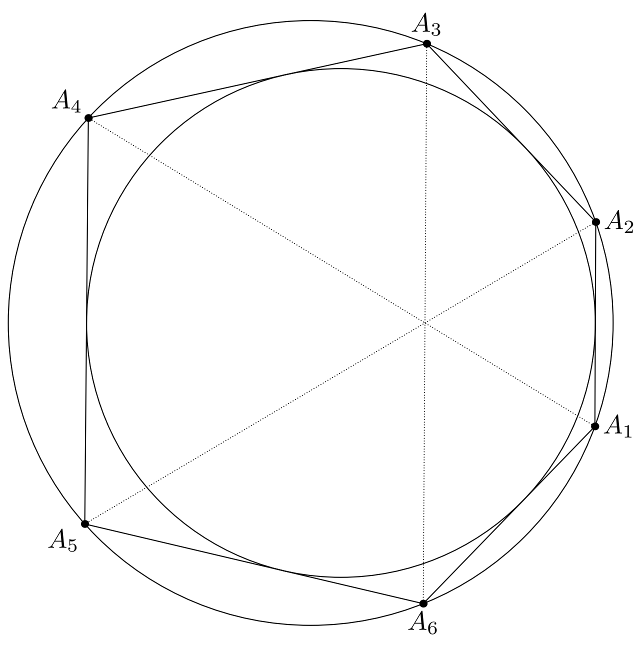


Figure 7: Now A_1 has moved, but the lines still match up.

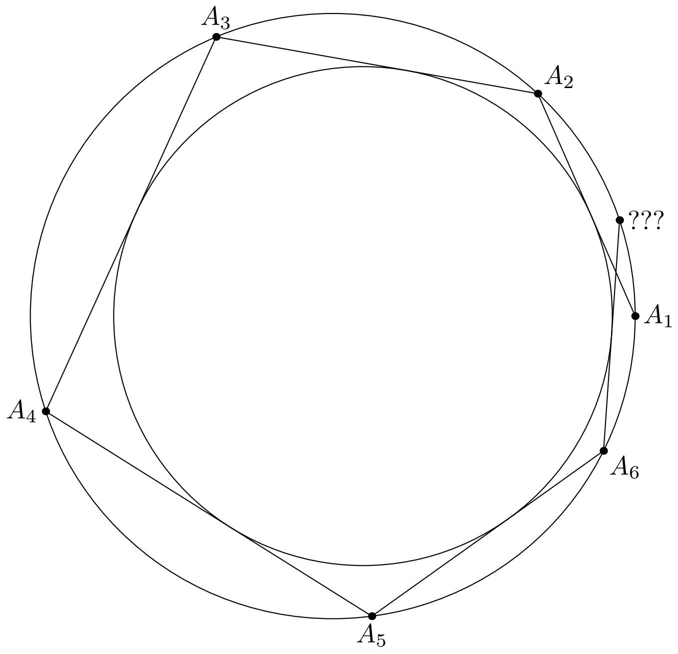
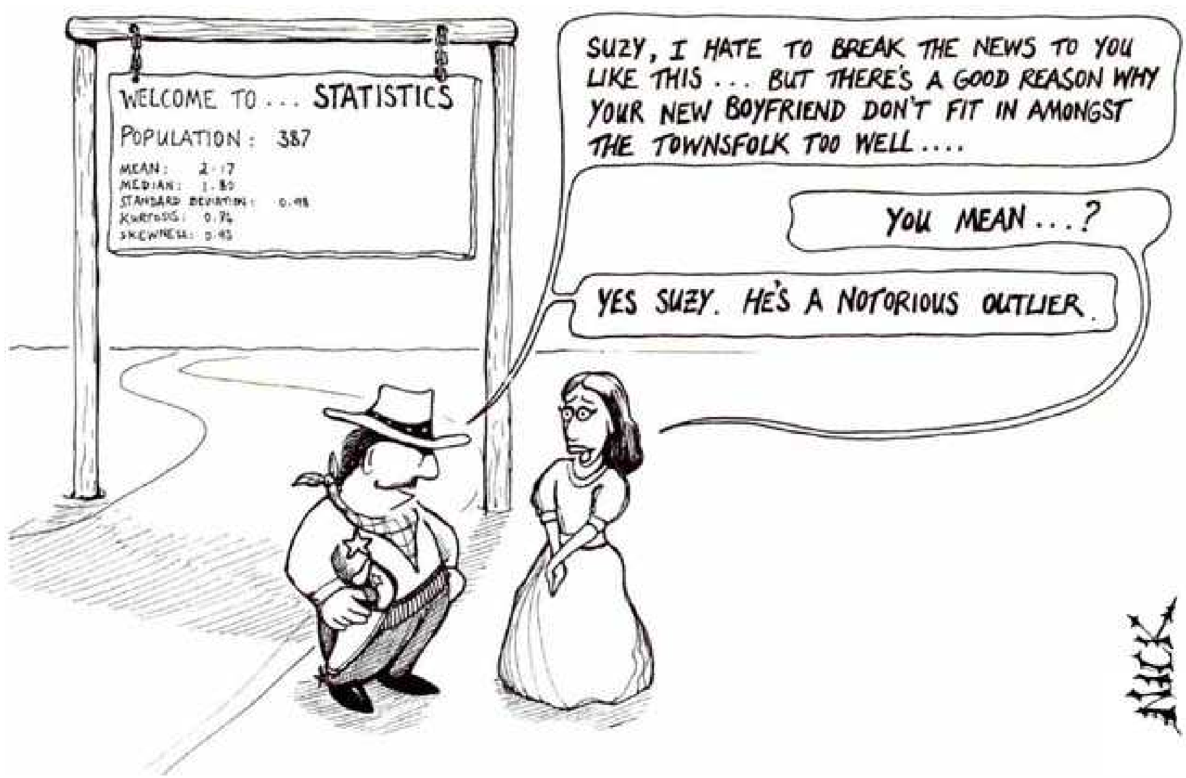


