SMO 2005 Questions and Solutions

Questions by Zhihong Chen, Tharatorn Supasiti and James Zhao
Solutions prepared by James Zhao

1. One bottle contains one litre of a mixture that is $\frac{1}{3}$ orange juice and $\frac{2}{3}$ carrot juice. Another bottle contains two litres of a mixture that is $\frac{3}{4}$ orange juice and $\frac{1}{4}$ carrot juice. These two bottles are poured into an empty jug. Find the final ratio of orange juice to carrot juice in this jug. Express your answer in the simplest form $a : b$ where a and b are integers.

   Solution:
   The first bottle contains $\frac{1}{3}$ litres of orange juice while the second bottle contains $\frac{3}{2}$ litres, giving a total of $\frac{1}{3} + \frac{3}{2} = \frac{11}{6}$ litres of orange juice in 3 litres of drink. This means there are $\frac{7}{6}$ litres remaining for carrot juice (which is disgusting and only Zhihong would drink), giving a ratio of $11 : 7$.

2. The roots of the equation $x^2 + 4x - 5 = 0$ are also the roots of the equation $2x^3 + 9x^2 - 6x - 5 = 0$. What is the other root of the second equation?

   Solution:
   Since the cubic contains both roots of the quadratic, we can factorise it into the product of the quadratic and a linear polynomial. This gives $(x^2 + 4x - 5)(ax + b) = 2x^3 + 9x^2 - 6x - 5$, and by equating powers of x, we see that $a = 2$ and $b = 1$. Thus, the other root is the solution to $2x + 1 = 0$, or $x = -0.5$.

3. Two cars are initially 500 metres apart. They speed towards each other, one at 30m/s and the other at 20m/s. How far apart, in metres, are they one second before crashing?

   Solution:
   In one second, one car moves 30m closer to the point of impact, and the other car moves 20m closer from the opposite direction, so one second before impact, they must be $20 + 30 = 50$ metres apart.

4. An island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. Andrew and James are inhabitants. Andrew says: 'James is a knave.' James says: 'Neither Andrew nor I are knaves.' List, respectively, what Andrew and James are.

   Solution:
   Use trial and error. If Andrew is a Knave, then he is lying so James is a

Knight, which means he is telling the truth and Andrew is also a Knight. This is impossible, so Andrew is a Knight, and is telling the truth, so James is a Knave.

5. A square box of side 5cm is leaning against a vertical wall with the bottom corner of the square 4cm from the wall. Find the height of the highest corner of the square in centimetres.

   Solution:
   Note that one corner of the square forms a 3-4-5 right-angled triangle with the wall and the floor, which means that the top corner of the square and the corner touching the wall also form two corners of a 3-4-5 triangle with the side of length 4 aligned vertically. The bottom of this side is in line with the corner touching the wall, which is 3cm above the ground, so the top corner is 7cm above the ground.

6. Nick's watch is 10 minutes fast but he thinks it is 5 minutes slow. Michael's watch is 5 minutes slow but he thinks it is 10 minutes fast. Thara's watch is 5 minutes fast but he thinks it is 10 minutes slow. Han's watch is 10 minutes slow but he thinks it is 10 minutes fast. Using their watches, each of them leaves for university at what each believes is the time to catch the 8:30am tram from Flinders Street. Who will miss their tram?

   Solution:
   Nick will arrive when his watch says 8.25, but it will actually be 8.15. Michael will arrive at 8.40 by his time, which is 8.45. Thara will arrive at 8.20 by his time, which is actually 8.15. Han will arrive at 8.40 by his time, which is actually 8.50. Hence Michael and Han will miss their train.

7. For integers a and b we define $a * b = a^b + b^a$. If $2 * x = 177$, find x.

   Solution:
   Since the 2^x term grows much faster than the x^2 term, we can get a rough estimate but finding $2^x < 177$, for which the largest solution is $x = 7$. This happens to work out immediately, but any method of guessing should yield the answer reasonably quickly.

