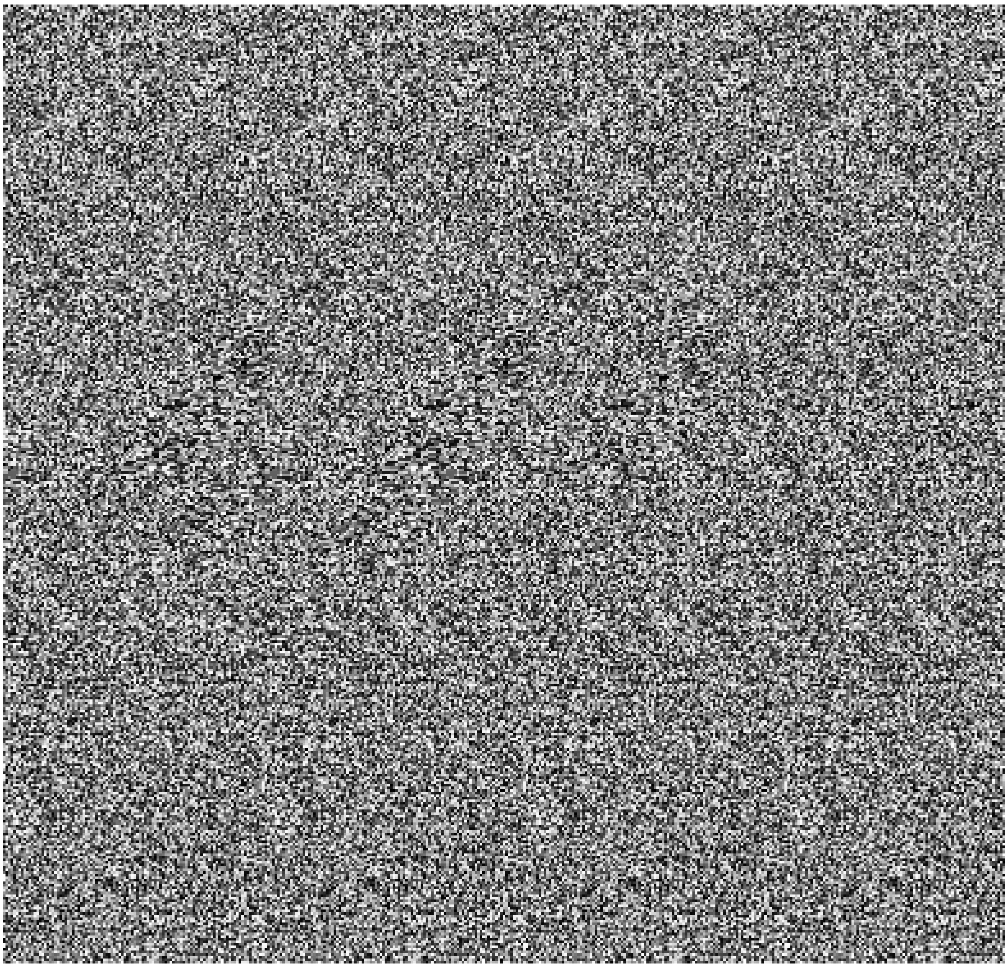
Issue 3, 1998

Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society



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Paradox

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Parting is such sweet sorrow

This is, sadly, the last edition of *Paradox* for 1998, and possibly the last edition ever to be put out by the current team of editors. For that reason, we've tried to make this into a "bumper" edition that will last right through the summer break, and keep you entertained for many hours. Of course, we are restricted by financial considerations that keep us to 24 pages, but we've tried to fit as many articles, puzzles and jokes into this edition as we possibly could.

Remember that even though the academic year is drawing to a close, MUMS still has activities planned. Aside from the regular series of Friday afternoon seminars about a number of mind-bogglingly varied topics, there is also the second *trivia evening* coming up in the last week of semester. The previous trivia afternoon held by MUMS was a great success, so be sure to keep on the lookout for information.

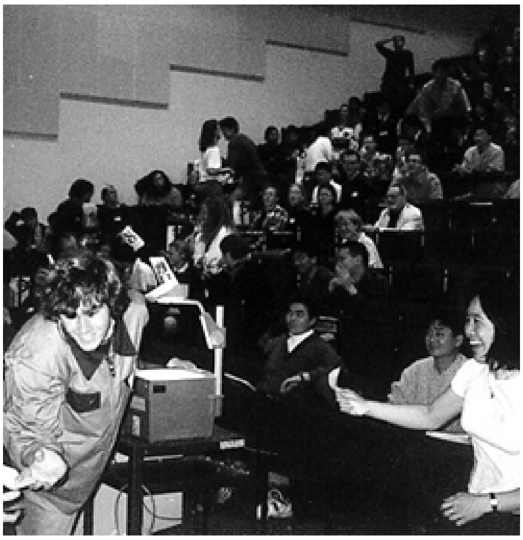
We've had a lot of fun putting together *Paradox* this year, and remind you that next year MUMS will be looking for a new editor or team of editors. Anyway, here it is, *Paradox* issue 3, 1998. Enjoy.

— Jeremy Glick, *Paradox* Editor

Scotch improves with age (but Grammar deteriorates)

On Friday 28th August, the Maths Olympics was staged by the Melbourne University Maths and Stats Society (MUMS) in Theatre A, Richard Berry Building.

The pace at the beginning was frenetic with 28 runners simultaneously rushing for that first problem. The *Paradox* team was streets ahead for much of the 45 minutes, but burned out when it really mattered. The lead changed several times, but ultimately Scotch College prevailed. It was the second time in three years that a school team has come first (better train harder next year!). Pipped at the post were the team of pure maths staff, who showed that worrying from 9 till 5 about whether you can turn a torus inside out *does* have advantages other than being able to claim doughnuts as a tax deduction.



Run, nerds, run!

The results are printed overleaf. The first number is the aggregate score, the second number is the number of the last question that was solved. In the event of equal scores, the second number was used to resolve ties. If this number was tied, the previous question solved was used, and so on. For example, 110/15 indicates that the team scored 110 points and that the last question they solved was question 15 (of 20).

(1)	Scotch	(Scotch College)	110/15
(2)	Pure Optimists	(pure maths staff)	100/15
(3)	These problems are making me thirsty	(ex-Olympians)	95/15
(4)	Totally UnKemped	(a big bunch of idiots)	80/15
(5)	Ellipsis	(physics/mech eng/maths)	65/12
(6)	$\neg\exists x \forall y ((Rxy \to \neg Ryy) \& (\neg Ryy \to Rxy))$	(*Paradox* people)	60/13
(7)	Black Tigers	(engineering)	60/13
(8)	Ormond Magic	(med/dental/sci)	60/12
(9)	Sons of Ho	(Mt Scopus Memorial College)	55/15
(10)	The Model Team	(applied maths staff)	50/11
(11)	On Edge	(maths and stats tutors)	45/11
(12)	IH	(International House)	45/9
(=13)	Lawrence Ip has gone — we have a chance	(commerce)	40/9
(=13)	University High School		40/9
(15)	Fat Chance	(statistics staff)	35/11
(16)	Pack of Idiots	(five people from various faculties)	35/10
(17)	Outliers	(3rd year med)	30/10
(18)	MACROB	(MacRobertson Girls High School)	30/9
(19)	LIFO	(elec eng)	30/7
(20)	Division by Zero	(mostly sci/eng)	25/9
(21)	Fast Tracker Team	(Trinity College, Trinity Foundation)	25/8
(22)	PLC	(Presbyterian Ladies' College)	25/7
(23)	The G08 Spot	(maths honours)	20/8
(24)	The Gurus	(working engineers)	20/8
(25)	Team 1	(1st year students)	20/7
(26)	Integration Sensation	(commerce/sci)	15/8
(=27)	α	(engineering)	10/7
(=27)	Mickey Mouse Desperadoes	(Melbourne Grammar School)	10/7

Our gratitude goes to Chaitanya Rao who organised the event. His effort paid off, with the competition being a success for MUMS, and thoroughly enjoyable for the competitors.

