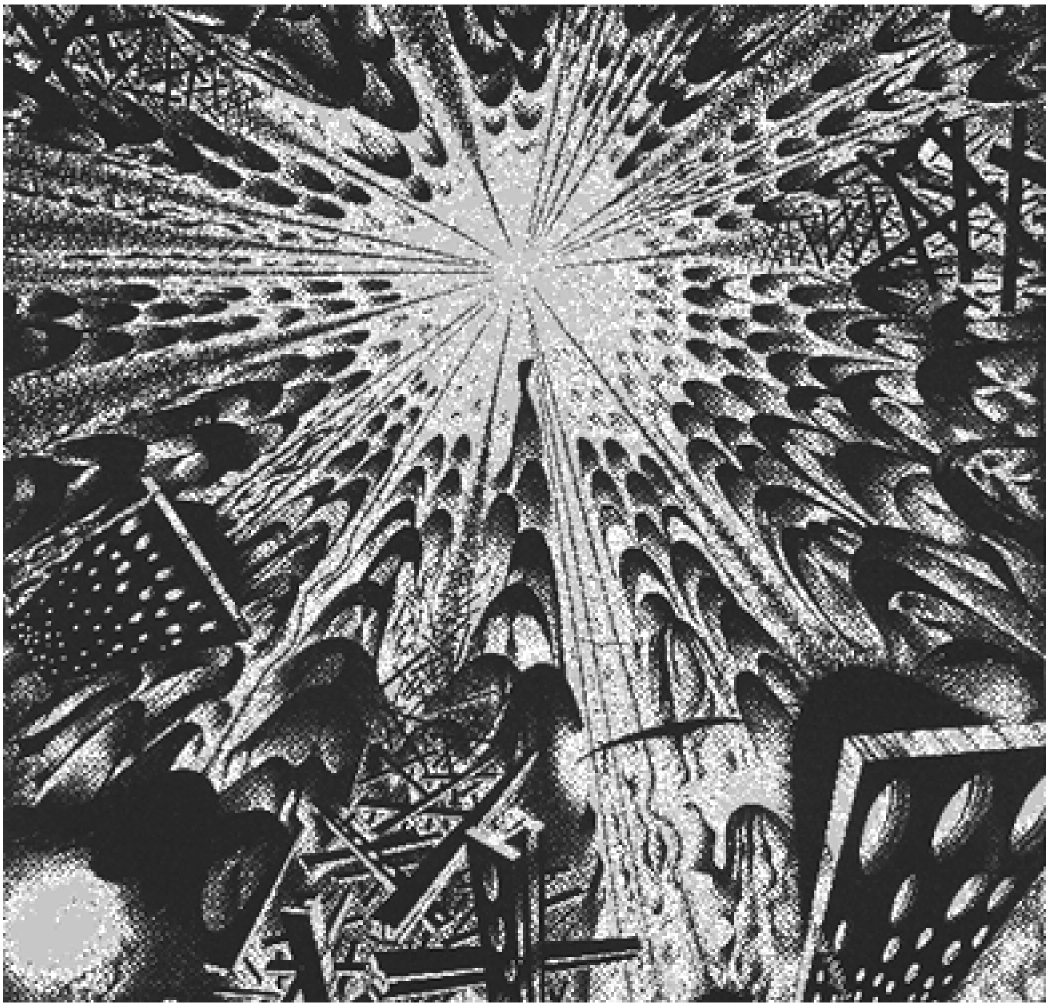
Issue 2, 1999

Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society



Limit Theorems . . . , Professor A. T. Fomenko.

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A warm welcome back

Welcome back to uni! I hope that you all survived the deadly combination of exams, *Star Wars*, and World Cup cricket. *Paradox* is up and running after a rather long break, and we have a new web page too! Point your browser to `http://www.ms.unimelb.edu.au/~paradox` for all the back issues and more.

Contributions are always welcome so please help out your overworked, underpaid (well, so to speak) editorial team by sending your comments, solutions to problems or other pearls of wisdom to `paradox@ms.unimelb.edu.au`.

Read on for further adventures of Paradox Kid, enlightenment from Tony and Chaitanya on the Duckworth-Lewis system and more spectacular failures of our quest for good maths jokes.

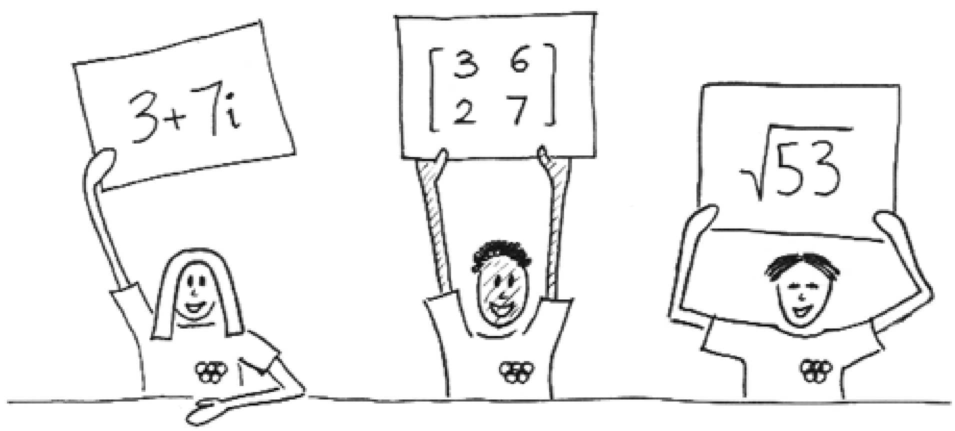
— Jian He, *Paradox* Editor

The Maths Olympics approaches once again

The Maths Olympics is just around the corner! MUMS plans to hold the event in September, so start forming teams and get a training schedule worked out soon. The MUMS web site is a useful resource for serious competitors with previous year's questions and a training guide by Lawrence Ip.

For those of you who don't know what the Maths Olympics is, try to imagine a combination of *Trivial Pursuit*, maths and logic, and a demolition derby. Form a team of five for possibly your only chance to literally and physically beat your lecturers and tutors. (Physical beating is not encouraged, but seems to be inevitable.)

Also coming up is the Schools Maths Olympics! Scheduled for Discovery Day (Sunday 15th August), the event will run from 11:30 a.m. to 12:30 p.m. in Theatre A of the Richard Berry Building. Approximately 20 school teams will compete, with the top six schools qualifying for the Maths Olympics in September. Spectators are most welcome, and snacks will be gratuitously distributed.



Resetting targets in one-day cricket matches — the Duckworth-Lewis method

One of the controversies of one-day cricket has always been how to set appropriate run targets in matches shortened due to rain or other interruptions. This is not a simple problem as there are several factors involved to estimate the state of the game at the time of the interruption(s), and then one needs to extrapolate it to a target that is fair for both teams. Many of you while following the recent World Cup would have heard about the *Duckworth-Lewis* method that was in place for the matches. This is quite different from other methods, for example it can mean that the team batting second (which we denote "Team 2") can chase a target that is *greater* than that set by the team batting first ("Team 1"). Does this sound fair? This article summarises some of the other approaches to target-resetting used in the past, shows the improvement introduced by the Duckworth-Lewis method, and points out that it too has its flaws.

