Paradox

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THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY

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Paradox

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Words from the Editor

Welcome to this year's final issue of Paradox. This issue is a bit unusual, as it contains an article on chess by our very own expert, Sam Chow. There is more material written on chess than any other games/sports/recreations, but Sam shows how a little mathematical knowledge can improve one's game. He goes on to discuss more about general strategies.

In this issue, we also have a review on some maths and physics books from our Sydney correspondent. Our regular columnist, Yi Huang, writes about the fascinating properties of the Catalan numbers.

If you are studying/teaching maths, does that mean you are, or will become, a mathematician? Take our test to find out.

And finally, there is an article about a numerical identity which at first sight seems to be the most evil conspiracy between algebra and geometry, if there is indeed such a thing.

The maths society has the need for a new T-shirt, and the design competition is open. See page 4 and 5 for more details, including the fantastic prizes to be won. Enjoy reading.

— James Wan

Words from the President

Dear reader,

As the year draws to a close, I would like to extend a welcome to the last issue of Paradox for 2007. Earlier this semester, we ran a successful Maths Olympics (UMO). For those who missed it, the questions are up on our site, though it's really no substitute for experiencing them first-hand as a competitor. I hope you'll remember to participate when the UMO comes around again next year. For what remains of the semester, we will be running our usual seminars, as well as the end of semester Trivia Night (which, as always, is free), so keep an eye out for those.

The MUMS committee is also working hard to prepare for next year's Puzzle Hunt, scheduled to take place early semester one, 2008. The Puzzle Hunt is our biggest annual event, attracting participants from all over the world, and you can do it all from the comfort of your computer chair, so don't miss it!

As always, you can check for these events on posters around the maths building, and on our website.

— Alisa Sedghifar

To give an indication of the standard of the UMO, here is a sample question:
A right-triangle has perimeter 14 and hypotenuse 6. What is its area?

Solution:
You can solve this by simultaneous quadratic equations, but this approach is time-costly. A better way is to let a and b be the two shorter sides. Then $a + b = 8$, $a^2 + b^2 = 36$. Squaring the first equation, we get $a^2 + b^2 + 2ab = 64$, and subtracting the second equation from it, we get $2ab = 28$. Hence the area, being $\frac{1}{2}ab$, is 7.

The MUMS T-shirt Competition!

You may have seen the official MUMS T-shirt at our regular events, such as the UMO. The back of the T-shirt is a cartoon of a man walking up to Heaven, the only thing barring his entry is an incredibly difficult maths problem. However, we have had the same design for many years, and it's time that we get a new one.

So we need your help! If you have any ideas for a T-shirt, please send it to us, either to the MUMS room (G06 in the Richard Berry building) or email it to the editor. The design for the shirt should signify our mutual bond as students of maths (or something like that, so if it's maths-related then it'll do), hopefully be humorous, and not offend the general public too much, or the wearer. For example, a shirt that says, "Punch me, I'm a maths geek!" won't be accepted.

To give you some ideas, we describe the designs of two legendary T-shirts. The first allegedly comes from the University of Sydney, and it features the slogan, "$\frac{22}{7}$, say π and die". The second comes from the University of Western Australia, and has printed on it, "Lipschitz Happens", followed by the Lipschitz condition, $|f(x) - f(a)| < c|x - a|$. Surely we can do better than that!

The **prize** for submitting the winning design is a free T-shirt with your design on it, a $30 book voucher, quality chocolate and wine.

$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{x^2+1} \mathrm{d}x$. In fact, if 1 is substituted for x in the denominator, one gets a lower bound on the integral; and 0 gives an upper bound ($\frac{1}{630}$). Hence $\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$.

Puzzle 1: Which question is its own answer?

What famous people have said about Paradox (actual quotes)...

"But human experience is usually paradoxical, that means incongruous with the phrases of current talk or even current philosophy." — George Eliot (1819 - 1880), English novelist

"Play not with paradoxes. That caustic which you handle in order to scorch others may happen to sear your own fingers and make them dead to the quality of things." — George Eliot

"The ocean is a place of paradoxes." — Rachel Carson (1907 - 1964), US biologist, writer

"We live in an age of prejudice, dissimulation and paradox, wherein, like dry leaves caught in a whirlpool, some of us are tossed helpless...ever struggling between our honest convictions and fear of that cruelest of tyrants." — Helena Petrova Blavatsky, (1831 - 1891) Russian author, translator, theosophist

"Two paradoxes are better than one; they may even suggest a solution." — Edward Teller (1908 - 2003)

"How wonderful that we have met with a paradox. Now we have some hope of making progress." — Niels Bohr (1885 - 1962)

"I learned to make my mind large, as the universe is large, so that there is room for paradoxes." — Maxine Hong Kingston (1940 -), US author

Should I Swap?[1]

Material and position are the two key considerations when playing chess; for most casual plays, material is the more important of the two. In fact, early chess programs, using a sound system of estimating the numerical values of pieces, were able to beat masters by almost material consideration alone. Humans too face the same task of evaluating the pieces, which is fundamental when one asks, 'should I swap'?

It's not just beginners who ask this. Many people play chess for years knowing little more about this than what they were taught as novices.

One of my friends the other day gave up his rook for a bishop and a pawn, then turned to me and said, "You see that, Sam? How awesome was that!?"

I couldn't believe that dude's ignorance, and told him a rook is way better than a bishop.

He argued, "I thought they were the same! They both move the same way, except one's diagonal..."