8. I have five matchsticks which have lengths 2, 3, 4, 5 and 6 cm respectively. How many different triangles can I make?

   Solution:
   The difference between the lengths of the two longest sides must be less than the length of the shortest side. If the shortest side is 2cm, then the other sides must differ by 1cm, so there are 3 choices, 3-4, 4-5 or 5-6. If the shortest side is 3cm, the other sides can differ by 1cm or 2cm, so the choices are 4-5, 4-6 or 5-6. Finally, there is the 4-5-6 triangle, giving 7 possible triangles.

9. Maurice has a standard die with six sides. Julian has a four-sided die with the numbers 0, 2, 4, 6 printed on each of the four sides. Maurice and Julian roll their own die. What is the probability that Julian rolls a higher number than Maurice? Express your answer as a fraction in the simplest form.

   Solution:
   There are $4 \times 6 = 24$ possible outcomes, each with equal probability. If Julian rolls 0, then he can't roll a higher number, while if he rolls some other number n, there are $n - 1$ ways to get a higher number. Hence there are $1 + 3 + 5 = 9$ ways out of 24 for Julian to roll higher. Thus, the probability is $\frac{9}{24} = \frac{3}{8}$.

10. A ball with radius 17cm is floating so that the top of the ball is 2cm above the smooth surface of a swimming pool. What is the circumference, in centimetres, of the circle formed by the contact of the water surface with the ball? Give exact answers.

    Solution:
    The centre of the circle, the centre of the ball and a point on the edge of the circle form a right angled triangle with hypoteneuse 17cm and sides 15cm and r where r is the radius of the circle. Using Pythagoras, $r = \sqrt{17^2 - 15^2} = 8$, so the circumference is given by $2\pi r = 16\pi$.

11. In a cubic room, a spider is at one of the corners in the ceiling. A fly is on the floor and at the midpoint of an edge that is on the opposite side of the room to where the spider is. If the shortest distance that the spider has to crawl in order to reach the fly is $\sqrt{13}$ metres, what is the sidelength of the room in metres?

    Solution:
    The trick is to see that the shortest distance is not to crawl along the ceiling then crawl down, but to crawl along two walls. The spider travels one side length vertically, and $\frac{3}{2}$ side lengths horizontally, so the shortest distance is given by Pythagoras as $\sqrt{x^2 + (\frac{3}{2}x)^2} = \frac{\sqrt{13}}{2}x$. This equals $\sqrt{13}$, since we're told what that distance is, so the side length of the room is given by $x = 2$.

12. Let x and y be real numbers such that:

    $$x^2 + 2y^2 - 2xy + 2y + 1 = 0$$

    What is x?

    Solution:
    Using the quadratic equation, for a real solution to exists for y, $(-2x + 2)^2 - 4 \times 2 \times (1 + x^2) \geq 0$. Simplifying this gives $-4x^2 - 8x - 4 \geq 0$, or $(x + 1)^2 \leq 0$, but a perfect square cannot be negative, hence $(x + 1)^2 = 0$ and $x = -1$.

13. Determine the smallest positive integer n such that $n^3 + 2n^2 = b$ where b is a square of an odd integer.

Solution:
We want $n^2(n+2)$ to be the square of an odd integer, so n must be odd and $n+2$ must be the square of an odd integer. Looking at the squares of odd integers, 1 obviously does not work, while the next one, 9, does. Hence $n = 7$.

14. Let a, b and c be the digits of a three-digit number satisfying the equation $49a + 7b + c = 286$. What is $a + b + c$?

Solution:
This is simply asking for the sum of the digits of 286 in base 7. a is the largest multiple of 49 less than or equal to 286, or 5, which leaves a remainder of 41. b is the largest multiple of 7 less than or equal to 41, which is 5, leaving $c = 6$, so $a + b + c = 16$.

15. What is the smallest integer greater than zero that is divisible by 225 and, when written in decimal representation, contains only zeros and threes as digits?