OTHER TARGET-RESETTING METHODS

The following is a commentary on some of the resetting methods that have been used in one-day cricket.

Average run rate (ARR) The traditional method has been to assume a constant rate of scoring and set a target in the proportion of overs faced or to be faced by each team. This is perhaps the first method one would think of and certainly is the simplest. However it does favour Team 2, having more wickets per over in hand to achieve a smaller target.

An important match that exposed this problem was the deciding 3rd final of the 1989 World Series Cup, played in Sydney between Australia and the West Indies. Australia batted first and was 2 wickets down for 93 in the 24th over, when the first rain interruption arrived, reducing the match to 38 overs per side. With just under 15 overs remaining and 8 wickets in hand, Australia then piled on the runs to set West Indies a total of 227 to win the final. West Indies made it to 2/46 in the 7th over when the heavens opened again, and 20 overs were lost. With the average run rate rule in place it meant they only had to score $\lfloor 18/38 \times 226 \rfloor + 1 = 108$ runs. West Indies of course were easily able to make the remaining 62 runs with 8 wickets in hand. A game in which Australia did little wrong became terribly one-sided, and this prompted rule changes. For example, the minimum length of a rain-reduced match was lifted from 15 to 25 overs per side.

(*Exercise*: After reading this article, approximate the 2 targets that would have been set for the West Indies using the Duckworth-Lewis method, one for 38 overs at the start of their innings, the other for 18 overs after the second rain interruption. A *\$10* prize will be awarded to the person with the first approximately correct solution with justification.)

Most Productive Overs (MPO) Suppose Team 2 has a reduced number of x overs in which to bat. This method combats the previous method by setting a target that is one more than the x *highest scoring* overs of Team 1. This method was in use during much of the early 1990s including the 1992 World Cup in Australia. Team 2 has a higher target to chase than with the ARR method, making it less easy for them to achieve. Note that this method depends on the manner in which Team 1 scored its runs, in addition to the final total.

However history has shown that Team 1 has in fact won more of these matches, since the positive aspects of Team 1's batting are enhanced while the positive aspects of Team 2's bowling are suppressed. The method also requires record-keeping of runs scored in each over, but is still relatively easy to understand and compute from the data.

One extreme example to illustrate the favouritism of the method to Team 1 is to consider this year's World Cup match between India and Sri Lanka. India batted first and scored 371 runs in its fifty overs. Imagine now if rain reduced Sri Lanka's innings to 25 overs. This would have given them a target of a whopping 276 runs!

Discounted Most Productive Overs This method has been used in Australia since the 1992/93 season. It is similar to the MPO method in taking the highest scoring overs, but to reduce the bias towards Team 1 it also deducts from the most productive overs 0.5% for each over remaining. This rule is based on results of the 1992 World Cup, and if implemented there would have resulted in matches won equally by Team 1 and Team 2. This method still carries some of the same disadvantages as the MPO method — it is dependent only on the number of overs remaining and on the batting of Team 1. The target set for Team 2 is therefore only a function of the number of overs remaining, and is independent of whether subsequent interruptions occur early or late in their innings.

The Parabola Method This rule has been used in recent seasons everywhere except Australia. The score to be chased by Team 2 is Team 1's score multiplied by a ratio, dependent on the number of overs bowled to each team. The ratios are tabulated and based on the parabola $y = 7.46x - 0.059x^2$, where y is the number of runs and x the number of overs remaining. According to the ICC it is derived "from a detailed mathematical analysis of a database of one day matches with the object of establishing 'normal' performance." This formula predicts that for example, with $x = 40$ overs remaining Team 2 can score $y = 204$ runs. The 50-over average score is 225. So if Team 1 scores 250 runs in 50 overs and Team 2 has 40 overs to reach their target, then they will be set a target of $\lfloor 204/225 \times 250 \rfloor + 1 = 227$ runs.

This method shows the diminishing nature of run rate versus total number of overs. Halving the number of remaining overs while keeping the number of wickets in hand constant should mean that more than half the number of runs can be scored. A different parabola equation would have to be used for matches that are for more than 50 overs — the parabola becomes decreasing at $x = 63$ overs!

The Clark-Samson Rule All of the above methods described so far are independent of the time of the interruption. The Clark-Samson rule, used in games in South Africa, was the first method to take into account the stage of the game at which the interruption takes place. There are six cases treated differently — whether the stoppage occurs at the start, during or after each innings. It is based on three graphs: two which give predicted scores given overs remaining, and one which gives a ratio of final score to current score based on the number of wickets that have fallen. A detailed description is given in the "About Cricket" section of the *CricInfo* web site [1]. A disadvantage of the method is that there are discontinuities where some of these six cases meet.

By now you can probably appreciate some of the difficulties in target-setting methods. Some tradeoff needs to be made between fairness and simplicity. The weaknesses of most of the methods listed above are that they do not take into account the point in the innings at which either the overs have been lost, or of the number of wickets that have fallen. We now turn our attention to the Duckworth-Lewis method.

THE DUCKWORTH-LEWIS METHOD

The Duckworth-Lewis method of resetting targets has been used for over two years now, though not yet in Australia. Despite being roundly criticized in Wisden 1998 for being too complicated for the average cricket fan to understand, the method was selected for the recently completed World Cup. Surprisingly, with the competition played in England, it was never actually used. During the few times that rain shortened play, it rained so much that the match had to be finished the next day (and one match was abandoned).

Setting up the method

The inventors, Frank Duckworth and Tony Lewis, wanted to create a method that:

1. was fair to both sides;
2. could be used in all situations;
3. didn't depend on Team 1's scoring pattern;
4. was relatively easy to apply — methods relying on scoring patterns tended to be tedious;
5. could easily be understood by players, officials and fans.

In our opinion Duckworth, a statistical consultant, and Lewis, a Mathematics lecturer, have done a commendable job on points 1, 3 and 4. We are concerned that there are some in which the method doesn't work very well.

Table 1: Relative resources remaining ($\times 1000$).

		Wickets lost			
		0	2	4	9
	50	1000	838	624	76
	40	903	776	598	76
Overs left	30	771	682	549	76
	25	687	618	512	76
	20	589	540	461	76
	10	341	325	298	75

The idea behind Duckworth-Lewis

The Duckworth-Lewis method views the target chase in terms of resources. There are two types of resource with which to score runs: overs and wickets. Duckworth and Lewis believe that only these quantities need to be known to make accurate predictions about the scores.