As is always the case in chess, there are many, many other factors to be taken into consideration. However, it is intuitive that a rook is better than a bishop.

Take an empty board, and place a rook on it:

			×				
			×				
			×				
			×				
×	×	×	R	×	×	×	×
			×				
			×				
			×				

Notice that wherever you place the rook, it attacks 14 squares.

Try the same thing with a bishop:

[1] The author, Sam Chow, was the Victorian junior chess champion, Australian under 16 chess champion, and currently is the best chess player in Australia aged under 20.

×						×	
	×				×		
		×		×			
			B				
		×		×			
	×				×		
×						×	
							×

×							
	×						
		×					
			×				
				×			
					×		
						×	
							B

The bishop's influence depends on its position: a bishop in the corner attacks 7 squares, while a bishop in the centre controls 13. The number of squares controlled by a bishop positioned on a particular square is summarised below:

7	7	7	7	7	7	7	7
7	9	9	9	9	9	9	7
7	9	11	11	11	11	9	7
7	9	11	13	13	11	9	7
7	9	11	13	13	11	9	7
7	9	11	11	11	11	9	7
7	9	9	9	9	9	9	7
7	7	7	7	7	7	7	7

On average, the bishop controls only $\frac{1}{64}(7\times28+9\times20+11\times12+13\times4) = 8.75$ squares. Compare this to 14 for a rook. Notice that $\frac{14}{8.75} \approx \frac{5}{3}$, the "traditional" relative value (see below) of a rook compared to a bishop.

However, such a method of determining piece values is surely too simplistic. A queen can move like a rook or a bishop, yet it becomes clear (from play) that a queen is far more valuable than a rook and a bishop together.

Most players learn the "traditional" piece values early on, which are:[2]

- Pawn (P) 1
- Knight (N) 3
- Bishop (B) 3
- Rook (R) 5
- Queen (Q) 9

[2] http://www.chess-theory.com/enthcct04_value_pieces_chess_learn_free_lesson.php

The King (K) can be assigned ∞. The first thing you might notice is that they're all integers. There you go, you've spotted the first problem – this 'rule of thumb' is too simplistic. That's not to say it shouldn't be taught to beginners – it should! The table provides a simple and memorable guide, and the players are merely told to 'take other factors into consideration'.

One ubiquitous error is the trading of two minor pieces (knights and bishops) for a rook and a pawn. I for one took some time to shake this habit. According to the above points system, R+P = B+N; however, the model is not quite linear and I learnt the hard way that B+N > R+P, especially earlier in the game.

The reason a rook is generally not worth $\frac{5}{3}$ of a bishop is because it is often very hard to find good files ('columns') for your rooks until the endgame, whereas bishops can usually become active earlier. Knights are perhaps the quickest pieces to get active, which is why players who trade their bishops for knights in the opening often gain a temporary advantage.

It follows that minor pieces are stronger in the opening and middlegame, and *depreciate* in value as the game progresses; the rooks are stronger in the endgame, hence *appreciate* in value as the game progresses. An extreme example is that K+R easily forces checkmate against K (with no other pieces on the board), while it is impossible to FORCE checkmate with K+N+N vs. K.

Pawns too appreciate in value. The reason for this is simple: pawns are much more likely to promote in the endgame. An astute coach once said that almost all decisive (where one side wins) games would involve a promotion, were neither side to resign. From experience, he was right.

The practical implications of this are now obvious: if you have a rook and 'some pawns' (usually 0, 1 or 2) against two minor pieces in the middlegame, trading pieces (in general) works in your favour! If you're on the other side of this material imbalance, you want to keep pieces on the board and blow your opponent away with your active army.

Still, even if we were to somehow 'average' the piece values over the life of the game, and over different games, the traditional values are still oversimplified.

A more realistic conclusion is reached by International Master Larry Kaufman.[3] Scrupulously gathering statistics from high level games, Kaufman arrives at the following values:

[3] http://mywebpages.comcast.net/danheisman/Articles/evaluation_of_material_imbalance.htm

- Pawn 1
- Knight 3.25
- Bishop 3.25
- Rook 5
- Queen 9.75
- Bishop pair +0.5

The final entry warrants some explanation. It has long been known that having two bishops (presumably they are on different-coloured squares, like the initial two bishops, and not artificially produced to be on the same colour through promotion) against other minor piece combinations confers some long-term advantage.

Some are willing to buy into the explanation that the bishops ‘complement’ one another, by being able to cover squares of both colour (as one bishop must always stay on the same coloured squares). Personally, however, I think the two bishops will eventually have a greater influence on the board than other minor piece combinations.

In fact, Kaufman’s statistics suggest that the bishop pair is worth an extra half a pawn – in higher level games at least. It is worth mentioning that beginners prefer knights because they are ‘trickier’ – and this idea is not without merit in play.

Another consideration regarding bishops is when you have one bishop on one colour, and your opponent has one bishop on the other colour: a phenomenon known as ‘opposite-coloured bishops’. The bishops cannot attack each other, and are therefore difficult to swap off.

Without going into too much detail, opposite-coloured bishops favour the attacker in the middlegame (because the defender’s bishop can often do little to defend), and the disadvantaged side in the endgame (they tend to make endings more drawish). One practical implication is that you have to consider when to allow or strive for a position with opposite-coloured bishops, and when you should avoid it.

Now might be a good time to touch upon general swapping. Many beginners lose time early in the game enforcing swaps which confer no real advantage to either side. Obviously, a seasoned opponent will make good use of the spare moves to gain an advantage. But let's consider a more difficult problem...