Figure 8: Now the circles have been changed, and the lines do NOT match up.

apply transformations to Steiner's Porism or Soddy's Hexlet, the problem is reduced to a nice symmetrical form, which makes it easy to solve. Unfortunately, despite the apparent elegance and simplicity of this problem, it doesn't reduce to such a simple problem under transformations. It is possible to construct a geometrical proof of this theorem, but all such proofs are highly complicated. There is, in fact, a deep reason for this difficulty, and it lies in the fact that, although there is an underlying symmetry at work in Poncelet's Porism, it is not a simple symmetry such as those we encountered in proving the other two porisms. This becomes most apparent when one analyses the problem using analytic geometry. That's right, this simple and elegant-looking geometrical problem is best explained by looking at it from an algebraic point of view. You use vectors and coordinates and construct a huge algebraic identity, which turns out to be true. By analysing the properties of this identity, you can find the fundamental symmetry that underlies this problem.

This is one of the unfortunate things about plane geometry. At its foundations, it produces some of the most elegant theorems in mathematics, and appears to promise even greater things as it becomes more complex. Yet, as you move into realms of geometry as complicated as Poncelet's Porism, you find that the proofs degenerate into lengthy and often messy algebraic monstrosities. The approach that yields such impressively simple elementary proofs becomes impotent in the face of more complex problems. Still, it's not all bad. Well before you reach that point, you will find that plane geometry is far cooler than any first-year course, and most second- and third-year courses. For those interested in reading more about the nine-point circle or inversion, read *Geometry Revisited* by Coxeter and Greitzer. It will be one of the most clearly written yet interesting books on maths that you'll ever read.

— Nick Sheridan



There are three kinds of mathematicians: those who can count and those who cannot.

There are 10 types of people in the world: those who understand binary and those who don't.

Q: What do you get when you cross an elephant with a banana?

A: Elephant banana sine theta in a direction mutually perpendicular to the two as determined by the right hand rule.

Q: What do you get if you cross an elephant with a mountain climber?

A: You can't do that. A mountain climber is a scalar.

Q: Divide 14 sugar cubes into 3 cups of coffee so that each cup has an odd number of sugar cubes.

A: 1,1,12

Riposte: 12 isn't odd!

A: It's an odd number of cubes to put in a cup of coffee (groan)

Thanks

Thanks must go to David Harvey, Nicholas Sheridan, Damjan Vukcevic and Geordie Zhang for contributing fantastic articles to this issue of Paradox. The Paradox team would also like to acknowledge *topdrawer*, the program used to draw all of the geometry diagrams in this issue.

Winners of Last Issue's Paradox Problems

The following people have won prizes from last issue's Paradox problems. Please drop by the MUMS room in the Richard Berry Building to receive your prize.

- Problem 1 — James Wan ($2)
- Problem 3 — T. Harvey ($5)
- Problem 4 — Sally Zhao ($5)
- Problem 5 — Geordie Zhang ($10)

Solutions to Last Issue's Paradox Problems

1. ($2) You are given a large supply of each of the standard denomination Australian coins: 5c, 10c, 20c, 50c, $1, $2. What is the smallest n such that it is impossible to select n coins that make exactly two dollars?