Solution:
To get rid of 5 in the last digit, it must be an even multiple of 225, so the number is a multiple of 450. Then again, to get rid of the 5 in the second last digit, it must again be an even multiple of 450, and hence is a multiple of 900, which is equivalent to saying the sum of its digits is a multiple of 9 and the last two digits are 0. We see that if all the digits are 3 or 0, the smallest number which works is 33300.

16. Two equilateral triangles of sidelength 1 are placed side by side to form a rhombus. Let the diagonals of this rhombus intersect at O. If the rhombus is rotated about O, what is the area which is always within the rhombus? Give exact answers.

Solution:
This area is given by a circle inscribed in the rhombus with centre O and radius r. Note that the line segment from O to a point of tangency is perpendicular to an edge of one of the triangles, and since that edge makes an angle of 60 degrees with the shorter diagonal, the radial line segment and the shorter diagonal intersect at an angle of 30 degrees. This gives $r = \frac{1}{2}\cos 30^\circ = \frac{\sqrt{3}}{4}$, so the area is $\pi r^2 = \frac{3\pi}{16}$.

17. Joanna and her grandmother have the same birthday. It is found that for six consecutive years, Joanna's age is a divisor of her grandmother's age. What is the age difference between Joanna and her grandmother in years?

Solution:

It's a lot easier to do these questions by trial and error. Since it's easier to be divisible by smaller numbers, make Joanna as young as possible, that is, starting from age 1. Call the age difference x. Then, $x+n$ is divisible by n for n between 1 and 6, which means x is divisible by the numbers from 1 to 6, which is the same as being divisible by 60. If we want to be sensible about ages, then the age difference is exactly 60.

18. How many positive integers less than one million have all their digits different?

    Solution:
    Most questions have a nice solution, but sometimes it's easier to just bash it out. Consider a n-digit positive integer with all its digits different. There are 9 choices for the first digit, since it can't be 0, then 9 choices for the second digit, 8 for the third, 7 for the fourth and so on until n digits have been chosen. Since this number has between 1 and 5 digits, the answer is $9+9\times 9+9\times 9\times 8+9\times 9\times 8\times 7+9\times 9\times 8\times 7\times 6 = 168570$.

19. Let $x_1 = 1$, and for $n \geq 1$, $x_{n+1} = \begin{cases} x_n^2 + n & n \text{ even} \\ \lfloor \sqrt{x_n} \rfloor & n \text{ odd} \end{cases}$
    What is x_{2005}?
    Note that $\lfloor x \rfloor$ means the largest integer less than or equal to x.

    Solution:
    Writing out a few terms, we see that $x_1 = 1$, $x_2 = 1$, $x_3 = 3$, $x_4 = 1$, $x_5 = 5$, $x_6 = 2$, $x_7 = 10$, $x_8 = 3$, $x_9 = 17$, $x_{10} = 4$, we see that from x_4 onwards, the odd terms are one more than a perfect square, while the even terms increment by 1. This is easy to prove by induction, and not much harder to form a motivated proof from the identity $(n+1)^2 = n^2+2n+1$, but here, all that is needed is the answer. Following the pattern, $x_{2004} = 1001$, so $x_{2005} = 1004005$.

20. Sally began collecting calendars in 2005 and will do so every year until the time when every subsequent year can be served by at least one of the calendars that she has already collected. When is the last year in which she must collect a calendar?

    Solution:
    It is reasonable to expect that the last year will be a leap year, since leap year calendars are 3 times rarer than standard year calendars. There are 7 possible calendars for leap years, so she will need at least 7 leap years worth of calendars. The seventh leap year after 2005 is 2032, which is the correct answer. It can be proven using either direct calculation or group theory, but why bother? Just guess!

21. A group of children share marbles from a bag. The first child takes one marble and a tenth of the remainder. The second child takes two marbles and a tenth of the remainder. The third child takes three marbles and

a tenth of the remainder. And so on until the last child takes whatever is left. Knowing that all the children end up with the same number of marbles, how many children were there?