Maths in ages past

Teaching maths in 1950: A logger sells a truckload of lumber for \$100. His cost of production is 4/5 of the price. What is his profit?

Teaching maths in 1960: A logger sells a truckload of lumber for \$100. His cost of production is 4/5 of the price, or \$80. What is his profit?

Teaching maths in 1970: A logger exchanges a set L of lumber for a set M of money. The cardinality of set M is 100. Each element is worth one dollar. Make 100 dots representing the elements of the set M. The set C, the cost of production contains 20 fewer points than set M. Represent the set C as a subset of set M and answer the following question: What is the cardinality of the set P of profits?

Teaching maths in 1980: A logger sells a truckload of lumber for \$100. His cost of production is \$80 and his profit is \$20. Your assignment: Underline the number 20.

Teaching maths in 1990: By cutting down beautiful forest trees, the logger makes \$20. What do you think of this way of making a living? Topic for class participation after answering the question: How did the forest birds and possums feel as the logger cut down the trees? There are no wrong answers.

Teaching maths in 1996: By laying off 40% of its loggers, a company improves its stock price from \$80 to \$100. How much capital gain per share does the CEO make by exercising his stock options at \$80? Assume capital gains are no longer taxed, because this encourages investment.

Teaching maths in 1997: A company outsources all of its loggers. They save on benefits and when demand for their product is down, the logging work force can easily be cut back. The average logger employed by the company earned \$50,000, had three weeks vacation, received a nice retirement plan and medical insurance. The contracted logger charges \$50 an hour. Was outsourcing a good move?

Teaching maths in 1998: A logging company exports its wood-finishing jobs to its Indonesian subsidiary and lays off the corresponding half of its Australian workers (the higher-paid half). It clear-cuts 95% of the forest, leaving the rest for the spotted owl, and lays off all its remaining Australian workers. It tells the workers that the spotted owl is responsible for the absence of fellable trees and lobbies Parliament for exemption from the Endangered Species Act. Parliament instead exempts the company from all federal regulation. What is the return on investment of the lobbying costs?

An epistemic puzzle: Hollis's Paradox[1]

Two people, A and B, think of positive integers and whisper the numbers chosen in a third's, C's, ear. C then asserts that neither A nor B can work out whose number is greater. He also adds that they did not both choose the same number. A reasons as follows: 'Clearly, B did not choose 1 since, if he had, he could deduce that my number would be greater than his. Further, B can reason similarly about me, which is correct since I picked 157. Hence 1 was not picked by either of us. But then B did not choose 2 since, now that 1 has been eliminated, B could again deduce that my number was the greater. Once again, B can reason analogously about me. Therefore, 2 could not have been picked by either of us. By continuing this chain of reasoning, it follows that 157 was not chosen by either of us, even though I chose it!'

[1] Glenn W. Erickson and John A. Fossa, *Dictionary of Paradox*, University Press of America, Inc., Lanham, 1998, pp. 85–86.

Hollis claims that the assertion that C was wrong about A and B's ability to decide which number is greater is also paradoxical, since, if he was, A must conclude that both numbers are greater than each other, which is obviously absurd. C's assertion thus eliminates the possibility that 1 was chosen by either A or B since, it being the smallest number, whoever chose 1 would know that the other's number was larger. But, then 2 becomes the smallest number that can be chosen and so the same argument applies to it. After 157 steps, A is forced to conclude that he could not have chosen the very number (157) that he did choose and an induction argument results in the conclusion that no number could have been chosen by either A or B.

Problems

The following are some problems for prize-money. The person who submits the best (i.e. clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number, and will have their solution published in the next edition of *Paradox*. Solutions may be emailed to `paradox@ms.unimelb.edu.au`. (LaTeX format would be appreciated though not demanded.) If you do not have access to email then drop in a hard copy of your solution to the MUMS pigeon-hole near the Maths and Stats Office in the Richard Berry building.

1. ($10) In the future, we expect that *Paradox* will be accepted as legal tender, so each copy will need to be individually numbered. As such, each copy will receive a six-digit number. However, legislation requires that each number differ from all others in at least two places (i.e. 123456 and 123457 could not both be used, but 123456 and 234567 could). What is the maximum number of *Paradox*es that could be printed, and why?

2. ($10) An arbitrary number of dodgem car drivers begin randomly placed around a circular rail. They all start with the same speed, however each driver randomly chooses to start going either clockwise or anticlockwise. After each collision each driver reverses direction without loss of speed (i.e. they bounce off each other). Prove that at some later time each of the cars will occupy their initial positions.

3. ($5) In a sock basket at home there are at most 1998 socks. These are made up of red and white socks. If you draw two socks (without replacement) then the probability of getting a matching pair is 1/2. What is the maximum number of red socks you could have? (Only solutions from first year students will be accepted for this problem.)

4. ($20) Consider an increasing sequence of natural numbers, $0 < a_1 < a_2 < \cdots$. Consider the set $A = \{a_i + a_j\}$ where $i, j \in N$. Show that there is an infinite subset of A, $\{b_n\}$, such that b_i does not divide b_j for all $i \neq j$.

Holiday fun

The following are some problems which do not attract any prize-money but which will hopefully provide you with boundless entertainment during the summer holidays.

1. Show that the perimeter of a triangle contained inside a polygon is less than the perimeter of the polygon.

2. Reduce
$$\frac{\sqrt{3-\sqrt{5}}}{\sqrt{2}+\sqrt{7-3\sqrt{5}}}$$
to the form $\sqrt{a/b}$, where a and b are relatively prime integers. *Hint*: Both a and b are less than 20.

3. A candidate at an examination writes four papers. If the maximum number of marks obtainable on each paper is m, show that the number of ways of obtaining a total of $2m$ marks is $\frac{1}{3}(m+1)(2m^2+4m+3)$.

4. A chocolate factory has produced twelve chocolates. Due to a machinery defect, one is of different weight from the other eleven. You are given a scale to assist in finding the defective one. Only a maximum of three weighings may be used however, as any more and the chocolates will be discarded by quality control. How would you determine which chocolate is defective, and whether it is heavier or lighter?

5. An odd number of engineers are standing in the fields such that no two pairs are the same distance from each other. Each takes out a water balloon and throws it at the nearest person. Prove that at least one engineer does not get wet.

There is no universe!

No doubt anyone who participated in this year's Maths Olympics would've been mightily impressed by the name of the *Paradox* team: $\neg\exists x\forall y((Rxy \to \neg Ryy)\&(\neg Ryy \to Rxy))$. Apart from being the best thing in an otherwise disappointing event for the brilliant *Paradox* editors, it also happens to be the first order logic formulation of Russell's Paradox. Literally it means: 'it is not the case that there exists a thing x such that for all things y, the relation Rxy implies not-Ryy and not-Ryy implies Rxy.' Here Rxy simply denotes a relation between two objects, for example 'x is greater than y', or 'x is to the left of y'. This doesn't seem too enlightening at first glance. If we talk about a universe where the only objects are sets however, then there is a very natural relation between two sets: a set A either belongs to (i.e. is a member of) another set B, or it doesn't. Now if we reformulate our rather cryptic statement above by taking x, y to be sets and Rxy to be the relation y

belongs to x, we'd get: 'there isn't a set x such that for all sets y, $y \in x$ implies $y \notin y$, and $y \notin y$ implies $y \in x$', or: 'there is no set of all sets that don't contain themselves'.