Potentially a team starts with 50 overs and 10 wickets (in hand) which it tries to convert into runs. Duckworth and Lewis try to estimate what proportion of resources are used at each stage of the game. Initially they model the number of runs $Z(u)$ that can be scored in u overs as

$$Z(u) = Z_0[1 - e^{-bu}],$$

where Z_0 is the (finite) number of runs that could be scored in an infinite number of overs, and b is some decay constant. Taking wickets into account, the asymptotic run score and the decay constant vary according to the number of wickets lost (w). Viz.

$$Z(u, w) = Z_0(w)[1 - e^{-b(w)u}].$$

Unfortunately the authors decided to sell their method and so their paper [2] doesn't explain how $Z_0(w)$ or $b(w)$ were determined. Perhaps of more concern, they don't explain why they selected this exponential decay model. Using these average scores they calculate the relative resources with u overs and w wickets to be

$$R(u, w) = \frac{Z(u, w)}{Z(50, 0)}.$$

The full table for these values would have 300×10 entries since the calculations are made per ball, not just per over. (We will only use overs in the notation for clarity.)

Again, the full table isn't available for free, but the paper [2] does include a small sample (Table 1). So if you have rain interruption in your game, make sure it occurs in one of these situations!

Applying the method

For the moment assume that Team 1's innings wasn't interrupted. Imagine that Team 2's innings stops with u_1 overs to go, when they have lost w wickets, and resumes with u_2 overs to go. The fraction of resources available to them is:

$$1 - (R(u_1, w) - R(u_2, w)).$$

As an example, imagine that the first twenty overs of Team 2's innings are washed out. Then the resources available are

$$1 - R(50, 0) + R(30, 0) = 1 - 1 + 0.771.$$

If Team 1 scored say 235 runs then Team 2 now has to score only $\lceil 0.771 \times 235 \rceil = 182$ runs to win. Note that if we use the average run rate method the score to chase is 142, quite easy. The Duckworth-Lewis calculation is more realistic because it acknowledges that Team 2 can afford to be aggressive since it has 30 overs in which to spend 10 wickets. Similarly, if overs 21 to 40 are lost, and Team 2 is 0/75 after 20 overs, then the appropriate calculation is

$$\lceil (1 - 0.771 + 0.341) \times 235 \rceil = 134.$$

This means the team must score a further 59 runs in ten overs with all wickets remaining. Given Team 2's solid start the run-rate method, which again sets 142 as the target, seems too tough.

Imagine now that there is some interruption to Team 1's innings. If this happens the match officials usually change the game so that both sides face the same number of overs. Say Team 1 makes 4/100 after 20 overs, then loses 10 overs, then ends up with 200 after 40 overs. The fraction of resources available is

$$1 - R(30, 4) + R(20, 4) = 1 - 0.549 + 0.461 = 0.912.$$

On the other hand, Team 2 has only $R(40, 0) = 0.903$ of their resources for their innings. This means their target is $\left\lceil \frac{0.903}{0.912} \times 200 \right\rceil = 199$. This example is interesting because Team 2 had fewer resources even though Team 1 lost overs later in their innings. The reason for this is that by having lost 4 wickets already, Team 1 didn't have that much chance to slog at the end anyway. Consider what happens if Team 1 plays more cautiously, scoring 2/60 by the interruption, and then finishes with 150. This time their resources fraction is

$$1 - R(30, 2) + R(20, 2) = 1 - 0.682 + 0.540 = 0.858.$$

Our guess is that Team 2 would face the target $\left\lceil \frac{903}{858} \cdot 150 \right\rceil = 158$. However, Duckworth and Lewis argue that if Team 2 has more resources than Team 1 we need to take a different approach. Their justification is the following example: say Team 1 scores 0/80 in 10 overs, using 0.097 of their resources, and there is so much rain that Team 2 also only gets 10 overs. Since Team 2 has 0.341 of their resources, they are expected to score

$\left\lceil \frac{341}{97} \cdot 80 \right\rceil = 282$ runs in 10 overs to win. Crazy! Duckworth and Lewis's solution, which they admit was chosen because it might be easy for fans to understand, but doesn't seem to be easy for mathematicians to understand, is to apply the excess resource (in this case $0.341 - 0.097 = 0.244$) to the average score 225 for one-day internationals, rather than the score made so far. In their example, the target becomes, $80 + \lceil 54.9 \rceil = 135$. Still pretty tough from 10 overs! In our example, with their modification the target is $150 + 11 = 161$ in this case slightly more than the standard scale-up method.

The problem with Duckworth-Lewis

Duckworth and Lewis's example in the last section shows the flaw in their method. The target they set for Team 2 in 10 overs (135) seems intuitively far too big. Our feeling is that a target between 90 and 100 is more appropriate. By *fixing* the fraction of resources used after 10 overs with 10 wickets in hand at 0.097, Duckworth and Lewis predict that Team 1 would have made 824 runs in 50 overs. This ridiculous prediction is made because the method fails to acknowledge the "regression towards the mean" effect. Even after such a good start, we would expect Team 1 to score say 280–320 runs. Similarly, we often see teams recover from 4/20 after 10 overs to make between 140 and 180 (Duckworth-Lewis predicts 50). The relative resources table only makes sense for average performances. After a bad start we expect a team to recover and after a great start we expect that there could be a few hiccups along the way.

The way to improve the method is to include some recognition of the team's performance until the interruption. Therefore the table should show that a team that is on 0/80 after 10 overs will probably make about 300 runs at the end. We then have a table which shows what sorts of scores teams chasing 300-odd runs make in the last 10 overs when they have all their wickets left. (This probably doesn't happen too often.) In the absence of any other knowledge we could still use the Duckworth-Lewis fraction of 0.341 which sets a target of 103. Constructing such a table would be tricky as there are now three (or perhaps four) factors: overs, wickets, and runs scored already and runs to make in total. Still, there is a lot of data around from the English domestic one-day season. After all, that's what the competition is for isn't it? It certainly isn't for developing new talent.

— Chaitanya Rao and Tony Wirth

REFERENCES

[1] *CricInfo — The Home of Cricket On The Internet*, [On-line], Available: `http://www.cricket.org` [1999, August 1].

[2] Duckworth, F. C. and Lewis, A. J. 1998, "A fair method for resetting the target in interrupted one-day cricket matches", *Journal of the Operational Research Society*, 49(3), pp. 220–7.

The geometric transformation of inversion

Geometric transformations such as translation, reflection, rotation and dilation can often produce neat and insightful shortcuts to seemingly complicated problems. Inversion is a lesser known transformation intimately related to hyperbolic geometry yet still has some amazing applications in the elementary Euclidean world.

Given a point O (the centre of inversion) and a positive real number r (the radius of inversion) we define the image P' of any point P other than O to be the point on the ray OP such that $|OP'| \cdot |OP| = r^2$ (where $|AB|$ denotes the distance between A and B).

Clearly the image of P' is P. They are inverses of each other, so performing the same inversion twice would map every point back to itself. The circle centred at O with radius r is called the circle of inversion, any point on it is a fixed point.

So what do straight lines and circles look like after an inversion? For that we have the following result.