Say your opponent gives you the opportunity to swap your queen for his queen. Suppose also that neither of these queens is especially strong or weak. Should you swap?

If one side has more material, the answer will usually be clear: you should swap the queens if you are ahead on material, and you should avoid swapping the queens if you are behind on material. This is because it is easier to realise a material advantage when there are less pieces on the board; with queens on the board, this task is substantially more complicated. Mathematically, say your pieces were worth a total of 28 points, and your opponents 26. Before the swap, your material was very roughly speaking $\frac{28}{26} \approx 1.08$ times that of your opponent, and after the swap you are $\frac{19}{17} \approx 1.12$ times better. Of course, if you have a material advantage and you're launching a decisive attack, you may want to keep your queen on and just finish your opponent off!

Also, with a material advantage, you want to keep pawns on the board, to give you more opportunities for promotion later (unless you're playing against your boss).

So the adage here is "swap the pieces, not the pawns", and it applies when you have a material advantage.

However, even this notion has complicated exceptions. Suppose you're a pawn ahead. Suppose also that both sides have two rooks and two minor pieces. Do you swap the minor pieces, or do you keep them on the board to help you to swap off the rooks?

Rather than just swapping blindly, you should keep the minor pieces on to swap off the rooks! The reason is that endings with rooks in them tend to be more drawish, so it will be easier to convert your material advantage if you swap off the rooks and go into a minor piece ending. Then, with only minor pieces on the board, swapping these off will make your task even easier.

The 20th century saw one particular 'rule of thumb' come into question: "swap pieces if you have less space". When I first came across this idea, I thought it made perfect sense. Having less space, you would want to swap off pieces so your position would be less cramped. Having more space, you would not want to swap off pieces because then you would not be able to control all your space, and gaps in your position would emerge and be exploited. And so had been the trend at top level play in the 19th and early 20th centuries.

However, the result was a rigid pawn structure and few pieces on the board for the player with less space. For many years, the disadvantaged player would dogmatically abide by the 'rule', while his opponent executed his plans without disturbance and achieved consistently strong results.

Following the wisdom of influential players such as Nimzowitsch, however, the game began to take on an entirely different nature.[4] Contrary to the traditional understanding of the game, the side with less space began to keep pieces on the board! This, combined with a fluid pawn structure, allowed them to constantly threaten to 'break out' with well-prepared pawn breaks, which would ideally unleash their pieces and overwhelm their space-hoarding opponent.

The notion of *dynamism* had emerged, and players such as (former world champion) Garry Kasparov began to win from all sorts of positions (with black pieces) that their ancestors would have rejected without a moment's thought. One can only imagine their reaction if they were to see today's top players seeking to keeping pieces on the board when they have less space!

In summary, it is useful to have some numerical sense of piece valuation. It helps also to be aware of the many factors that can affect whether or not it is beneficial to enforce or allow a particular exchange, or series of exchanges.

However, nothing will aid you more than experience. The experienced player will 'look ahead' to assess the long-term implications of an exchange or series of exchanges.

So next time you ask me whether or not you should swap, expect nothing more than "Dude, it's obvious."

— Sam Chow

Puzzle 2: Find an integer polynomial $P(x)$ that gives primes values for all integers x.

Solution to puzzle 1: that question in the puzzle is also the answer.

[4] John Watson, Chess Strategy in Action (2003)

Maths Jokes

Q: What's a dilemma?

A: A lemma that produces two results.

∞

A topologist is a man who doesn't know the difference between a coffee cup and a doughnut.

∞

An evil psychologist put an engineer, a physicist, and a topologist in an experiment: each of them is locked in a room for a day, with a can of food, a pencil and paper.

At the end of the day, the psychologist opens the engineer's room first. Pencil and paper are unused, but the walls of the room are covered with dents. The engineer is eating from the open can: he threw it against the walls until it cracked open.

The physicist is next. The paper is covered with formulae, there is one dent in the wall, and the physicist is eating: he calculated how exactly to throw the can against the wall.

When the psychologist opens the topologist's room, the paper is also full of formulae, the can is still closed, and the mathematician has disappeared. But there are strange noises coming from inside the can.

He gets an opener and opens the can. The topologist crawls out. "Damn! I got a sign wrong..."

∞

When considering the behaviour of a howitzer:

- A mathematician will be able to calculate where the shell will land
- A physicist will be able to explain how the shell gets there
- An engineer will stand there and try to catch it

∞

According to a recent poll, 51% of all Americans are in the majority.

∞

Q: What is the difference between numbers and women?

A: For numbers, they are rational if they have a period.

∞

All generalisations are false.

∞

The law of the excluded middle either rules or does not rule.

∞

Zenophobia is the irrational fear of convergent sequences.

Settlers of Catalan

First, an aside.

Contrary to what the title might suggest, this isn't an article about the noble peoples of Catalonia, nor is it about the number 16. Every article has to start somewhere, and it just so happens that this one begins with a rant about the number 16. Did you know that 16 is the only number expressible as m^n and n^m using distinct natural numbers n and m? Moreover, 16 is the smallest positive integer with exactly five positive divisors, the smallest square whose reverse is a prime, and the smallest number that is the sum of two distinct odd primes in two ways. In light of these awesome properties, it is of little wonder that I have been inspired to write the following questions – each making use of the number 16.