To see why this is true we consider the question of whether x belongs to itself. Of course this is a valid question only if x exists. Well if $x \in x$ then since all members of x are sets that don't contain themselves, and x belongs to itself, it follows that x is also a set that doesn't contain itself, but this is a contradiction. If $x \notin x$ then since all sets that don't contain themselves are members of x, but x isn't, so x must contain itself, also a contradiction. Therefore no matter what we do, we arrive at a contradiction, hence there cannot be such a set x.

We know however, that there are sets which don't contain themselves (for example the empty set for it contains nothing), so if a set is more or less a collection of things, then why can't we somehow collect all these sets together and form the required set x? Russell's Paradox shows that even with concepts as simple as sets and belonging, intuition can be a very dangerous guide. We must be very careful about how we can construct sets.

A major principle of set theory, the Axiom of Specification, states that: 'To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.' For example given the set of natural number and the condition '2 divides x', we would obtain the set of all even natural numbers. This is exactly our intuitive idea of collecting things that share a common property to form a set. If we let $S(x)$ to be $x \notin x$, then we can apply the Axiom of Specification to get the set B of all sets that don't contain themselves, provided A, the set of all sets, exists. But we already saw that this is impossible. Hence there is no set of all sets. In other words, 'nothing contains everything', or more spectacularly/ambiguously, 'there is no universe'. Here "universe" is used in the sense of 'universe of discourse', meaning, in any particular discussion, a set that contains all the objects that enter into the discussion.

The idea behind Russell's Paradox is not only useful in showing that the universe doesn't exist and thus laughing at all those poor physicists, but also useful in proving some very important results. One such result is Cantor's Theorem: 'For any set X, let $P(X)$ denote the set of subsets of X, then there is no surjective function from X into $P(X)$.' (It is another axiom of set theory that the subsets of a set A form a set.) Remember a function f is surjective if for any member S of $P(X)$ we can find $a \in X$ such that $f(a) = S$. The idea of the proof is to find a subset of X, say B, that is not equal to $f(a)$ for any $a \in X$. Note that B will depend on the function f. We'll use the Axiom of Specification. Let A in the axiom be X, let $S(x)$ be the condition $x \notin f(x)$. This condition make sense because $f(x)$ belongs to $P(X)$ and hence is a subset of X, so the element x of X either belongs to $f(x)$ or doesn't. Thus by the Axiom of Specification we have a set B of all x such that x doesn't belong to $f(x)$. If $B = f(a)$, then either $a \in B$ or $a \notin B$. If $a \in B$, then since all elements of B are those that satisfy $x \notin f(x)$, we conclude that $a \notin f(a)$, but then $a \notin B$ as $B = f(a)$, hence a contradiction. If $a \notin B$, then since B is the set of all x such that $x \notin f(x)$, we conclude that $a \in f(a)$, or $a \in B$, also a contradiction. Hence there is no a such that $B = f(a)$. This argument very much resembles Russell's Paradox.

Cantor's Theorem shows that in a sense there are more subsets of a given set A than the number of elements of A (which may be infinite), since to each element x of A we can associate a subset of A to it, namely $\{x\}$, the one that contains only that element, but we can't associate to each subset of A an element of A. So the "number" of subsets must be somehow "more" than the "number" of elements.

— Jian He

Ramanujan

The British mathematician G. H. Hardy once visited Srinivasa Ramanujan in a sanatorium where he lay dying. By way of making conversation, Hardy said that the number of his taxicab was 1729 which was, in his opinion, a very dull number. Ramanujan, immediately exclaimed, 'No, Hardy! No, Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways.' Ramanujan was an enigmatic figure. He spent much of his life in isolation and was plagued by poor health. But Ramanujan's frail body held the mind of a great mathematical genius which places him with giants like Euler and Gauss.

Ramanujan was born into an impoverished family in Southern India in 1887. His mathematical ability was recognised early and, at the age of seven, he was awarded a scholarship to the Kumbakonam Town High School. It is said that he often recited mathematical formulae to his school mates. By 1903, Ramanujan was granted a place at a government college. Despite his brilliant mathematical insight, Ramanujan failed his examinations, being totally absorbed by his own "mathematical diversions".

By 1912, Ramanujan had compiled a considerable number of mathematical results. Working as a clerk for the Madras Port Trust, he was encouraged to communicate these results to three prominent British mathematicians by the Trust's English chairman, Sir Francis Spring. Only one — G. H. Hardy, replied to Ramanujan's letter.

Hardy was used to receiving prank mail but upon closer examination of the 120 formulae appended to Ramanujan's letter, he saw that this was the work of a true genius. Later, Hardy devised a "pure-talent" scale. On this scale Ramanujan was rated 100, himself a mere 25. Even the most influential mathematician of the time, David Hilbert, merited a modest 80. Some of Ramanujan's formulae defeated Hardy completely. He wrote '...yet they must be true, because if they were not true, no one would have had the imagination to invent them.'

Upon Hardy's invitation, Ramanujan travelled to England in 1914. Over the next few years, he worked closely with Hardy at Trinity College. His brilliant insight combined with Hardy's technical knowledge produced some astonishing progress in studies of various arithmetic functions which gave the foundation for questions like: How many ways can one express a number as a sum of smaller positive numbers?

In 1917, Ramanujan was made a Fellow of Trinity College and of the Royal Society of London, the first Indian to be granted either of these memberships. His health deteriorated due to difficulties in maintaining a traditionally vegetarian diet in war-rationed Britain.

Ramanujan was in and out of hospitals but continued to produce new results. In 1919, he returned to India where he became a hero to young Indian intellectuals. He died aged 32.

Much of Ramanujan's work lay in the field of series and sequences. He wrote results in his so-called "notebooks" but used his own nomenclature. This made it difficult for future mathematicians to interpret his work. Even now, his "notebooks" are often published in non-annotated form.

Like many mathematicians, from Archimedes to Leibniz, Ramanujan was fascinated by calculations of the constant π. Much of his work on π comes from his investigations on modular equations. These are algebraic relations between $f(x)$ and a combination of terms consisting of $f(x^n)$ where n, a positive integer, is known as the "order" of the equation. The simplest modular equation is of second order and is given by

$$f(x) = \frac{2\sqrt{f(x^2)}}{1 + f(x^2)}.$$

There is a class of functions, called modular functions, that satisfy such a relationship. Ramanujan was unparalleled in coming up with solutions to modular equations that also satisfy other conditions. These are called singular values and some of their natural logarithms coincide with constant multiples of π. Ramanujan treated these values in his paper *Modular Equations and Approximations of* π.

AN EXAMPLE OF RAMANUJAN'S FORMULAE

$$\pi = \frac{k_6\sqrt{k_3}}{S}$$

where

$$\begin{aligned} S &= \sum_{n=0}^{\infty}(-1)^n \frac{(6n)!(k_2 + nk_1)}{n!^3(3n)!(8k_4k_5)^n}, \\ k_1 &= 545140134, \\ k_2 &= 13591409, \\ k_3 &= 640320, \\ k_4 &= 100100025, \\ k_5 &= 327843840, \\ k_6 &= 53360. \end{aligned}$$

Although its implementation had to wait for modern computers, efficient algorithms and new ways to multiply numbers, Ramanujan's work holds the key ingredients to some of the most recent methods of approximating π.