Theorem 1 *Let (O, r) be an inversion.*

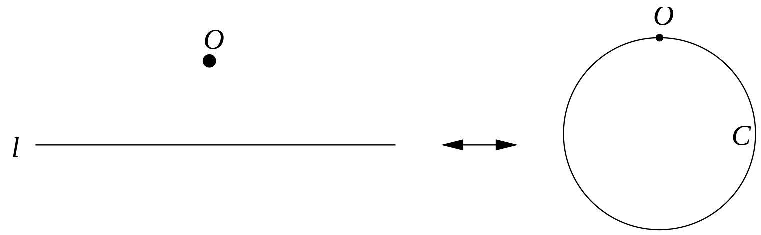
a. *The circle of inversion inverts to itself, points inside it invert to points outside and vice versa.*

b. *A line through O inverts to itself.*

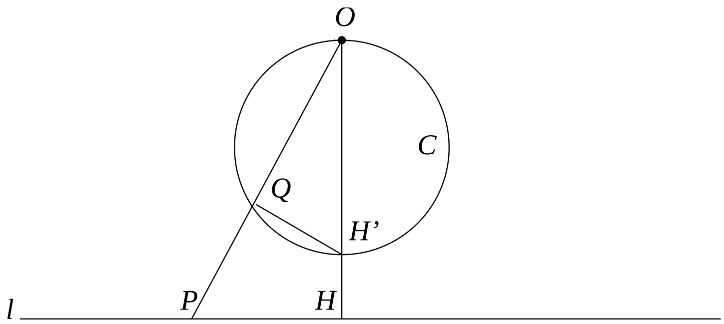
c. *A line l not through O inverts to a circle C through O.*

d. *A circle C through O inverts to a line l not through O.*

e. *A circle not through O inverts to another circle not through O.*

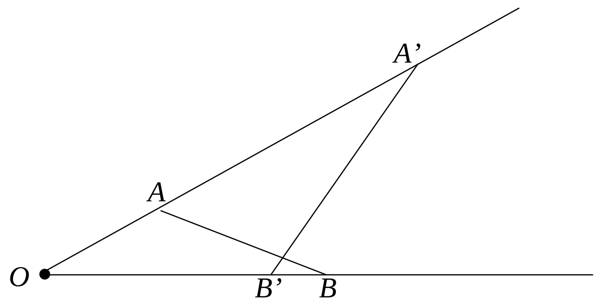


Parts a and b follow straight from the definitions. For part c, consider H, the foot of the perpendicular from O to l, and its image H'. We claim that the circle with OH' as its diameter is the image of l. For a point P on l, PO intersects C at Q. Since OH' is the diameter of C, $\angle OQH' = 90° = \angle OHP$. Hence $\triangle OH'Q$ and $\triangle OPH$ are similar and $|OP| \cdot |OQ| = |OH| \cdot |OH'| = r^2$, so Q is indeed the image of P. Part d can be shown in exactly the same way and part e is left as an exercise.

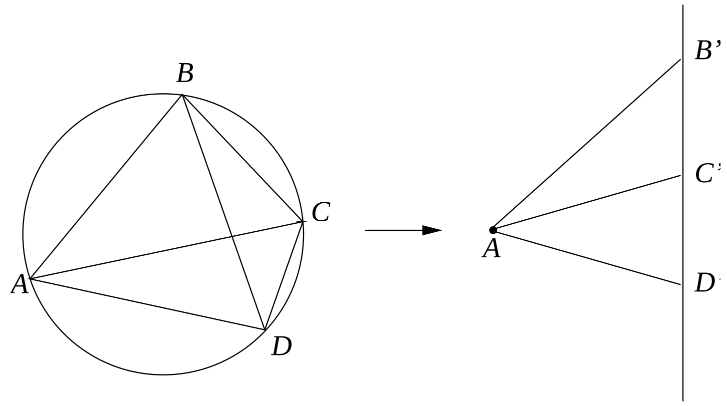


Another natural question to ask is how distances between points change under inversions. Let A', B' be the images of A, B respectively. Then $\angle A'OB' = \angle BOA$ and $|OB'|/|OA| = (r^2/|OB|)/|OA| = (r^2/|OA|)/|OB| = |OA'|/|OB|$. Hence $\triangle A'OB'$ and $\triangle BOA$ are similar. Thus $|A'B'|/|AB| = |OB'|/|OA| = |OA'|/|OB| = r^2/(|OA||OB|)$, so

$$|A'B'| = |AB| \cdot \frac{r^2}{|OA||OB|}. \tag{1}$$



Now consider four points A, B, C, D lying on a circle C_0 in that order. Invert with centre A and radius r. By part c of theorem 1, C_0 becomes a straight line as it passes through the centre of inversion. Since B', C' and D' lie on a line, $|B'D'| = |B'C'| + |C'D'|$.



Using equation 1, this becomes

$$|BD| \cdot \frac{r^2}{|AB||AD|} = |BC| \cdot \frac{r^2}{|AB||AC|} + |CD| \cdot \frac{r^2}{|AC||AD|}.$$

Continued page 12


ISSUE #2
The Adventures Of Paradox Kid
by Jeremy Glick & Sally Miller
IT WAS AN UNCHARACTERISTICALLY WARM DAY, AND P.K. WAS BASKING IN THE SUN WITH SOME FRIENDS...
HOW'S EXAM STUDYING GOING, P.K.?
IT'S GOING OKAY. I'M A BIT WORRIED ABOUT 132, BUT OTHERWISE...
UCB
SUDDENLY...
WHOOSH
HEY YOU!
WHOOSH
WHOOSH
WHOOSH
HEY, WHAT'S GOING ON, STEPHAN?
MY FRIENDS AND I WERE STUDYING FOR EXAMS AND THE MASKED INVIGILATOR CAME AND STOLE ALL OUR BOOKS - IF WE DON'T GET THEM BACK, WE'LL FAIL FOR SURE!
OH MY GÖDEL!
THE MASKED INVIGILATOR - WHO'S THAT?
SHE USED TO INVIGILATE MELBOURNE UNI. EXAMS, UNTIL ONE DAY SHE SWAPPED THE '98 COMP. ENG. 1 EXAM WITH THE '97 ONE. ADMINISTRATION THREW HER OUT IN DISGRACE. SHE CLAIMS SHE'S INNOCENT, AND FOR REVENGE HAS VOWED TO WREAK HAVOC ON EXAMS TILL THE END OF TIME.
I CAN'T STAND BY AND WATCH WHILE MY FRIENDS ARE HAVING THEIR ACADEMIC REPUTATIONS JEOPARDISED - THIS SOUNDS LIKE A JOB FOR PARADOX KID!