- How many ways are there to write 16 left brackets and 16 right brackets in a row such that the brackets match up correctly? (E.g.'()((())())' is correctly matched, whilst ')(()))((' is not.)

- Given 32 points drawn on a circle, how many ways are there of joining up all of these points with 16 non-intersecting chords?
- On a 16×16 grid, how many ways are there of going from the bottom left corner $(1, 1)$ to the top right corner $(16, 16)$, given that you're only allowed to go either up or right and you can't go above the diagonal line stretching from $(1, 1)$ to $(16, 16)$?
- How many 16-term non-decreasing integer sequences are there that start with 1, and have the property that the second term is at most 2, the third term is at most 3, etc?

Well, some simple sixteen second calculations should reveal that the answer to all of the above questions is 35357670 – the 16th Catalan number.

What are Catalan numbers?

Stigler's law of eponymy asserts that "No scientific discovery is named after its original discoverer." Hence we may rest assured that the Catalan numbers were not first discovered by Eugène Charles Catalan. Sources vary as to the origins of the sequence. It is generally claimed that Euler was the first to describe the sequence, although it is known that others such as Segner had already investigated the numbers before Euler.

Without further ado, the nth Catalan number (for $n \geq 0$) is given by:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

And just to give you a feel for the sequence, here are the first 16 terms (not including C_0): 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845 and 35357670 – which we've all seen before.

The property

One of the most important properties of the Catalan numbers is that they satisfy the following recursion for $n \geq 0$:

$$C_0 = 1, C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Indeed, it is this very recursion that results in the Catalan numbers being the solution to the above problems.

Consider the first of the four questions. We will prove by induction that when we're given n pairs of brackets, there will be C_n ways to arrange them in a row, so that the brackets match up correctly. Notice that when we have no brackets, there is only one way to arrange these 0 pairs of brackets, so we have the base case for our induction proof. Let's assume that it's all true up to some n. For $n+1$ pairs of brackets arranged so that they match, the highlighted leftmost bracket and its corresponding closing bracket separate all the other n pairs of brackets into two strings of brackets (0 strings are permitted) – each string with brackets correctly matched. The following is an example for $n = 6$:

(()(())**)**()()()

If there are k pairs of brackets in the left string, then there will be $n-k$ brackets in the right string. Therefore, there will be $C_k \times C_{n-k}$ ways to have k pairs of brackets in the left string. Which mean that the total number of ways of arranging $n+1$ pairs of brackets in a row is:

$$\sum_{i=0}^{n} C_i C_{n-i}.$$

And this is the formula for C_{n+1}, which completes our induction proof. The careful and astute reader may notice that we've skipped over a simple but important step showing that we haven't counted the same arrangement twice – an exercise I leave to the careful reader.

All of the above is well and good, but how do we know that the recursion actually gives the Catalan numbers? There are many ways to prove this; for example, students who have done 620-374 Discrete Maths will have seen the generating function proof. We will use a different method.

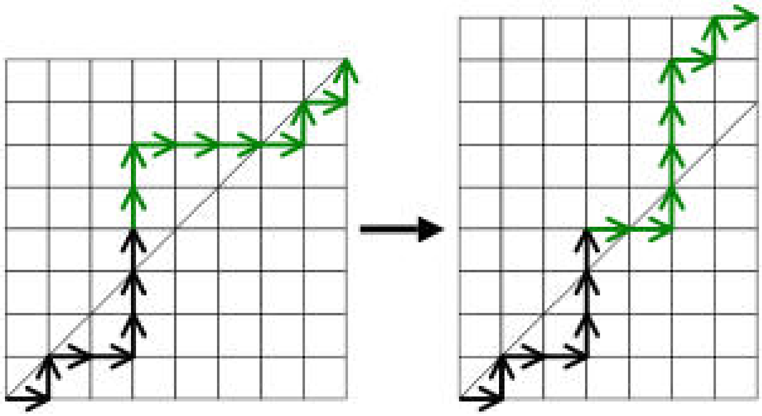
The proof

Let us turn to the third question that I posed at the beginning. We call a path that only goes to the right or up "monotonic". It can be shown that the Catalan number recursion is satisfied by the number of monotonic paths getting from $(1,1)$ to (n,n), with the additional constraint of not crossing the main diagonal. We will look at an alternative method of computing this number, but to do so we shall need the following result:

Given a $n \times m$ grid, there are $\binom{n+m}{n} = \binom{n+m}{m}$ monotonic paths going from one corner to the other.

We can prove this by noticing that each monotonic path can be uniquely represented by a sequence of n r's and m u's, respectively denoting 'rights' and 'ups'. Conversely, any line of n r's and m u's will give a monotonic path from $(1, 1)$ to (n, m). Therefore, using baby maths, we see that the total number of monotonic paths is just the number of ways to arrange n r's and m u's in a row, hence the above result.

Now comes the actual proof. Consider an $n \times n$ grid, and some monotonic path that goes above the diagonal. If we take the part of the path after it first goes above the main diagonal, and flip it along the diagonal, then we have a monotonic path going from $(1, 1)$ to $(n - 1, n + 1)$.