— Alex Grinstein

References

[1] Borwein and Borwein, *Ramanujan and Pi,* in Ferris (ed.), *The World treasury of physics, astronomy and mathematics,* Clifton Hardiman, Boston, 1991.

[2] Ashis Nandy, *Alternative sciences: Creativity and authenticity in two Indian scientists,* Allied Publishers Private Ltd, New Delhi, 1980.

A couple of mathematical gems

THE PIGEON-HOLE PRINCIPLE

The Pigeon-Hole Principle (PHP for short) is one of the more easily understood but also more powerful results of mathematics. In its general form it reads:

Given n pigeon-holes containing a total of $mn + 1$ pigeons, there is at least one pigeon-hole containing at least $m + 1$ pigeons.

After a moment's thought, the reason this is true becomes clear. If each pigeon-hole had at most m pigeons, there would be no more than mn pigeons in total, a contradiction.

Using the PHP, one can establish a plenitude of facts involving collections of objects. One easy example is that amongst 25 people, at least three of them were born in the same month of the year. Here the pigeons are the 25 people, the 12 months are the pigeon-holes, and $25 > 12 \times 2$.

A more interesting example is that at a given moment there exist twenty-one Melbournians, each with the same non-zero number of hairs on their head! This is found by estimating that there are more than three million people in Melbourne, each with between 1 and $150,000$ hairs on their heads!

However there are harder types of problems which use the PHP in more ingenious ways. Consider the following two for instance:

1. *Prove that in any set of* 14 *different odd numbers, all less than* 50*, there is a pair of numbers with sum* 52*.*

To solve this problem, we match up the odd numbers to give sums of 52 where possible. This gives the pairs $\{3, 49\}, \{5, 47\}, \ldots, \{25, 27\}$, with $\{1\}$ left behind. These 13 sets represent the pigeon-holes. By the PHP, if we choose 14 different odd numbers, we are forced to choose both elements of one of the sets $\{i, 52 - i\}$. giving sum 52, as required.

2. *An athlete trains for fifty hours during the month of January. She trains every day, for a whole number of hours each time. Prove that there is a succession of days in which she trains for a total of* 11 *hours exactly.*

Here it is not at all obvious how the numbers 50, 31 (the number of days in January) and 11 are related, or how to apply the PHP. The solution below makes a clever construction of the pigeon-holes.

Let x_i be the total number of hours the athlete has trained up to and including the ith day of the month. Then $1 \leq x_1 < x_2 < \ldots < x_{31} = 50$.

The trick is to consider the 62 numbers $x_1, x_2, \ldots, x_{31}, x_1 + 11, x_2 + 11, \ldots, x_{31} + 11$. We have 62 natural numbers, which take on possible values in the set $\{1, 2, \ldots, 61\}$ (as $x_{31} + 11 = 61$ is the largest of the numbers). By the PHP, two of the 62 numbers must be equal. But since $x_s < x_t$ whenever $s < t$, the only way two of the numbers can be equal is if $x_i + 11 = x_j$, for some i and j. This tells us that she trained for exactly 11 hours from the $(i + 1)$th day until the jth day (inclusive), as required.

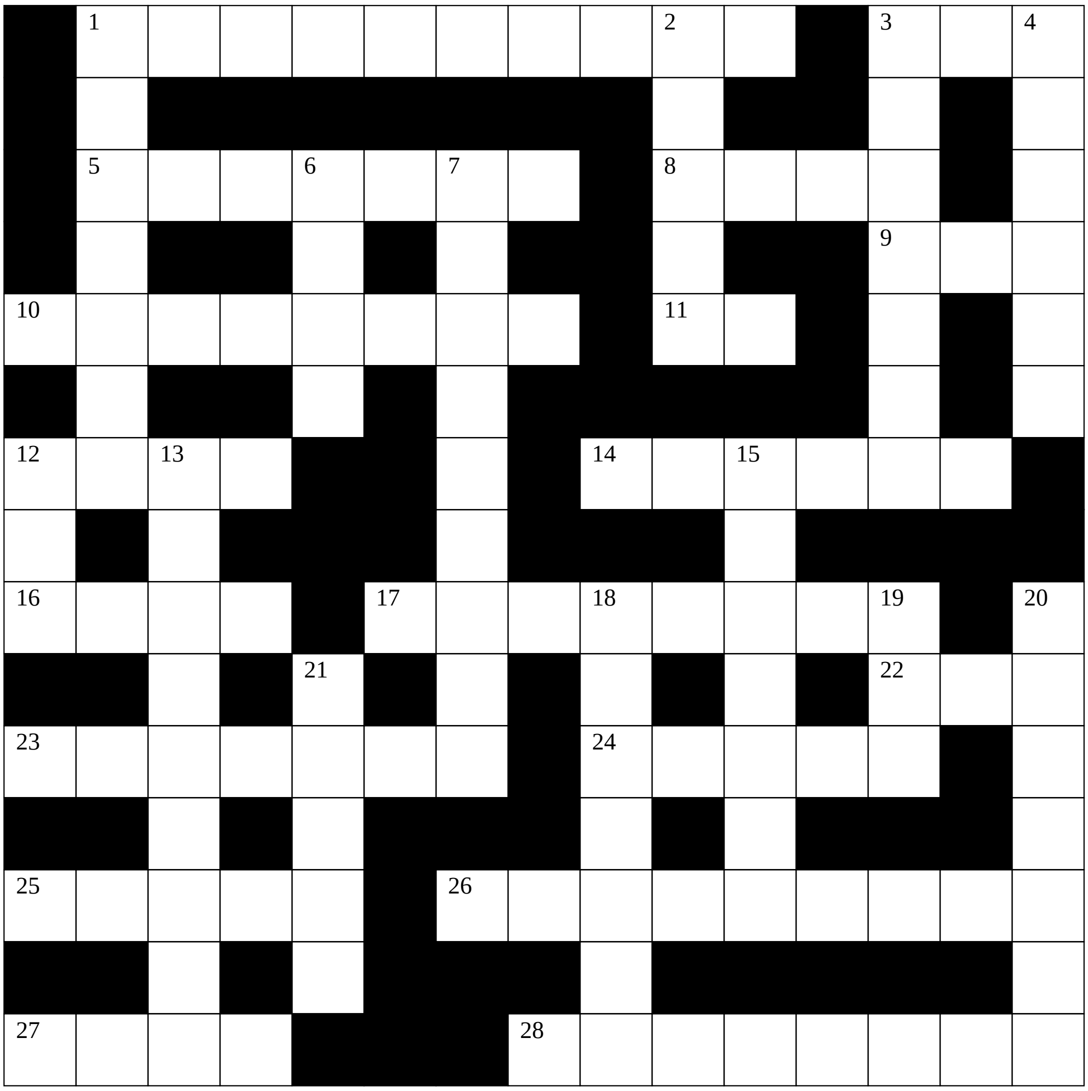
Continued page 12

Paradox October centrefold: A cryptic crossword

Admittedly, cryptic crosswords are somewhat less mathematical in nature than activities featured in previous *Paradox* centrefolds, but they are by no means less enjoyable. The following is a cryptic crossword with a mathematical slant. The person who submits the first correct entry will receive $5 as a reward for their incredible wit and skill. Submissions can be made by filling in the following details and depositing your completed centrefold in the MUMS pigeon-hole. A solution will be printed the next edition.