P.K. DONS HIS SUPER VISOR AND IS TRANSFORMED INTO PARADOX KID, WITH
MORE POWER THAN A QUINTIC (5),
MORE BRAINS THAN THE BASKET BEHIND A GUILLOTINE, AND
MORE SEXUAL ALLURE THAN A HYPERBOLIC ATTRACTOR
DON'T LET THE MASKED INVIGILATOR GET AWAY!
THERE SHE GOES, PARADOX KID - SHE'S RUNNING INTO THE LIBRARY!
DON'T WORRY - WITH THESE INITIAL CONDITIONS, IF I TAKE THE GEODESIC* TO THE LIBRARY I'LL REACH HER IN 12·8 SECONDS.
* LOCALLY DISTANCE-MINIMISING PATH
GIVE UP, MASKED INVIGILATOR!
I KNEW YOU'D FIND ME, PARADOX KID, BUT I'M READY FOR YOU!
SHHH
PREPARE TO BE TORN INTO AN ARBITRARILY LARGE NUMBER OF PIECES OF SIZE EPSILON AND RANDOMLY DISTRIBUTED ACROSS THE UNIVERSITY!
THE LIBRARY IS A PLACE OF SILENCE!
HOORAY FOR PARADOX KID!
HOORAY FOR THE LIBRARIAN!
HEY PARADOX KID, CAN YOU SORT THROUGH THESE BOOKS AND FIND MINE?
AND MINE?
AND MINE?
AND MINE?
SURE STEPHAN - IN FACT THE QUICKSORT ALGORITHM IS O(nlogn)
AND THE MASKED INVIGILATOR? WELL, EVERY NOW AND THEN, A MYSTERIOUS MASKED FIGURE IS SEEN HELPING WITH CHORES AROUND THE MATHS LIBRARY...
TO BE CONTINUED...

From page 9

Multiplying by $|AB||AC||AD|/r^2$ yields

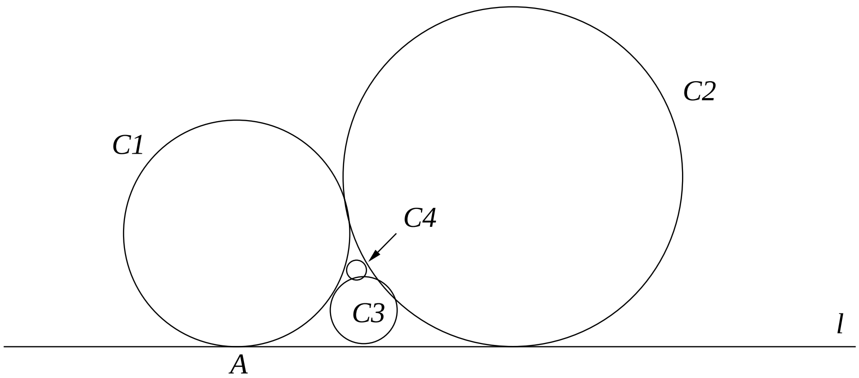
$$|AC||BD| = |AD||BC| + |AB||CD|.$$

This is known as Ptolemy's Theorem which states that the product of the diagonals of a cyclic quadrilateral is equal to the sum of the products of the opposite sides. In fact we can do even better than that. Suppose the four points aren't cyclic to begin with; then, after the inversion, B', C' and D' are not collinear, but we still have the triangle inequality, $|B'D'| \leq |B'C'| + |C'D'|$.

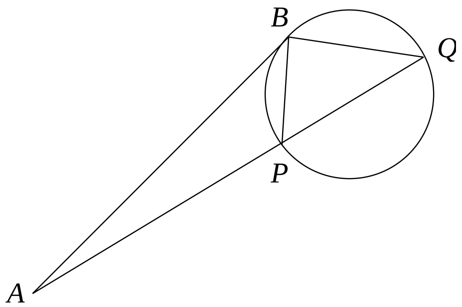
This translates to $|AC||BD| \leq |AD||BC| + |AB||CD|$ for any four points in the plane, with equality occurring if A, B, C, D lie on a circle in that order.

Since inversion is an almost one-to-one map (with the exception of the centre of inversion), two tangential circles/lines will invert to circles/lines with exactly one point in common, i.e. they are still tangential after the inversion. There is one exception to this and that is the case if their point of tangency is the centre of inversion, in which case they will be mapped to straight lines with no points in common, i.e. parallel lines.

Consider the following situation: l is a common tangent of circles C_1 and C_2; C_3 is tangential to C_1, C_2 and l; and C_4 is tangential to C_1, C_2 and C_3. If we scale the diagram so that the radius of C_4 is 1, then amazingly the distance from the centre of C_4 to l is constant no matter how C_1 and C_2 change!

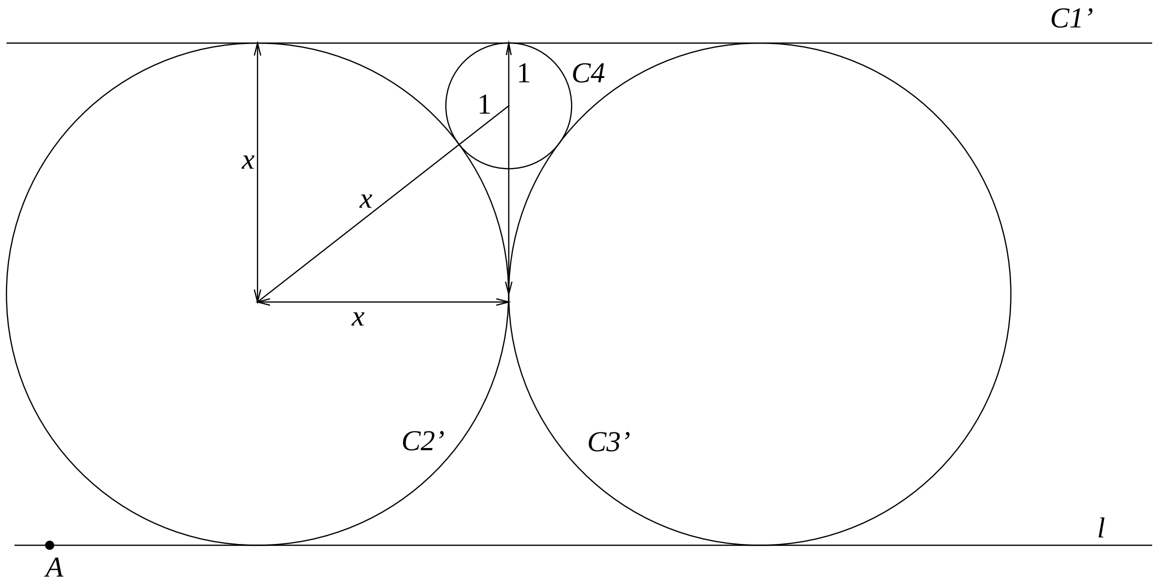


We'll invert the diagram with centre of inversion A, the point where C_1 touches l. We'll choose a radius of inversion r so that C_4 is also fixed by the inversion (l is fixed as well, hence the distance from the centre of C_4 to l will not change if we can fix C_4). This is possible if we take r to be $|AB|$ where B is on C_4 and AB is tangential to C_4.



For any other point P on C_4, AP intersects C_4 again at Q, and as $\triangle ABP$ and $\triangle AQB$ are similar, $|AP||AQ| = |AB||AB|$. Thus the image of P is again a point on C_4 which means C_4 is fixed by the inversion.

Now C_1 passes through the centre of inversion, so it inverts to a straight line, and as it is tangential to l, it maps to a straight line parallel to l. C_2, C_3 and C_4 invert to tangential circles as shown.