Notice that each monotonic path generated this way is unique. Moreover, every monotonic path going from $(1, 1)$ to $(n-1, n+1)$ must cross the diagonal from $(1, 1)$ to $(n - 1, n - 1)$ at some point, and by doing the opposite flip we'll get a monotonic path on an $n \times n$ grid which goes above the diagonal. Therefore, the total number of monotonic paths from $(1, 1)$ to (n, n) that don't go above the main diagonal is simply the number of monotonic paths from $(1, 1)$ to (n, n) minus the number of monotonic paths from $(1, 1)$ to $(n-1, n+1)$. That is:

$$C_n = \binom{n+n}{n} - \binom{n-1+n+1}{n-1} = \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

Concluding tidbits

Catalan numbers do occasionally make guest appearances in counting problems, in areas such as discrete mathematics or computer science, but mostly they're just studied recreationally. If you've worked out (and proved) the solutions to all of the questions I asked at the start, perhaps you might be interested in the following:

- How many pairs of non-intersecting $n+1$-step long monotonic paths are there, that start from $(1,1)$ and end at the same point?
- How many ways are there of stacking cylindrical wooden logs so that the bottom row consists of n consecutive logs lying side by side?
- How many n-term long sequences starting with 0 have the property that $0 \leq a_{i+1} \leq a_i + 1$?

— Yi Huang

Awesome quotes:

John von Neumann: "Young man, in mathematics you don't understand things, you just get used to them."

Aaron Levenstein: "Statistics are like a bikini - what they reveal is suggestive, but what they conceal is vital."

George Bernard Shaw: "Statistics show that of those who contract the habit of eating, very few survive."

Edward Bulwer-Lytton (1803-1873): "Fate laughs at probabilities."

Joe Mattis: "Fast cars, fast women, fast algorithms... what more could a man want?"

Richard Hamming: "The purpose of computing is insight, not numbers."

Leo Tolstoy: "A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction."

Useless number crunch:

$12^2 + 33^2 = 1233, 88^2 + 33^2 = 8833$

$153 = 1^3 + 5^3 + 3^3$ (the smallest non-trivial narcissistic number, i.e. an n-digit number that is the sum of the nth powers of its digits)

$2646798 = 2^1 + 6^2 + 4^3 + 6^4 + 7^5 + 9^6 + 8^7$

$111111111^2 = 12345678987654321$

$12345679 \times 8 = 98765432 \times 1$ (because $12345679 = 11111111 \div 9 = 111111111 - 98765432$)

Law student:	I can get a good job by studying law.
Maths student:	And why do you want a good job?
Law student:	I will earn a lot of money with a good job.
Maths student:	And why do want to earn a lot of money?
Law student:	I can buy a car and a house, marry a woman and start a family.
Maths student:	And why do want all those things?
Law student:	They make me happy.
Math student:	I study mathematics and by doing so I reach the same in one step, as doing mathematics makes me happy directly.

Test: are you a Mathematician?

Add up your score for these questions (you score when you answer 'yes').

- (2) You make mistakes... but they are really interesting mistakes.
- (1) You wonder how Euler pronounced "Euclid".
- (7) You understand all the mathematics Gauss produced... through age 15.
- (2) You know all of the Greek alphabet, but not a word of Greek.
- (3) You visit Earth primarily for lectures and family obligations.
- (4) You think the Cantor Set is beautiful.
- (4) Your retirement plans include solving the twin prime conjecture.
- (2) You spend time helping people you don't know with their homework.

- (1) You know what an Erdös number is.
- (5) You have an Erdös number.
- (3) You write emails in LaTeX.
- (6) When being interrogated as a suspect by the police, and asked to prove your innocence, you started off by saying "assume instead that I'm guilty...", at which point you were promptly arrested.
- (1.02) You try to find a pattern in the scoring scheme of this test.
- (3) You answer "Do you know the time?" with "Yes".
- (3) You wear a jumper with (an approximation to) the Mandelbrot Set on it.
- (3) You say "there exists" in lieu of "there is".
- (e^5) You have a theorem named after you.
- (1) You have seen a version of this test somewhere on the Internet.
- (1) You use the word "trivial" often.
- (2) You have strong views on the least natural number.
- (3) You know a few ways to rigorously construct the real numbers.
- (3) You write mathematical articles for Wikipedia.
- (4) You know what Tits groups are.
- ($-\infty$) You have found a pattern in the scoring scheme of this test.
- (3) You are unable to read your handwriting.
- (4) You are either hopelessly incompetent or frighteningly competent at mental arithmetic.
- (4) You know what to do with Napier's bones.
- (5) You have done something merely to improve your score in this test.

If you get 40 or higher on this test, then you might just consider yourself a mathematician, or will probably become one. What the community thinks of you is a very different matter.

A Curious Identity

It may surprise some people that $1^3 + 2^3 + 3^3 + 4^3 = 100$. However, this is simply a trivial case of the more general identity,

$$\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2$$

This aesthetically pleasing identity seems too good to be true. Since it does not generalise to higher powers, some people dismiss it as a coincidence, degrading it to the realm of induction:

Proof 1

We use an incredibly inelegant method to deduce this beautiful result. Firstly we note that the right hand side equals $\frac{1}{4}n^2(n+1)^2$ by the sum of an arithmetic series. For $n = 1$, the identity is clearly true. Suppose it is true for some m, then we have:

$$\sum_{k=1}^{m+1} k^3 = \sum_{k=1}^{m} k^3 + (m+1)^3 = \frac{1}{4}m^2(m+1)^2 + \frac{1}{4}(4m+4)(m+1)^2 = \frac{1}{4}(m+2)^2(m+1)^2.$$

And so it holds for $m + 1$, and by induction we are done.