Name: ______________________________

Email address: ______________________________



ACROSS

1 Expression of many names but lacking direction (10)
3 What was required was the 500th monarch (1,1,1)
5 22-across starts making blobs in clay, shapely like diamond (7)
8 Sounds like little thanks is due to letter from 24-across (4)
9 Beam from the center (3)
10 True statement was itself complete after identification (8)
11 Melbourne University is in smut (2)
12 Cuts reference lines (4)
14 Little bears eat an integrated circuit for some 1-across (6)
16 Police takes you first for image (4)
17 Almost all backwards after mid-easterner inside post office showed conic section (8)
22 Seventeenth letter started rasping a short laugh (3)
23 Ring when they cancel out America (7)
24 Nerd keeps right of Greece (5)
25 Trig in resin esters (5)
26 Lie on a few hundred, and one number sequence (9)
27 Situation smashes aces (4)
28 Copper found in some cruel surroundings using 7-down (8)

DOWN

1 Even commercial bull is a contradiction (7)
2 The basic truth of destroyed communist ditched right around Jupiter's moon (5)
3 Volume over leaderless state of fourth degree (7)
4 Differentiate dux, not you, next to odd day (2,2,2)
6 Make American subject sweep Harry's head under the rug (4)
7 Infinite sums in largest mixup (9)
12 Curve of endless march (3)
13 Show upset nest around northern powers (9)
15 No ebola affected 18-down, logically (7)
18 Stormy barrage, heartlessly, took last pupil of symbolic 6-down (7)
19 12-down, I hear, was Noah's (3)
20 Town bias, u for a strip (7)
21 Set crown initially on girl (5)

RESULTS OF LAST ISSUE'S CENTREFOLD

In the last issue, we printed a map consisting of many regions, and our readers' task was to use only four colours, but colour each region such that no two regions sharing a border have the same colour. Although the submission made by Jolene Koay was not completely correct, her neatness and judicious choice of colours impressed us so much that we awarded her the prize.

From page 9

Here now are some more examples to try:

- Prove that of any 10 points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most 1/3.
- Prove that in a group of six people at a party there are either three people who mutually know each other, or three who are strangers to each other.
- Show that given any 37 integers it is possible to choose 7 whose sum is divisible by 7.

THE AM-GM INEQUALITY

Inequalities arise in almost all areas of mathematics. They can be used to compare quantities for either simplifying or optimising problems with constraints. One of the more useful inequalities is the so-called Arithmetic Mean - Geometric Mean (AM-GM) inequality. It simply states that the arithmetic mean of a positive set of numbers is no less than their geometric mean. That is, if $a_1, a_2, \ldots, a_n$ are positive, then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1.a_2.\ldots.a_n)^{\frac{1}{n}} \tag{1}$$

The left and right sides of this inequality are equal if and only if the n numbers are equal, otherwise the left side is strictly greater than the right side.

We now look at a couple of examples of how this tool can be used instead of calculus to solve some optimisation-type problems.

1. We shall show that the total surface area S of a box having a fixed volume V is minimised when the box is a cube.

Let the edge lengths be x, y and z. Then it is easy to see that $V = xyz$, and $S = 2(xy + yz + xz)$ by summing the areas of each of the six rectangular faces. Then

$$\begin{aligned} S &= 2(xy + yz + xz) \\ &= 6\left(\frac{xy + yz + xz}{3}\right) \\ &\geq 6\left((xy).(yz).(xz)\right)^{\frac{1}{3}} \text{ by (1)} \\ &= 6(xyz)^{\frac{2}{3}} \\ &= 6V^{\frac{2}{3}} \end{aligned}$$

So the surface area S is bounded below by the fixed number $6V^{\frac{2}{3}}$, and achieves this value if and only if $xy = yz = xz$, or equivalently, when the box is a cube.

2. Here is a more sophisticated example of an AM-GM application. We shall find the maximum value of the function $f(x) = (x + 1)^2(4 - x)^3$ in the domain $(-1, 4)$ without using calculus. The interested reader may later like to try to generalise the problem to other expressions of this form.

We see that the function is in the form of a product of five terms, which reminds one of the right side of equation (1). One may then be tempted to let $a_1 = a_2 = x + 1$ and $a_3 = a_4 = a_5 = 4 - x$. Substituting this into equation (1) however, will lead to

$$(x+1)^2(4-x)^3 \leq \left(\frac{14-x}{5}\right)^5,$$

which is not very enlightening. The trick is to make the x's cancel when summing the a_i's. This can be done by letting $a_1 = a_2 = \frac{x+1}{2}$ and $a_3 = a_4 = a_5 = \frac{4-x}{3}$ to give

$$\begin{aligned} f(x) &= (x+1)^2(4-x)^3 \\ &= 2^2.3^3\left(\left(\frac{x+1}{2}.\frac{x+1}{2}.\frac{4-x}{3}.\frac{4-x}{3}.\frac{4-x}{3}\right)^{\frac{1}{5}}\right)^5 \\ &\leq 2^2.3^3\left(\frac{1}{5}\left(\frac{x+1}{2}+\frac{x+1}{2}+\frac{4-x}{3}+\frac{4-x}{3}+\frac{4-x}{3}\right)\right)^5 \text{ by (1)} \\ &= 2^2.3^3\left(\frac{1}{5}.(5)\right)^5 \\ &= 2^2.3^3.1 \\ &= 108 \end{aligned}$$

So $f(x)$ has a maximum value of 108 when $-1 < x < 4$, and this value is attained when $\frac{x+1}{2} = \frac{4-x}{3}$, that is, when $x = 1$.

The AM-GM inequality is a special example of another inequality called *Jensen's Inequality*, which deals with convex functions. A convex function is simply one whose graph appears to "bulge" downwards. If the function is twice differentiable, it is equivalent to the function having a positive second derivative. For example, $f(x) = x^2$ is convex because $f''(x) = 2 > 0$.

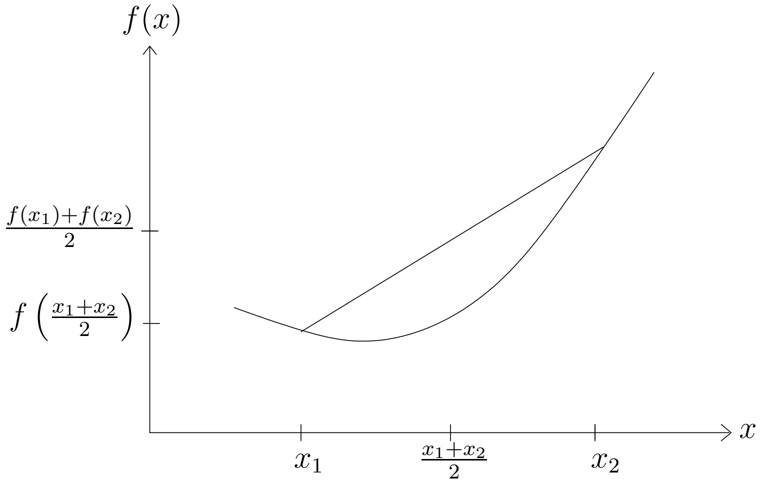
Suppose we have some numbers in an interval over which a function is convex. One may consider a "mean value" of the function firstly by taking the mean of the numbers and then finding the function's image of that mean. Alternatively, one may first find the image of each number under the function, then take the mean value of each image. Jensen's Inequality states that the mean obtained in the first method gives a smaller answer than the second method, unless the numbers are all equal in which case the two answers are equal.

In mathematical parlance, if $x_1, x_2, \ldots, x_n$ are numbers in an interval over which f is convex then:

$$\frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n} \geq f(\frac{x_1+x_2+\ldots+x_n}{n}) \tag{2}$$

with equality if and only if the x_i's are equal.

This inequality is illustrated in the case $n = 2$ below. It follows simply from the fact that the mid-point of a straight line joining two points on a convex graph lies above the graph. This can be then be generalised to any n by induction.