Now if we let the common radius of C'_2 and C'_3 be x, then by Pythagoras' Theorem, $(x+1)^2 = x^2 + (x-1)^2$, which implies that $x = 4$. Hence the distance between the two parallel lines is 8, so the distance from the centre of C'_4 (same as C_4) to l is $8 - 1 = 7$ which is independent of the initial radii of C_1 and C_2!

If we represent points on the plane by complex numbers, the inversion of radius r centred at the origin is just the same as the function $z \to \overline{r/z} = r/\overline{z}$. A transformation of the form $z \to (az+b)/(cz+d)$ is called a Möbius Transformation. If we allow reflection in the real axis (i.e. taking conjugates) then $z \to (a\overline{z}+b)/(c\overline{z}+d)$ is called an extended Möbius Transformation. Inversion then is just a special extended Möbius Transformation. It turns out that if we define distance between two points slightly differently then the collection of real Möbius/extended Möbius Transformations is exactly all the distance preserving transformations.

— Jian He

Problems

The following are some problems for prize-money. The person who submits the best (i.e. clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number, and will have their solution published in the next edition of *Paradox*. Solutions may be e-mailed to `paradox@ms.unimelb.edu.au`. (LaTeX format would be appreciated though not demanded.) If you do not have access to e-mail then drop in a hard copy of your solution to the MUMS pigeon-hole near the Maths and Stats Office in the Richard Berry building.

1. ($5) Divide an 8×8 grid into four identical shapes, such that cell (8,8), cell (7,7), cell (6,6) and cell (5,5) are each in one of the pieces.

2. ($15) Three identical snowploughs are responsible for keeping a certain strip of road clear of snow. The first leaves at 9 a.m., the next at 10 a.m., and the last at 11 a.m. The snowploughs all have the following properties: they travel at a speed inversely proportional to the amount of snow on the road before them, and they leave the road free of snow after they travel over it. The snow is falling at a constant rate. If all snowploughs (taken to be points) collide simultaneously, what was the speed of the first snowplough when it left the depot at 9?

3. ($5) Des has a pack of ten cards, numbered one through ten. He shuffles them, then takes the front card. It is a six. He places the six at the back of the deck, and takes the *sixth* card from the front. It is a three. He places the three at the back of the deck, and takes the *third* card. This game will loop if the card he takes is a ten, but are there any other loops possible?

Apology: We would like to apologise for the appalling quality of last issue's problems; those responsible have been rooted out and subjected to Japanese water torture.

Paradox **from the past**

In 1965, Walter D. Neumann, now a Professor here, wrote an article for *Matrix*, the predecessor of *Paradox*. In the second part of our *Paradox* from the past series (for part 1 see the article by Frank Barrington in issue 1, 1999) we reprint Professor Neumann's article — fully digitally remastered. Enjoy hexaflexing!

HEXAFLEXAGONS

Flexagons come in all shapes, however the most interesting are the square "tetraflexagons" and the hexagonal "hexaflexagons". They were first discovered by accident in 1939 by Arthur H. Stone, then a graduate student at Princeton. A committee was then formed to investigate them and in 1940 J. W. Tukey and R. P. Foynman completed a mathematical theory of flexagons. This has never been published, though various people have published papers on the simpler flexagons. I shall here give the general method of constructing a desired hexaflexagon as I learned it from Prof. Stone in 1960. The method of constructing a tetraflexagon is much more complex, though similar.

To illustrate exactly what a flexagon is I give an example of the simplest hexaflexagon and its construction from a strip of paper as in Fig. 1.

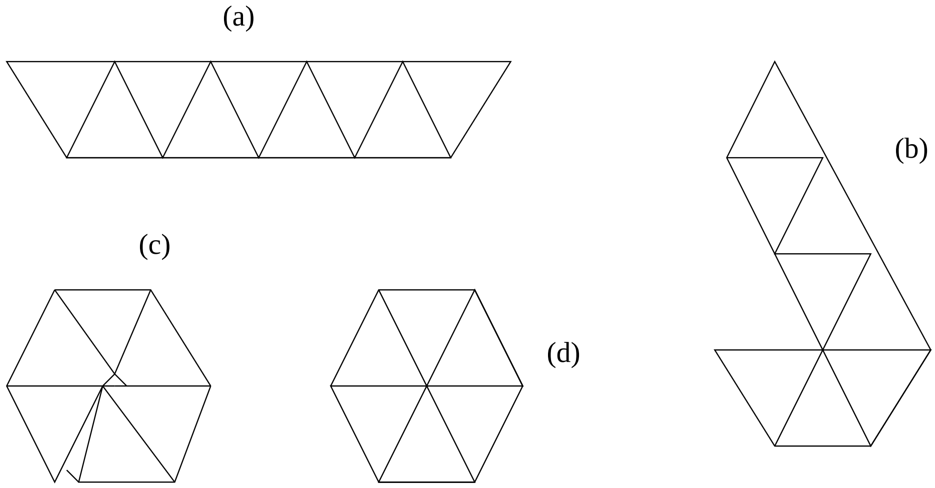


Figure 1

Figure 1(d) shows the completed flexagon, rather inaccurately made in order to show its construction clearly.

The flexagon is "flexed" by folding alternate vertices down as in Fig. 2(a). The flexagon should then open from the top to reveal a new face, Fig. 2(b). If it does not open the other set of alternate vertices should be folded down instead. This is hard to imagine and the reader is advised to make the flexagon described above and try it out.

Figure 2

A flexagon may be constructed with any desired number of "faces", where a face is defined as the set of six triangles showing on one side of an open hexaflexagon at any one time. These faces may be numbered by giving a different number to each set of six triangles.

The construction of a specific hexaflexagon with five faces will be given to illustrate the general method.

(1) A polygon is drawn with as many vertices as the flexagon is to have faces. The vertices are numbered according to the faces of the flexagon and the edges are numbered by ordered pairs as in Fig. 3(a). The polygon is then divided into triangular cells by joining

vertices of the polygon by non-intersecting lines as in Fig. 3(b). Draw the "dual" figure by joining midpoints of the sides of each cell as in Fig. 3(c). We may now forget about the original figure and consider only the dual as in Fig. 3(d).