   Solution: The maximum possible number of coins that can make exactly two dollars is 40, when they are all 5c coins. Amazingly enough, for every positive integer n less than or equal to 40, it is possible to select n coins that make exactly two dollars. Thus, the smallest n such that it is impossible to do so must be 41.

2. ($5) A confused bank teller transposed the dollars and cents when he cashed a cheque for Ms Smith, giving her dollars instead of cents and cents instead of dollars. After buying a newspaper for 50 cents, Ms Smith noticed that she had left exactly three times as much as the original cheque. What was the amount of the cheque?

   Solution: Let d be the number of dollars and c be the number of cents Ms Smith is supposed to get (so she should have received $100d + c$ cents). Then we have the equation $100c+d-50 = 3(100d+c)$. After some manipulation we have $97c-299d = 50$. This is a Diophantine equation with restrictions $d \geq 0$, $0 \leq c \leq 99$. Using Euclid's Algorithm on the numbers 97 and 299 gives $97 \times 37 - 299 \times 12 = 1$. Hence, after multiplying the equation by 50, we have $97 \times 1850 - 600 \times 299 = 50$. We now subtract $299 \times 6 \times 97$ from both terms on the left hand side. This yields $97 \times 56 - 299 \times 18 = 50$. Here the coefficients of 97 and 299 both satisfy our restrictions. Hence $d = 18$ and $c = 56$, and the original cheque was worth $18.56.

3. ($5) A rectangular sheet of paper is folded so that two diagonally opposite corners come together. If the crease formed is the same length as the longer side of the sheet, what is the ratio of the longer side of the sheet to the shorter side?

   Solution: Let the rectangle be $ABCD$ where AB is the shorter side. Let P and Q be points on BC and AD respectively, where PQ is the crease. Then PQ and AC will meet at a point O, the centre of the rectangle. Note that the crease formed is perpendicular to the diagonal of the rectangle. Therefore, the line PQ in the diagram is perpendicular to AC. Suppose that the length $AB = 1$ and $BC = x$. Then by Pythagoras' Theorem, we have $AC = \sqrt{1+x^2}$. Also, since the crease formed is the same length as the longer side of the sheet, we have $PQ = x$. Now note that the two right-angled triangles $\triangle ABC$ and $\triangle POC$ are similar. So we have the following equal ratios:

$$\begin{aligned}
\frac{AB}{BC} &= \frac{PO}{OC} = \frac{2PO}{2OC} = \frac{PQ}{AC} \\
\frac{1}{x} &= \frac{x}{\sqrt{1+x^2}} \\
x^2 &= \sqrt{1+x^2} \\
x^4 &= 1+x^2 \\
x^4 - x^2 - 1 &= 0
\end{aligned}$$

This last equation is a quadratic in x^2, and the only positive solution is

$$\begin{aligned}
x^2 &= \frac{1+\sqrt{5}}{2} \\
x &= \sqrt{\frac{1+\sqrt{5}}{2}}
\end{aligned}$$

So the required ratio is $\sqrt{\frac{1+\sqrt{5}}{2}}$, which is the square root of the golden ratio.

4. (\$10) Prove that for every integer $n > 0$ there exists an integer $k > 0$ such that $2^n k$ can be written in decimal notation using only the digits 1 and 2.

   Solution: We will prove the following stronger result by induction: for every positive integer n, there exists a positive integer A_n which is divisible by 2^n and has n decimal digits, all of them 1 or 2. For $n = 1$, we can simply take $A_1 = 2$. Now let us assume that such an A_m exists satisfying the given conditions for some positive integer m. That is, A_m is divisible by 2^m and has m decimal digits, all of them 1 or 2.

   Suppose then that $A_m = 2^m s$. If s is even, then we can write $s = 2t$ for some positive integer t and define:

$$A_{m+1} = 2 \times 10^m + A_m = 200\ldots00 + A_m.$$

   Note that A_{m+1} has decimal expansion with $m+1$ digits, all of which are 1 or 2. Furthermore, $A_{m+1} = 2 \times 10^m + A_m = 2^{m+1}(5^m + t)$, so that A_{m+1} is divisible by 2^{m+1} as required.

   Now suppose that $A_m = 2^m s$ where s is odd. Then we can write $s = 2t+1$ for some positive integer t and define:

$$A_{m+1} = 10^m + A_m = 100\ldots00 + A_m.$$

   Note that A_{m+1} has decimal expansion with $m+1$ digits, all of which are 1 or 2. Furthermore, $A_{m+1} = 10^m + A_m = 2^{m+1}\left(\frac{5^m+2t+1}{2}\right)$, so that A_{m+1} is divisible by 2^{m+1} as required.