Given this, one can invent a huge class of inequalities: simply choose any convex function, and substitute it into (2). For example the AM-GM inequality may be derived from Jensen's Inequality by considering the function $f(x) = -\log x$, $x > 0$. This is a convex function because $f''(x) = \frac{1}{x^2} > 0$. The remaining details are left to the reader.

We conclude with some exercises that can be solved using the AM-GM or Jensen's Inequality.

- Find the minimum value of $4x + \frac{9}{x}$ for $x > 0$, and the value of x for which this occurs.
- If A, B and C are the three angles of a triangle, show that
$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$
- Prove the Root Mean Square - Arithmetic Mean Inequality: if $a_1, a_2, \ldots, a_n$ are real numbers, then
$$\sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

— Chaitanya Rao

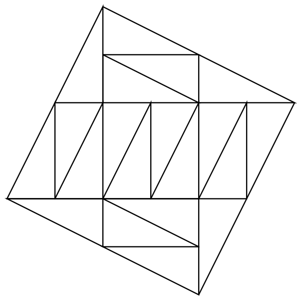
References

[1] D. A. Holton, *Combinatorics 1: Pigeon-hole principle, Some basic counting*, University of Otago, Dunedin, 1988.

[2] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

Solutions to last issue's problems

Problem 1: ($10) We are given tiles in the form of right angled triangles having perpendicular sides of length 1cm and 2cm. Form a square from 20 such tiles.
Solution:



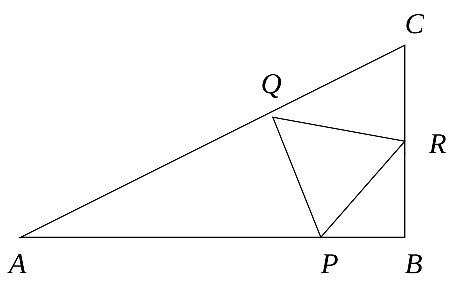
— Corey Plover

Problem 2: ($10) On the island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colours meet, they both simultaneously change colour to the third colour (e.g. if a grey and brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be same colour?
Solution: Consider the number of chameleons of each colour, modulo 3. The initial configuration is [1, 0, 2]. If a brown and a grey chameleon meet, the new configuration will be [0, 2, 1]. Similarly, if brown and crimson meet, or if grey and crimson meet, the new configuration will also be [0, 2, 1]. By symmetry, we conclude that the configuration after any number of moves will always comprise the numbers 0, 1, and 2 in some order.

If all 45 chameleons were of the same colour then the configuration would be [0, 0, 0]. By our reasoning above this will never happen, so the answer to the question is no.

— Mercury group, Dept. of Computer Science
`http://www.cs.mu.oz.au/mercury`

Problem 3: ($10) In $\triangle ABC$, $AB = \sqrt{3}$. $AC = 2$ and $BC = 1$. What is the smallest equilateral triangle, $\triangle PQR$, that can be inscribed (one vertex on each side) in $\triangle ABC$?



Solution: Let $\theta = \angle RPB$, and $l = PQ = QR = RP$. Then

$$BR = l\sin\theta \Rightarrow CR = 1 - l\sin\theta.$$

In $\triangle CQR$,

$$\frac{l}{\sin 60^\circ} = \frac{1 - l\sin\theta}{sin(90^\circ - \theta)} = \frac{1 - l\sin\theta}{\cos\theta}$$

by the sine rule. Hence

$$\begin{aligned} l &= \frac{1}{\frac{2}{\sqrt{3}}\cos\theta + \sin\theta} \\ &= \frac{1}{r\sin(\theta + \alpha)} \end{aligned}$$

for some r, α to be determined.

Now,

$$\begin{aligned} \frac{2}{\sqrt{3}}\cos\theta + \sin\theta &= r\sin(\theta + \alpha) \\ &= (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta. \end{aligned}$$

Therefore,

$$r\sin\alpha = \tfrac{2}{\sqrt{3}} \quad \text{and} \quad r\cos\alpha = 1.$$

Thus,

$$r^2 = \tfrac{7}{3} \Rightarrow r = \sqrt{\tfrac{7}{3}} \quad \text{and} \quad \alpha = \arctan\tfrac{2}{\sqrt{3}}.$$

Hence

$$l = \frac{1}{\sqrt{\frac{7}{3}}\sin(\theta + \alpha)}$$

where $\alpha = \arctan\frac{2}{\sqrt{3}}$.

So l is minimised when $\sqrt{7/3}\sin(\theta + \alpha)$ is maximised, and $\sqrt{7/3}\sin(\theta + \alpha) \le \sqrt{7/3}$. Therefore, the minimal value of l is $\sqrt{3/7}$, i.e. the smallest equilateral triangle that can be fitted into $\triangle ABC$ is one with side length $\sqrt{3/7}$.

— Geoffrey Kong

Alternative solution: Let $x = BR$, we will express the side length of $\triangle PQR$ in terms of x.

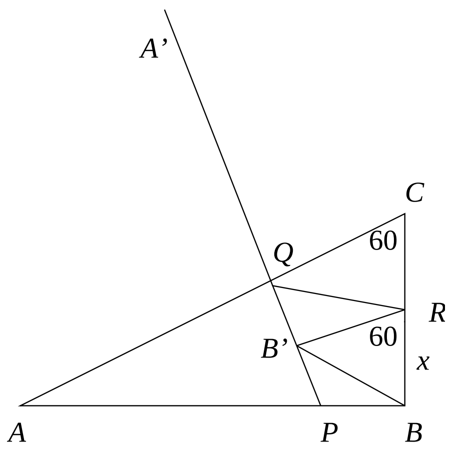
Since $PQ = QR$ and $\angle PQR = 60^\circ$. If we rotate P about R by 60°, the image is Q. Hence if we rotate the entire line AB about R by 60°, Q would be the intersection of the image of AB ($A'B'$) and AC.

So,

$$\angle QB'R = \angle PBR = 90^\circ$$
$$QR^2 = B'Q^2 + B'R^2 = B'Q^2 + BR^2 = B'Q^2 + x^2.$$

$\angle B'RB = 60°$ since B' is the image of B under a rotation of $60°$ about R. Therefore,

$$\angle B'RB = \angle ACB$$
$$B'R \parallel QC$$
$$\angle B'QA = \angle QB'R = 90°.$$



Hence

$$\begin{aligned} B'Q &= \text{distance from } R \text{ to line } AC \\ &= RC\sin 60° \\ &= (1-x)\frac{\sqrt{3}}{2}. \end{aligned}$$

Therefore, $QR^2 = \frac{3}{4}(1-x)^2 + x^2$. Now we just have to minimise a quadratic with $0 \leq x \leq 1$. $QR^2 = \frac{7x^2-6x+3}{4}$ is minimum when $14x = 6$, i.e. $x = 3/7$. Thus,

$$QR^2 = \left(\frac{4}{7}\right)^2\left(\frac{3}{4}\right) + \left(\frac{3}{7}\right)^2 \Rightarrow QR = \sqrt{\frac{3}{7}}.$$

— Jian He, *Paradox* Problems Editor

Problem 4: (\$5) Show that for all distinct real numbers a, b and c,

$$\left(\frac{1}{b-c}\right)^2 + \left(\frac{1}{c-a}\right)^2 + \left(\frac{1}{a-b}\right)^2 = \left(\frac{1}{b-c} + \frac{1}{c-a} + \frac{1}{a-b}\right)^2.$$

Solution: For convenience (and neatness), let $x = \frac{1}{b-c}$, $y = \frac{1}{c-a}$, and $z = \frac{1}{a-b}$. Then

$$\begin{aligned} \text{RHS} &= (x+y+z)^2 \\ &= (x^2+y^2+z^2) + 2xy + 2yz + 2xz \\ &= \text{LHS} + 2(xy+yz+xz). \end{aligned}$$

But

$$\begin{aligned} (xy+yz+xz) &= \frac{1}{(b-c)(c-a)} + \frac{1}{(c-a)(a-b)} + \frac{1}{(b-c)(a-b)} \\ &= \frac{(a-b)+(b-c)+(c-a)}{(a-b)(b-c)(c-a)} \\ &= 0 \end{aligned}$$

if $(a-b)(b-c)(c-a) \neq 0$. i.e. if $a-b \neq 0$ and $b-c \neq 0$ and $c-a \neq 0$. But all three are distinct, so $(a-b)(b-c)(c-a) \neq 0$, and hence $2(xy+yz+xz) = 0$. Therefore, RHS = LHS.