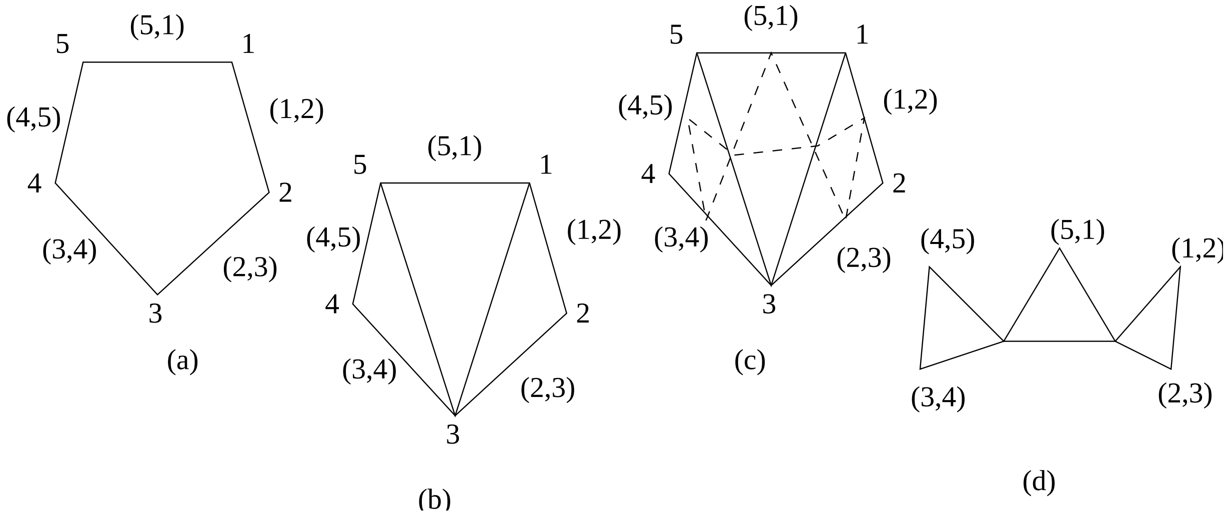


Figure 3

(2) Starting from the point $(1,2)$ of the dual, follow the lines of the dual, crossing over any line of the dual that you meet on the way. Write down each number pair as you come to it, assigning it a sign in the following way. Start by giving $(1,2)$ the sign $+$. From then on change the sign every time a line of the dual is crossed. In our example this gives us the "path sequence" $(1,2)+$, $(2,3)+$, $(5,1)-$, $(3,4)+$, $(4,5)+$. The next number in the sequence would be $(1,2)+$ again. This gives a check that no errors have been made so far. A further check is given by the fact that the difference in number of $+$ signs and $-$ signs should be divisible by three.

The "net sequence" is now formed by reversing the order of every second number pair of the path sequence and changing every second sign from $+$ to $-$ or from $-$ to $+$. In our example this gives the net sequence

$$(1,2)+,\ (3,2)-,\ (5,1)-,\ (4,3)-,\ (4,5)+\,.$$

In the following we use only the net sequence.

(3) Draw a triangle and assign it the number pair $(1,2)$. Mark one edge and call this the "entrant edge" of the triangle. As the sign of $(1,2)$ is $+$ we call the next edge of the triangle (proceeding in a clockwise direction) the "exit edge" of triangle $(1,2)$ and the entrant edge of the next triangle $(3,2)$. The sign of $(3,2)$ in the net sequence is $-$, so we take the edge adjacent to the entrant edge of $(3,2)$, in the anticlockwise direction, as the exit edge of $(3,2)$ and the entrant edge of triangle $(5,1)$. In general, working from the net sequence, we take as the exit edge the edge next to the entrant edge in the clockwise

direction if the sign is +, and anti-clockwise if the sign is −. This is also the entrant edge of the next triangle.

This gives us the net of Fig. 4.

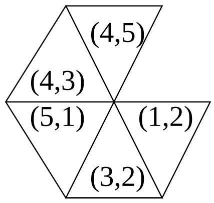


Figure 4

We now number each triangle on top according to the first number of its number pair and on the bottom according to the second. This gives us a strip numbered as in Fig. 5(a) on top and Fig. 5(b) underneath.

Figure 5

(4) Make three strips as in Fig. 5. Fold one of these strips by folding triangles with the same number onto each other until a parallelogram is reached, as in Figs. 6(a), (b), (c), (d).

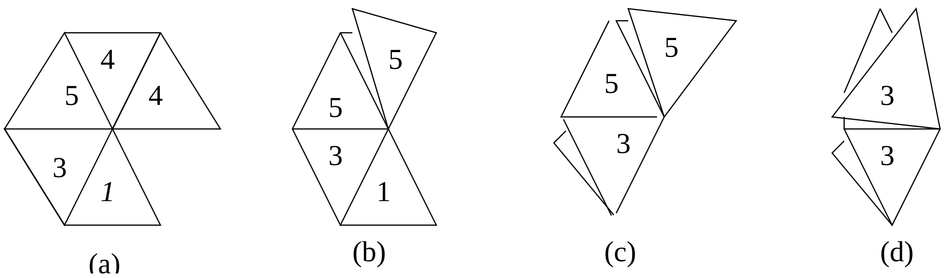


Figure 6

Fold the other two strips in exactly the same way and arrange them as in Fig. 7, such that the entrant edge of the first triangle of each touches the exit edge of the last triangle of each. Stick together the corresponding edges either by leaving flaps or by cellulose tape.

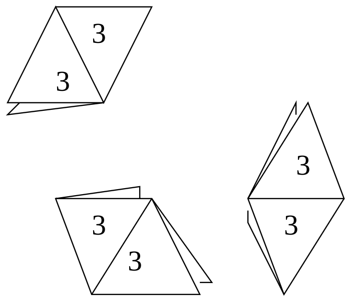


Figure 7

I recommend stiff paper and $1\frac{1}{2}$ to 2 inch triangles for practical construction, though small trial models may be made from ordinary note paper.

Fig. 3(b) actually gives the exact properties of the flexagon constructed. For instance if faces 1 and 3 are showing on opposite sides of the flexagon, the flexagon may be flexed to bring either face 2 or 5 out. However two flexes are needed to bring face 4 out.

— Walter Neumann

Poor pure Percy P

Percy P was a mathematician whose "pureness" was never denied.
But he found one day, to his sorrow, that his theorems had been applied!
He had used all the standard precautions; his papers were pointedly dry!
But his own esoteric notation had been solved by a physicist spy!
The colloquium buzzed with the gossip; he could offer no valid excuse.
Percy P was a traitor of traitors, for his work was of *practical use*!
Nobody dared to defend him. Could it be that he'd plead the crime
That his work was just then needed to effect quantisation of time?
Ignored when he joined conversations; one would think that he poisoned the air.
And he felt on his way to the office — a new man might be in his chair.
A committee was in operation, working twenty-four hours a day,
Deleting his name from the journals, and throwing his reprints away.
He knew where his future was leading, no sense in prolonging the pain;
He left with a handful of papers, and never was heard from again.
So take heed all you mathematicians who pretend your endeavor is pure;
Tho' your luck may hold for a decade, in the end you can never be sure.

Book reviews

Epidemic Modelling: An Introduction
by Daryl J. Daley and Joe Gani
Cambridge University Press
ISBN 0 521 64079 2

Epidemic Modelling: An Introduction discusses a wide range of probability models for the spread of diseases. The first chapter gives an outline of the historical development of the analysis of epidemiological data. It provides a context for the aim of the book, which is to introduce the mathematics needed for understanding, predicting and controlling the spread of diseases. This historical focus remains throughout the book.