Proof 2

Induction is technically correct, but it does not show why it is true, or how to deduce it in the first place. To prove the identity from scratch, we use the method of telescoping. We look at

$$\sum_{k=1}^{n} (k^4 - (k-1)^4) = n^4,$$

because when we write out the terms, a massive feat of cancellation occurs, and we immediately obtain n^4. However, if we expand the summand, we get

$$\sum_{k=1}^{n} (4k^3 - 6k^2 + 4k - 1) = n^4.$$

Now we know how to sum k; we also know that the sum of $k^2 = \frac{1}{6}n(n+1)(2n+1)$ (if not, then we can use the telescoping trick here, but with $k^3 -$

$(k-1)^3$ instead, to figure it out!). Doing the algebra, we get $\sum_{k=1}^{n} 4k^3 = n^4 + n - 2n(n+1) + n(n+1)(2n+1) = n^2(n+1)^2$, as before.

Proof 3

But the algebra from the above proof is messy, and we may need to apply telescoping a few times if one of the sums is not known. The following proof was first published in 1974. It was produced by Jeannette Hilton, who was 13 years of age when she first came up with the proof.

We note that $\sum_{k=1}^{n} k$ is defined to be the nth triangular number, T_n. Pictorially, T_n may be represented as a right-triangular array of dots, with n dots in the bottom row, $n-1$ in the second last row, $\ldots$, and 1 dot in the first row. From this picture, it is trivial to see that $T_n - T_{n-1} = n$. Almost as easy is that $T_n + T_{n-1} = n^2$ – this can be seen by combining the two triangles to form a square. Hence,

$$k^3 = k^2 k = (T_k + T_{k-1})(T_k - T_{k-1}) = T_k^2 - T_{k-1}^2.$$

Summing this from 1 to n, we see that most terms cancel out and we end up with T_n^2, the desired result.

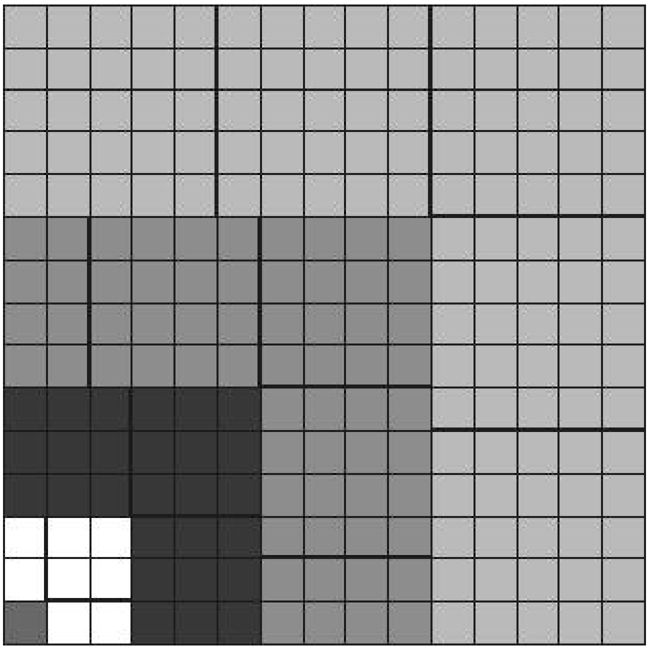
Proof 4

The above proof is remarkable for its grace and simplicity; what's more, it begins to reveal the identity's true root in geometry. However, a small amount of algebra is needed to understand it. Inspired by the ancient Greeks and in search of true enlightenment, yours truly accidentally stumbled upon a purely geometric proof, with no words at all (due to the monochromatic nature of Paradox, some explanation is given. However, it is a one line explanation if we start the line far enough to the left on a piece of A4 paper). So we look at the picture on the next page; the patterns can be extended in the obvious way.

We count the number of small squares and get T_n^2. By stacking blocks of squares of the same shading on top of one another, we also get sum of the first n cubes.

Hence, our curious little identity reduces to nothing more than some squares. Trivial, after all.

— James Wan



Book Reviews

Theoretical Physics, Georg Joos & Ira Freeman, Third Edition, Dover Publications.

For many years, Halliday and Resnick has been the standard textbook for many universities. It is a wonderful textbook, with plenty of historical anecdotes and challenging exercises aimed at the freshman who wants something a bit more. The problem, however, is that Halliday and Resnick fails to cover mathematical concepts required to understand the physical concepts at a greater depth. It also has a steep learning curve for those from a mathematical background.

That's where this nice little book, Theoretical Physics, comes in. If you are a person who would like a no-fuss approach to learning theoretical physics in a ridiculously short amount of time, then this book is ideal for you. This book truly cuts down on any history about physics (so it may be boring for some people), has hardly any exercises and concentrates on the mathematics and the derivations of the many laws of physics within a surprisingly short volume of only 862 pages.

The first chapter concentrates on fundamental calculus and linear algebra theory, so the book is accessible to any high school graduate. This book covers pretty much all areas of physics that any undergraduate would cover, and much more! Even for those who have completed undergraduate physics and would like a brief review, or to step back and take a look at the big picture, this book would also be ideal.

How To Read and Do Proofs, Daniel Solow , Third Edition, John Wiley & Sons.

The mathematical thought process is elusive for grasp, and even many child prodigies fail to understand proofs, let alone appreciate their beauty, until they enter university. This little gem focuses on the full range of techniques used in proofs, examples, exercises and logic using ordinary language rather than symbolic language. I would highly recommend this to those who would like to get a bit more out of university lectures, or simply to sharpen one's mind.