— Alex Lubansky

Problem 5: (\$15) For the sequence $a_n = [n\sqrt{2}]$, show that there are infinitely many a_n which are powers of 2. ($[x]$ is the largest integer $\leq x$.)

Solution: To show that the set $a_n = 2^k, k \in N$ is infinite we first assume it is finite with largest element $a_m = 2^{k-1}$. It is then sufficient to prove that there must exist some $a_n = 2^k - 1$ and that it follows directly from this that there must be a further $a_{2^i n+1} = 2^{k+i}$. This proves inductively that the set is infinite in magnitude.

We may write $a_n = \left[\sqrt{2}n\right] = \sqrt{2}n - 1 + h_n$ where $h_n = a_n + 1 - \sqrt{2}n$ is the deviation of $\sqrt{2}n$ from $a_n + 1$ with $0 < h_n \leq 1$ and use this to prove two fundamental ideas.

1. $a_{n+1} = a_n + \langle \begin{array}{ll} 1, & h_n > (\sqrt{2}-1), h_{n+1} = h_n + (1-\sqrt{2}) \\ 2, & h_n \leq (\sqrt{2}-1), h_{n+1} = h_n + (2-\sqrt{2}) \end{array}$.

   Proof.

$$\begin{aligned} a_{n+1} &= \sqrt{2}(n+1) + h_{n+1} - 1 \\ h_{n+1} &= a_{n+1} - \sqrt{2}(n+1) + 1 \\ &= a_{n+1} - a_n - \sqrt{2} + h_n. \end{aligned}$$

   If $h_n \leq (\sqrt{2}-1)$, then $h_{n+1} \leq (a_{n+1} - a_n) - 1$. For $0 < h_{n+1} \leq 1$ we must have

$$\begin{aligned} (a_{n+1} - a_n) &= 2 \\ h_{n+1} &= h_n + (2-\sqrt{2}). \end{aligned}$$

   If $h_n > (\sqrt{2}-1)$, then $h_{n+1} > (a_{n+1} - a_n) - 1$. For $0 < h_{n+1} \leq 1$, we must have

$$\begin{aligned} (a_{n+1} - a_n) &= 1 \\ h_{n+1} &= h_n + (1-\sqrt{2}). \end{aligned}$$

2. $a_{2n} = 2a_n + \langle \begin{array}{ll} 0, & h_n > \frac{1}{2}, h_{n+1} = 2h_n - 1 \\ 1, & h_n \leq \frac{1}{2}, h_{n+1} = 2h_n \end{array}$.

   Proof.

$$\begin{aligned} a_{2n} &= \sqrt{2}(2n) + h_{2n} - 1 \\ h_{2n} &= a_{2n} - 2\sqrt{2}n + 1 \\ &= (a_{2n} - 2a_n) - 1 + 2h_n. \end{aligned}$$

   If $h_n \leq \frac{1}{2}$, then $h_{2n} \leq (a_{2n} - 2a_n)$. For $0 < h_{n+1} \leq 1$, we must have

$$\begin{aligned} (a_{2n} - 2a_n) &= 1 \\ h_{2n} &= 2h_n \end{aligned}$$

   If $h_n > \frac{1}{2}$, then $h_{2n} > (a_{2n} - 2a_n)$. For $0 < h_{n+1} \leq 1$ we must have

$$\begin{aligned} (a_{2n} - 2a_n) &= 0 \\ h_{2n} &= 2h_n - 1. \end{aligned}$$

We are now ready to tackle the real problem, that of proving that the set of $a_n = 2^k$ is infinite.

If we assume that $a_m = 2^{k-1}$ is the largest known case, it follows that there must exist an $a_n = 2^k - 1$, $a_{n+1} = 2^k + 1$ (from 1) since any successive a_n's differ by 1 or 2 where $h_n \leq (\sqrt{2} - 1)$.

From this we see that

$$\begin{aligned} a_{2n} &= 2a_n + 1 \\ &= 2^{k+1} - 1 \\ h_{2n} &= 2h_n. \end{aligned}$$

Inductively, we can generalise this to

$$\begin{aligned} a_{2^i n} &= 2^{k+i} - 1 \\ h_{2^i n} &= 2^i h_n \end{aligned}$$

and given that $2^i h_n \leq (\sqrt{2} - 1)$, it follows that $a_{2^i n+1} = 2^{k+i} + 1$.

However, for some $i' = \left\lceil \log_2 \frac{\sqrt{2}-1}{h_n} \right\rceil$ we find that

$$h_{2^{i'} n} = 2^{i'} h_n > (\sqrt{2} - 1)$$

and

$$a_{2^{i'} n+1} = (2^{k+i'} - 1) + 1 = 2^{k+i'}.$$

And thus we have a new case of $a_{n'} = 2^{k'}$ and we see by induction that the series is infinite.

— Tim Gould

Alternative solution: We will assume that the series is finite and conclude that $\sqrt{2}$ is rational, a contradiction.

Assume that $a_m = 2^k$ is the last power of 2. Since $[2\sqrt{2}] = 2$, $m \geq 2$. Then

$$\begin{aligned} 2^k \leq &\quad m\sqrt{2} &\leq 2^k + 1 \\ 2^{k+1} \leq &\quad 2m\sqrt{2} &\leq 2^{k+1} + 2. \end{aligned}$$

$2m\sqrt{2} \geq 2^{k+1} + 1$, otherwise $[2m\sqrt{2}] = 2^{k+1}$. So

$$\begin{aligned} 2^{k+1} + 1 \leq &\quad 2m\sqrt{2} &\leq 2^{k+1} + 2 \\ 2^{k+1} + 1 - \sqrt{2} \leq &\quad (2m-1)\sqrt{2} &\leq 2^{k+1} + 2 - \sqrt{2}. \end{aligned}$$

$(2m - 1)\sqrt{2} \leq 2^{k+1}$, otherwise $[(2m - 1)\sqrt{2}] = 2^{k+1}$, as $2 - \sqrt{2} \leq 1$. So

$$2^{k+1} + 1 - \sqrt{2} \leq (2m - 1)\sqrt{2} \leq 2^{k+1}.$$

We can then show by induction that $2^{k+i} + 1 - \sqrt{2} \leq (2^i m - 2^{i-1})\sqrt{2} \leq 2^{k+i}$ for all $i \geq 1$. Assume that $2^{k+t} + 1 - \sqrt{2} \leq (2^t m - 2^{t-1})\sqrt{2} \leq 2^{k+t}$. Then

$$\begin{aligned} 2^{k+t+1} + 2 - 2\sqrt{2} \leq \quad & (2^{t+1}m - 2^t)\sqrt{2} & \leq 2^{k+t+1} \\ 2^{k+t+1} + 2 - \sqrt{2} \leq \quad & (2^{t+1}m - 2^t + 1)\sqrt{2} & \leq 2^{k+t+1} + \sqrt{2}. \end{aligned}$$