Therefore, by the principle of mathematical induction, we have shown that A_n is a positive integer whose decimal expansion consists only of the digits 1 or 2 and which is divisible by 2^n.

Paradox Problems

The following are some maths problems for which prize money is offered. The person who submits the best (i.e. clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number. Solutions may be emailed to

`paradox@ms.unimelb.edu.au`

or you can drop a hard copy of your solution into the MUMS pigeonhole near the Maths and Stats Office in the Richard Berry Building.

1. ($5) You have a glass cube with a small hole at the centre of one of its faces. If the cube has no scale provided, how can you accurately fill the cube with water occupying one third of its volume?

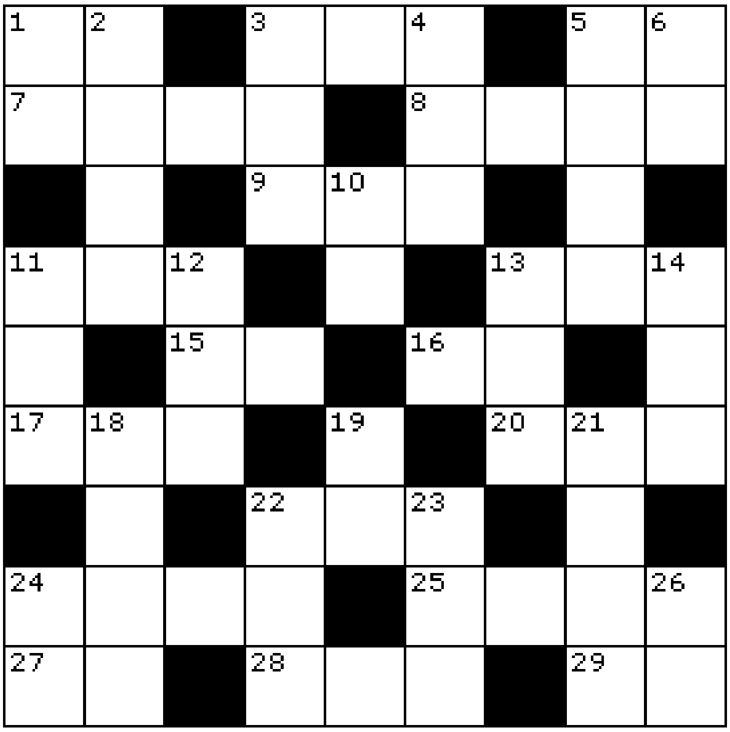
2. ($5) The plane is divided into regions by straight lines. Show that it is always possible to colour the regions with two colours so that adjacent regions are never the same colour.

3. ($5) There are five houses in a row, each of a different color, and inhabited by 5 people of different nationalities, with different pets, favorite drinks, and favorite sports. Use the clues below to determine who owns the monkey and who drinks water.

   (a) The Englishman lives in the red house.
   (b) The Spaniard owns the dog.
   (c) Coffee is drunk in the green house.
   (d) The Russian drinks tea.
   (e) The green house is immediately to the right of the white house.
   (f) The hockey player owns hamsters.
   (g) The football player lives in the yellow house.
   (h) Milk is drunk in the middle house.
   (i) The American lives in the first house on the left.
   (j) The table tennis player lives in the house next to the man with the fox.
   (k) The football player lives next to the house where the horse is kept.
   (l) The basketball player drinks orange juice.
   (m) The Japanese likes baseball.
   (n) The American lives next to the blue house.

4. ($5) If an infinite arithmetic progression of positive integers contains a perfect square and a perfect cube, show that it must contain a perfect sixth power.

5. ($5) Solve the following crossword.



Across Clues

1. A square number
3. 2 down divided by eight
5. A prime number
7. 8 across plus 140
8. 29 across times 5 across
9. 4 down plus ten
11. 9 across plus eighty-eight
13. 1 across times eight
15. 13 across divided by six
16. 12 down divided by eleven
17. 12 down minus 150
20. 13 down minus 167
22. 11 across minus thirty-nine
24. 1 down times 28 across
25. A square number
27. A prime number
28. Five times 26 down
29. 1 across plus thirty-four

Down Clues

1. 10 down minus fifteen
2. Six times 25 across
3. Twelve times 27 across
4. 3 across plus eighty
5. Eleven times 3 down
6. 26 down minus thirty
10. 28 across divided by four
11. 8 across divided by five
12. 5 down divided by twelve
13. 28 across plus ninety-one
14. 13 across times three
18. 5 down plus 1221
19. 6 down times two
21. 25 across plus 1003
22. 14 down plus thirty-eight
23. 8 across divided by seven
24. 19 down plus fifty-five
26. 27 across minus nineteen