Theoretical Concepts in Physics, Malcolm Longair, Second Edition, Cambridge University Press

If you are looking for an unorthodox approach to combat concepts in physics, and have already read other texts such as Feynman's Lectures, then you may be interested in this highly original book. It is intended to be used as a supplement to the final years of an undergraduate course and assumes the reader has a knowledge of university physics.

After reading the first few pages, you will notice Longair's enthusiasm and emphasis on the excitement of research and its eventual development of the theories of Physics that we have taken for granted.

— Thuc Tran

Think your professor is pompous? Check these out.

Serge Lang (1927 – 2005) was a French-born American mathematician. He was known for his work in number theory and for his textbooks, including the influential *Algebra.* Lang almost never used computers, typing exams and problems on a typewriter.

In class, he would often become irate and throw chalk when students asked silly questions. When attending others' maths lectures, he would tell the lecturer "your notation sucks".

He maintained that the prevailing consensus that HIV causes AIDS has not been backed up by reliable research.

Kurt Gödel (1906 – 1978) was one of the greatest logicians of all time, famous for his incompleteness theorems, formulated at the age of 25. He was also quite eccentric. He avoided human contact, for instance, handshakes. He would weave through crowds in a strange dance to avoid touching people. When reading the US constitution, he discovered troubling logical loopholes and this cast him into deep distress. He was convinced that people were trying to poison him, and starved himself to death.

The Hungarian mathematician Frigyes Riesz (1880 – 1956) was fundamental in developing functional analysis. When giving a lecture, he would march into the room followed by an Associate Professor and an Assistant Professor. The Associate Professor would read to the class from Riesz's famous book and the Assistant Professor would copy the words and symbols onto the blackboard. Professor Riesz would stand, hands behind his back, and nod sagely.

Frigyes' brother, Marcel Riesz (1886 – 1969), who worked on analysis and partial differential equations, once gave a series of lectures at Stanford. After he had filled up the board, he motioned imperiously to Gabor Szegö to wash the blackboard while he stood by. Szegö at the time was a very distinguished faculty member (head of department) at Stanford, regarded in awe by students.

Solutions to Problems from Last Edition

We had a large number of correct solutions to the problems from last issue. We are pleased to announce that collectively, all problems were solved. Below are the prize winners. The prize money may be collected from the MUMS room (G06) in the Richard Berry Building.

Jess McClintock may collect $2 for solving problem 1.

Adrian Khoo may collect $2 for solving problem 1.

Quynh-Chi Nguyen may collect $2 for partially solving problem 3.

Michael Peter Nair may collect $4 for solving problems 1 and 2.

Chang Yang Yew may collect $4 for solving problems 1 and 3 (partial).

Ben Fleming may collect $6 for solving problems 1 and 3.

Alex Hua may collect $9 for solving problems 1, 2 and 4.

Sally Zhao may collect \$10 for solving problem 5.

Ben Lansdell may collect \$10 for solving problem 6.

Dion Gory may collect \$10 for solving problem 6.

Rosemary Nguyen may collect \$12 for solving problems 3 (partial) and 6.

Keng Lon Lam may collect \$16 for solving problems 3 (partial), 5 and 6 (partial).

We thank everyone who submitted solutions and gave us feedback on the problems!

I am thinking of one of three numbers: 1, 2 or 3. You may ask me exactly one yes-no question to find out what number it is, and I will answer truthfully (yes, no or I don't know). What do you ask?

First Solution: many people found this solutions. "I am thinking of a number between 1.5 and 2.5. Is your number larger than mine?"

Second Solution: we can use other properties of number apart from ordering. For instance, "I am thinking of an odd number. Is it divisible by yours?"

You are blindfolded before a table,[5] and on the table there are some coins, exactly 247 of which are head up. How can you divide all of the coins into 2 piles such that each has the same number of heads facing up?

Solution: take 247 coins to one pile and turn all of them over. So if there are n heads in the other pile, then there were originally $247 - n$ heads in this pile, which by turning over become $247 - (247 - n) = n$ heads.

You have two candles, each burns for exactly 1 hour at uneven rates. Measure 45 minutes with the candles and some matches.

Solution: this is a very well known problem and hence we were looking for some justification with the solution.

We simultaneously light the first candle at both ends and the second one at one end. When the first candle burns out, let the time passed be x minutes. Now suppose the first candle had been lit from one end only, then after burning for x minutes, it would take another x minutes to finish burning, because it takes

[5] As our perceptive reader Daniel Yeow noted, "why would you blindfold a table?"

this much time to burn from the other end to the x minute mark. Hence x = 30. Thus, the second candle has 30 minutes left in it when the first one burns out, at which point we light its other end. This candle will burn out in 15 minutes, so the total time passed is 45 minutes.

Let a, b and c be the side lengths of a triangle with fixed perimeter $2s$. As a, b and c vary, what is $\sup\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$?

Solution: (from Alex Hua) let $a = x + y, b = y + z, c = z + x$, so $x, y, z > 0$ and $x + y + z = s$. So we want $\sup\{(x-y)^2 + (y-z)^2 + (z-x)^2\}$.

Now

$$s^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$> x^2 + y^2 + z^2 - xy - yz - xz = \frac{1}{2}((x-y)^2 + (x-z)^2 + (y-z)^2).$$

So $2s^2 > (x-y)^2 + (x-z)^2 + (y-z)^2$. We may get arbitrarily close to equality by letting $x \to 1$ and $y, z \to 0$. Hence the supremum is $2s^2$.