As $0 < 2 - \sqrt{2} < 1$ and $[(2^{t+1}m - 2^t + 1)\sqrt{2}] \neq 2^{k+t+1}$,

$$\begin{aligned} 2^{k+t+1} + 1 \leq \quad & (2^{t+1}m - 2^t + 1)\sqrt{2} & \leq 2^{k+t+1} + \sqrt{2} \\ 2^{k+t+1} + 1 - \sqrt{2} \leq \quad & (2^{t+1}m - 2^t)\sqrt{2} & \leq 2^{k+t+1}, \end{aligned}$$

and we have finished the inductive step. Hence for all $i \geq 1$,

$$\begin{aligned} 2^{k+i} + 1 - \sqrt{2} \leq \quad & (2^i m - 2^{i-1})\sqrt{2} & \leq 2^{k+i} \\ \frac{2^{k+i} + 1 - \sqrt{2}}{2^i m - 2^{i-1}} \leq \quad & \sqrt{2} & \leq \frac{2^{k+i}}{2^i m - 2^{i-1}}. \end{aligned}$$

Now we let $i \to \infty$ and we obtain

$$\frac{2^k}{m - \frac{1}{2}} \leq \sqrt{2} \leq \frac{2^k}{m - \frac{1}{2}}.$$

Hence $\sqrt{2} = \frac{2^k}{m - \frac{1}{2}}$ which is rational, resulting in a contradiction.

— Jian He, *Paradox* Problems Editor

We have been reduced to absurdity ...

We're all familiar with the "proof by contradiction" or "proof by induction", but what about these less frequently used but equally effective proofs? Watch for them soon in a lecture near you.

Proof by convection: Lots of hot air and hand waving.

Proof by reduction: At each step, ignore some detail of the original problem. Continue the reduction process until the original problem has been reduced to something trivial, at which point, the proof is complete.

Proof by obfuscation: Generously apply Greek letters, sequences, series, partial derivatives, complex numbers, π, and e. Distance your solution from the original problem as much as possible. Once the audience looks sufficiently perplexed, cancel everything, write '= 0, Q.E.D.', and smile confidently.

Proof by trepidation: Write down something which is obviously wrong, and then loudly and angrily defy anyone to prove otherwise.

Proof by delegation: 'This proof is left as an exercise.'

Proof by truncation: Cleverly leave your proof until the end of class. When the class time is up and you have not yet finished the proof announce, 'We'll continue this proof next class' and then promptly forget about it.

Proof by intoxication: Every time you write an equal sign, take a shot of Jack Daniels. By the time you're done the proof, it makes sense to you even if nobody else gets it.

Proof by prediction: Write down what you want to prove, then predict that one day it will be proven. Whip out some tarot cards and support your claim.

Proof by post-hypnotic suggestion: Begin proof by 'You are growing verrrry sleepy ... '

Proof by condescension: Do lots of things "by inspection", use the phrase 'it is blatantly obvious that' and scoff at any questions that arise.

Proof by intervention: Somewhere in the middle of your proof a miracle occurs.

Proof by religion: Describe your proof, tell your students that they have to take it on faith, and instruct them to go home and pray to the all powerful Lambda that they, mere mortals, may be granted the insight to comprehend even a smidgen of this wisdom of the ages.

Proof by bovine excretion: Reference Spivak to support the part of your proof you have absolutely no idea how to do yourself. Use exact page numbers.

[Taken from the office window of Dr. David Dickson, Hons. Convenor, Actuarial Studies.]

Big thank-yous to ...

The following people should be given a pat on the back (if you see them walking down the corridor):

> Andrew Oppenheim (available for proof-reading, room G08, very reasonable rates), Kirsten Raynor (our agent in the Maths Office), Vanessa Teague (for not getting [too] offended), Chaitanya Rao (where would we be without him?), Cardozo, Deane J, Luke Skywalker, J. & E. Blues, Sporty Spice, Chester the Cockerspaniel, the people at Melbourne Central Station who hand out free breakfast cereal (mmm, fruity nut bran fibre flakes), Macintosh Classic — a cross between a computer and Ned Kelly's helmet, pppd and all his little daemon friends, those soft comfortable seats outside the maths library, the NetHack development team for a different/better(?) way of life, everyone who submitted a solution or has spent any time on the problems, JJJ, the *SOED*, and the trusty and ever faithful computational device that is Ickis.

paradox (ˈpærədoks), *sb. (a.).* Also 6–7 **-oxe**. [ad. (perh. through F. *paradoxe*, 14th c. in Hatz.-Darm.) L. *paradoxum*, *-on*, sb., properly neuter of *paradox-us*, Gr. *παράδοξ-ος* adj. contrary to received opinion or expectation, f. *παρά* past, beyond, contrary to + *δόξα* opinion; in Gr. and L. also used subst., esp. in pl. *παράδοξα* Stoical paradoxes: cf. Cicero *Paradoxa*, proœ m. 4. In Fr. and Eng. the sb. is earlier and more important.]

A. *sb.* **1. a.** A statement or tenet contrary to received opinion or belief: often with the implication that it is marvellous or incredible; sometimes with unfavourable connotation, as being discordant with what is held to be established truth, and hence absurd or fantastic; sometimes with favourable connotation, as a correction of vulgar error. (In actual use rare since 17th c., though often insisted upon by writers as the proper sense.)

†**b.** *Rhet.* [repr. L. *paradoxum.*] A conclusion or apodosis contrary to what the audience has been led to expect. *Obs.*

2. a. A statement or proposition which on the face of it seems self-contradictory, absurd, or at variance with common sense, though, on investigation or when explained, it may prove to be well-founded (or, according to some, though it is essentially true). *spec.* in *Literary Criticism.*

b. Often applied to a proposition or statement that is actually self-contradictory, or contradictory to reason or ascertained truth, and so, essentially absurd and false.

c. *Logic.* A statement or proposition which, from an acceptable premise and despite sound reasoning, leads to a conclusion that is against sense, logically unacceptable, or self-contradictory; freq. distinguished by name, esp. of its propounder or of the type of problem it raises. Cf. LIAR I d, *Russell's paradox* s.v. RUSSELL.

[*The Oxford English Dictionary* 2nd Ed., prepared by J. A. Simpson and E. S. C. Weiner, Oxford University Press, New York, 1989.]

'I was so excited after I read *Paradox*, that I had to have a cold shower.' *Anonymous Esso executive*

'*Paradox* clearly supports a G.S.T. [Good Solution Technique].' *John Howard*

'Last issue's jokes were so funny, I'm *still* laughing!' *Jon Faulkner*

'I refuse to read a magazine that has an article about the four colour theorem.' *Pauline Hanson*

'Great fonts, great layout, great symbols. Which version of Microsoft Word™ did you use?' *Bill Gates*, still in denial concerning our use of LaTeX.

'O.K., so it's intellectually stimulating, but what is its revenue generating capacity?' *Alan Gilbert*

'... a thoroughly arousing experience ...' *The Herald-Sun*

'The English-speaking world is divided into those who have read *The Hobbit* and *The Lord of the Rings* and those who are going to read them.' *The Sunday Times*

'I have never read *Paradox*. Not a single time.' *Bill Clinton* under oath. This claim has since been withdrawn, after intense media speculation that he had in fact read MUMS's magazine.

'We don't get it.' *Anonymous Physics Students Society member*