Solution:
Each child takes a tenth of the marbles as their final move, so at the start of each move, the number of marbles in the bag is nine-tenths of something, and hence a multiple of 9. Since this does not necessarily apply for the first child, take the second child, which starts with a multiple of 9, takes 2 away and ends up with a multiple of 10, so a reasonable guess might be 72. The third child then has 63, the fourth 54, and so on down the multiples of 9, until the 9th child has 9 marbles to take. Note that each child has exactly 9 marbles, so 9 children is the answer.

22. Find the smallest n such that if $10^n = X \times Y$, then at least one of X or Y must contain the digit 0.

Solution:
The only way to have neither X and Y contain the digit 0 is by having $X = 2^n$, $Y = 5^n$, or vice-versa, since any number containing both a power of 2 and a power of 5 must end with a 0. The powers of 2 are 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024 so the smallest n for which the power of 2 must contain a 0 is 10. On the other hand, the powers of 5 are 5, 25, 125, 625, 3125, 15625, 78125, 390625, so $n = 8$ guarantees the digit 0 in either X or Y.

23. Let $ABCD$ be a convex quadrilateral such that $AB = 8$, $BC = 6$ and $BD = 10$. Also, $\angle BAD = \angle CDA$ and $\angle ABD = \angle BCD$. What is CD?

Solution:
Let E be the intersection of BC and AD extended. With the equal angles given, we see that $\triangle CED \simeq \triangle BDA$, so $\angle BED = \angle BDA$ and $BE = BD = 10$. This gives $CE = 16$, and from the similarity above, $\frac{CD}{CE} = \frac{BA}{BD}$ which gives $CD = \frac{16 \times 8}{10} = \frac{64}{5}$.

24. Maurice and Geordie arrive at a cafe independently at random times between 9am and 10am and each stay for m minutes. What is m if there is a 51% chance that they are in the cafe together at some moment.

Solution:
Plot the time Maurice arrives on the x-axis, and the time Geordie arrives on the y-axis, both from 0 to 60 representing the time in minutes after 9am they each arrive. With a little understanding of probability, or a little imagination, one can see that the probability of any event is equal to the proportion of the area of this square which corresponds to that event. For Geordie and Maurice to meet, they must arrive within m minutes of each other, so this is the area $|x - y| \leq m$, which is the square with two triangles cut out, each with base and height both $60 - m$, and hence the valid area is $60^2 - (60 - m)^2 = 120m - m^2$. Then,

$\frac{120m-m^2}{3600} = \frac{51}{100} \Rightarrow m^2 - 120m + 1836 = 0$; solving the quadratic gives $m = 18$ or $m = 102$, so the answer is 18 minutes since $m > 60$ gives a 100% chance of meeting.

25. Triangle ABC has $AB = AC$ and $\angle BAC \leq 90$. P lies on AC, and Q lies on AB such that $AP = PQ = QB = BC$. Find ratio of $\angle ACB$ and $\angle APQ$.

Solution:
Let $\angle BAC = x$ and $AP = 1$. Then, $AB = 1 + 2\cos x$, $2AB \sin\frac{x}{2} = 1$, so $(1 + 2\cos x)\sin\frac{x}{2} = \frac{1}{2}$.

$$\begin{aligned} 2\cos x \sin\frac{x}{2} &= \cos x \sin\frac{x}{2} + (2\cos^2\frac{x}{2} - 1)\sin\frac{x}{2} \\ &= \cos x \sin\frac{x}{2} + \sin x \cos\frac{x}{2} - \sin\frac{x}{2} \\ &= \sin\frac{3x}{2} - \sin\frac{x}{2} \end{aligned}$$

Therefore, $\sin\frac{3x}{2} = \frac{1}{2}$, so $x = 20°$. Then $\angle ACB = 80°$, $\angle APQ = 140°$, so the ratio is $\frac{7}{4}$.