In a regular heptagon $A_1A_2 \cdots A_7$, prove that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$.

Note: A_1, A_2, A_3 form the sides of what's known as the heptagonal triangle, which has many amazing properties (see, for instance, the MathWorld website).

First Solution: (from Keng Lon Lam) without loss of generality, let $A_1A_2 = 1$. Applying the sine rule after some angle chasing, we get $A_1A_3 = \frac{\sin 5a}{\sin a}$, and $A_1A_4 = \frac{\sin 4a}{\sin a}$, where $a = \frac{\pi}{7}$. So we have $\sin 5a = \sin 2a$ and so on.

So we have

$$\frac{1}{A_1A_3} + \frac{1}{A_1A_4} = \frac{\sin a \sin 2a + \sin a \sin 3a}{\sin 2a \sin 3a}.$$

We now apply the identity $2\sin x \sin y = \cos(x-y) - \cos(x+y)$ to the previous equation, and we see quickly that it simplifies to $1 = \frac{1}{A_1A_2}$.

Second Solution: we use Ptolemy's theorem, which says that in a cyclic quadrilateral $ABCD$, $AB \cdot CD + AD \cdot BC = AC \cdot BD$. The theorem can be proved using similar triangles.

Apply this to $A_1A_2A_3A_5$, noting that due to symmetry, we have $A_2A_3 = A_1A_2, A_1A_5 = A_2A_5 = A_1A_4, A_3A_5 = A_1A_3$, we see that $A_1A_2 \cdot A_1A_3 + A_1A_2 \cdot A_1A_4 = A_1A_3 \cdot A_1A_4$. Dividing by $A_1A_2 \cdot A_1A_3 \cdot A_1A_4$ gives the result.

Third Solution: (from Sally Zhao) clearly $A_1A_4//A_2A_3, A_3A_7//A_2A_1$ and $A_1A_3//A_7A_4$. Let A_1A_4 and A_3A_7 intersect at X. Then $A_1X = A_1A_2, A_7X = A_7A_3 - A_3X = A_1A_4 - A_1A_2, A_4A_7 = A_1A_4$. Using the similar triangles A_1A_3X and A_4A_7X, we get $\frac{A_1X}{A_7X} = \frac{A_1A_3}{A_4A_7}$. The result follows.

We have n letters with n corresponding envelopes. Suppose we put each letter in an envelope randomly, what is the probability that none of them go into the correct envelope?

First Solution: this problem is also well known. The usual solution goes like this: we have $n!$ arrangements, from which we subtract $n(n-1)!$ ways that at least 1 letter is in the correct envelope. However, the case of at least 2 letters in the correct envelope has been subtracted twice, so we add it back: $n! - n(n-1)! + \binom{n}{2}(n-2)!$. We then subtract the case where at least 3 letters in the correct envelope ($\binom{n}{2}(n-3)!$), as it has not been counted. Continuing this way, we arrive at the answer, $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

Second Solution: Let the number of 'derangements' of the n letters be a_n. For a_{n+1}, let the first letter go into envelope x, so there are n choices for x. For the other n letters, two things can happen:

If the xth letter does not go into envelope 1, we swap the labels 1 and x, and rearrange the n letters (not including the new x). We see there are a_n ways.

If the xth letter goes into envelope 1, then we rearrange the remaining $n-1$ letters (not including 1 and x), and there are a_{n-1} ways.

So $a_{n+1} = n(a_n + a_{n-1}), a_1 = 0, a_2 = 1$. Now it is easy to verify (e.g. by induction, inspection, series manipulation or generating function) that $a_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \frac{\Gamma(n+1,-1)}{e}$.

So the probability is $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$, which interestingly tends to $\frac{1}{e}$ as $n \to \infty$. There is a deeper connection. The probability that all the cycles of the arrangement all have length $> k$ tends to e^{-H_n}, where H_n is the nth harmonic number.

Solution to puzzle 2: we never said the polynomial couldn't be constant, so $P(x) = 2$ will do.

Paradox **Problems**

"Paradox problems are the only maths I learn" – dedicated problem solver.

Below are some puzzles and problems for which cash prizes are awarded. Bear in mind that anyone who submits a clear and elegant solution may *each* claim the indicated amount (unless two solutions are the same, in which case only the first submission will be rewarded). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G06) in the Richard Berry Building; please include your name.

These two puzzles require no real maths:

1. ($2) You have a compass whose legs are set at a fixed distance apart. How can you draw 2 circles of different radii on paper?

2. ($3) Can you tile an 18×18 board (i.e. one divided into 324 squares) using only T-shaped pieces each made of 4 squares?

   And some real problems:

3. ($3) Prove that for reals $x + y + z = 1$, $xy + yz + zx \leq \frac{1}{3}$.

4. ($5) A bug crawls along the edges of a cube; at each vertex it has probability $\frac{1}{3}$ of going to any adjacent vertex. When it reaches the vertex opposite its starting one, enlightenment is achieved. What is the average number of edges it must crawl on to do this?

5. ($6) Any right triangle contains an isosceles triangle whose area is at least α times the area of the original triangle. Find the maximum value for α.

6. ($6) Find

$$\int_0^\pi \frac{x}{1+\cos^2 x} \mathrm{d}x.$$

Paradox would like to thank Sam Chow, Daniel Yeow, Yi Huang, Alisa Sedghifar, Kate Mulcahy, Chris Loo, Lu Li, and Julia Wang for their contributions to this issue.