There is good discussion and justification for the probabilistic models considered in this book. Whenever new models are introduced or specific epidemics referred to, an effort is made to demonstrate relevant insights into the physical and biological nature of disease. Some examples of epidemics and the types of models used are: the deterministic model for the common cold in a classroom of schoolchildren; a carrier model for the spread of malaria; chain infection probability models varying between households applied to measles epidemics; the chain binomial model with replacement for HIV infections; and the interesting study of the spread of rumours in a population by the Law of Mass Action.

The authors emphasise the importance of choosing and validating models. They give a number of worked examples, in particular they fit the simple Reed Frost model. In the more theoretical sections, many results are derived that also emphasise this point, and numerous graphical displays accompany the theory and fitting of different models.

Following every chapter is a set of exercises designed to extend specific results in the chapter. These require a good understanding of the probability theory being used. The level of mathematics is about that of an advanced graduate student or researcher, and seems quite a bit higher than an Australian Honours level. In summary, this book is a nice challenging introduction to the probabilistic models used in epidemic modelling.

— Natalie Roberts

Introductory Statistics with Applications in General Insurance (2nd ed.)
by I. B. Hossack, J. H. Pollard, and B. Zehnwirth.
Cambridge University Press
ISBN 0 521 65234 0 (hardback)
ISBN 0 521 65534 X (paperback)

The title should have been "Introductory Statistics and then its Applications in General Insurance". This is not a hefty text, which is not surprising considering that it is obviously directed at an audience which would probably be sent running at the sight of a statistics text the size of one of those from economics or accounting (which are often mistaken for telephone books). Indeed, it is so light that it might appeal to anyone, from the commerce student looking to make inroads into the world of assurances to general insurance practitioners, or even a mathematics or statistics enthusiast who is interested in the particular area of applied statistics which is found in general assurance.

The first six chapters, together with the later chapter on simulation, are dedicated to explaining the fundamentals of the probability and statistics needed in the assurance industry. For students, the first thing to note is that it does not assume any prior knowledge of mathematics or statistics except for a (very) basic high-school level. Hence this is not, nor was it designed to be, equivalent to an undergraduate text in statistics. I imagine the authors envisaged people within the industry (and a long time out of school) tackling this text. With this in mind, it is a useful, concise introduction to the theory needed later on when they discuss the issues involved in pricing general insurance products. Also, it is a good refresher before applying the concepts. However, students looking for exercises to supplement other texts will find that they only offer eight to ten in each chapter, although they are accompanied with full solutions.

The remaining four chapters deal purely with topics in general insurance. The manner in which they are presented is cogent and easily understood. More importantly, they are not too verbose, which enables one to quickly grasp the ideas presented. For those whose mathematical background is not strong, there is always a relevant example. And for others, this abundance of examples can be ignored without losing the thread of the topics presented. In fact, most of the general insurance topics are fully developed through examples, but for the more theoretically minded, references for further reading are given at the end of each section. Thus I am sure that the reader looking for an explanation of the terms, practices and theory underpinning general insurance will be satisfied, able to avoid being bogged down too much in the examples. At the same time, I am equally confident that the concepts presented are expounded more than enough to enlighten someone with a weak knowledge of statistics.

The structure of the book makes it easy to read and navigate. In particular, there is a summary of the topics contained in each chapter at its commencement. Particular concepts can be easily found. This is relevant for those students doing 300-321 Risk Theory II, as the section on no claim discount is rarely found in other statistical texts.

This text offers a concise introduction to the statistics involved in general insurance and would be a prudent read for those looking for insight into actuarial studies. It adopts a practical approach which makes it approachable to those who have not studied for a while. This point, however, will put off certain mathematically minded individuals, who will surely find the level of the statistics too low. And this is a statistics text. But let us not allow this to detract from a very accessible book successfully designed to introduce the statistical concepts involved in general insurance.

— Will James

University Physics
by Ronald Lane Reese
Brooks/Cole Publishing Company
ISBN 0 534 24655 9

Ronald Reese's *University Physics* is the latest addition to the cut-throat world of first-year textbooks. This places it in direct competition with texts such as *Fundamentals of Physics* (5th ed.) by Halliday, Resnick and Walker (HRW), which is the current recommended text for 141 (standard first-year physics) at Melbourne Uni. Reese's book and HRW both obey the golden rules of first-year textbook writing: "thou shalt contain as many glossy photos and pictures as possible", and "thou shalt use a breathtaking array of colours on every page". When Isaac Newton declared that white light was composed of *seven* colours, he could scarcely have imagined the multitude of colours fighting for centre stage in these books. In contrast, O'Hanian's somewhat more advanced *Physics* (2nd ed.), used in 121 (advanced first-year physics), uses only black and pale blue: it is a relic from an era when students did not demand to be continually entertained ("or else I'll hold my breath until I turn blue").

Reese's book and HRW are also similar in many other respects. Both have a large number of sample worked problems, with Reese introducing the innovation of often including alternative solutions to these. Both include "Problem Solving Tactics" in an attempt to help students develop problem-solving skills, as well as a summary at the end of each chapter. To stir the reader's interest, HRW opens each chapter with a "Chapter Opening Puzzler", which is answered later in the chapter in one of the sample problems. For instance, the chapter on rotation starts with the question: "What advantages does physics offer in judo throws?" Reese, on the other hand, uses quotations and humour to make the text an enjoyable read. The chapter on kinetic theory begins with this pearl:

> Ludwig Boltzmann (1844-1906), who spent much of his life studying statistical mechanics, died in 1906, by his own hand. Paul Ehrenfest (1880-1933), carrying on the work, died similarly in 1933. (So did another disciple, Percy Bridgman [1882-1961].)
>
> Now it is our turn to study statistical mechanics.
>
> Perhaps it would be wise to approach the subject cautiously.

Both of these approaches seem to work. Certainly, such approaches are necessary, since a book cannot help students unless they read it and first-year students are notoriously reluctant readers/workers.

One criticism which could be made of first-year physics books in general is that most of their contents refer to physics at least a century old. This is especially sad when one considers that many (most?) of the students taking general physics courses choose never to do another physics subject for the rest of their lives (although one colleague has suggested to me that such people deserve to be stuck back in the 19th century). However, one cannot fit everything into a first-year course (Reese is already 1267 pages long), and the edict "Do not teach too many subjects, and what you teach, teach thoroughly" is generally good advice.

In conclusion, HRW and Reese are both very good elementary textbooks, with little to make the reviewer prefer one over the other. (This probably means that the status quo will prevail in unis now using HRW.) Surveys by the National Science Foundation in the US show that less than half of all Americans know that electrons are smaller than atoms. Locally, surveys reveal that only one in six Australians can correctly interpret bar and pie charts. Thus it seems safe to assume that science educators face significant challenges. If meeting these challenges requires millions of colours per page, then so be it.

— Keith Lee

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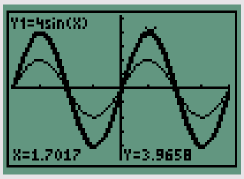

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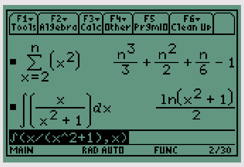

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