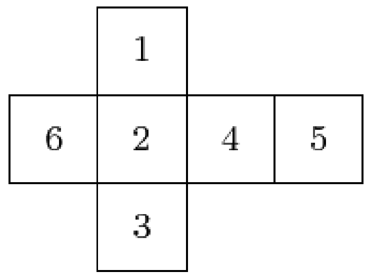
Question 1 **10 marks**

The following figure may be folded along the lines shown to form a number cube. What is the largest sum of three numbers whose faces come together at a corner?

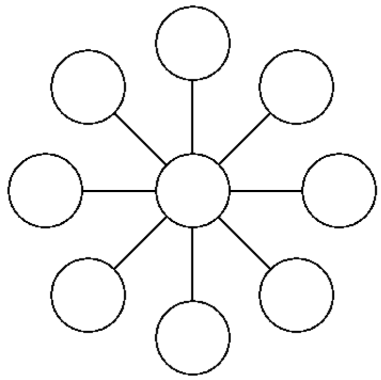


Answer: 14

The maximum sum of any three numbers would be $4 + 5 + 6 = 15$. But 4, 5 and 6 do not come together at a corner. On the other hand, 3, 5 and 6 do and so the maximum sum is $3 + 5 + 6 = 14$.

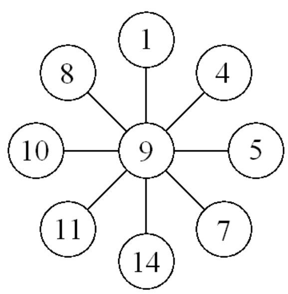
Question 2 **10 marks**

One rainy afternoon, Adib arranged the numbers $1, 4, 5, 7, 8, 9, 10, 11, 14$ into the circles shown, so that every line added up to the same value. What number was in the middle?



Answer: 9

Notice that $1 + 14 = 4 + 11 = 5 + 10 = 7 + 8 (= 15)$. This leaves 9, which has to be in the middle. Thus each line adds up to 24. Here's a possible solution:



Question 3 **10 marks**

Walking through the University of Melbourne during Open Day, the seven dwarves stumble upon a \$10 note. They decide to put the money to good use and buy themselves seven decaf soy lattes. There is a short wait while the barista works her magic. When the coffees arrive the dwarves pay using the note and receive one coin in change. Since the coffees are so excellently brewed, they leave their one coin as a tip. What is the price of a decaf soy latte?

Answer: \$1.40

There are only 6 different Australian coins to try. Of these, only the 20-cent coin gives a number divisible by 7, that is $\$9.80 \div 7 = \$1.40$.

Question 4 **10 marks**

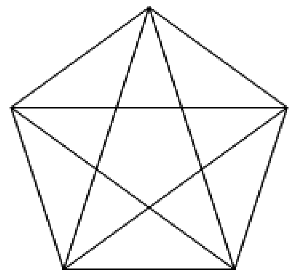
If it takes Tim two minutes to transcribe three trick questions, and it takes Tom three minutes to transcribe two trick questions, then if the two (that's Tim and Tom) tried transcribing thirteen trick questions, how long would it take them?

Answer: 6

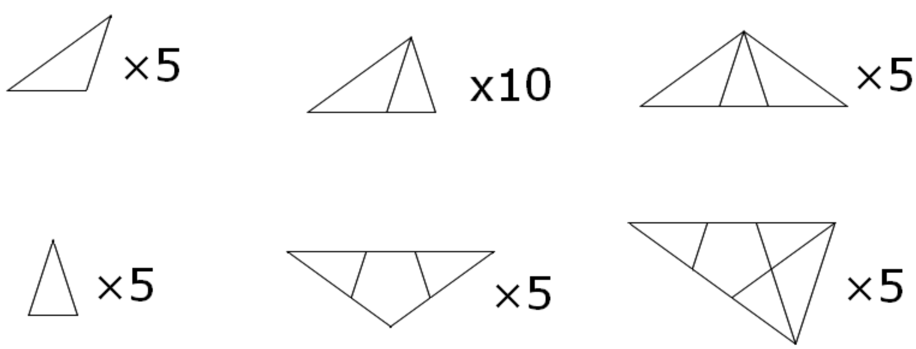
Each minute, Tim transcribes $\frac{3}{2}$ questions and Tom $\frac{2}{3}$ questions. So they can transcribe a total of $\frac{3}{2} + \frac{2}{3} = \frac{13}{6}$ questions per minute. This gives 6 minutes for 13 questions.

Question 5 **CHANGE RUNNER NOW** **10 marks**

How many triangles are there in the following diagram:



Answer: 35



Question 6 **10 marks**

If n points are placed at random inside a unit cube, what is the lowest n guaranteeing that two of these points are no more than $\frac{\sqrt{3}}{2}$ apart?

Answer: 9

If you partition the cube into 8 smaller half-cubes, the pigeonhole principle states with 9 points, there must two points in the same half-cube, and thus no more than $\frac{\sqrt{3}}{2}$ apart. On the other hand, 8 points does not guarantee the condition since you could place a point at each corner of the cube.

Question 7 **10 marks**

Evaluate $2009 - 2008 + 2007 - 2006 + ... + 3 - 2 + 1$.

Answer: 1005

Each pair of numbers (e.g. $2009 - 2008$) gives 1, and there are 1004 such pairs. So $(1 + 1 + 1 + ... + 1) + 1 = 1005$.

Question 8 **10 marks**

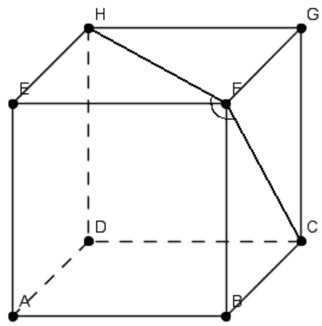
In how many ways can you arrange the letters of the word MATHS so that the letter M is always before (but not necessarily adjacent to) the letter S?

Answer: 60

There are $5! = 120$ ways of arranging the letters. By symmetry, M comes before S exactly half the time, so $120 \times \frac{1}{2} = 60$.

Question 9 **10 marks**

If ABCDEFGH is a cube, find the angle $\angle CFH$.



Answer: $60°$

CF, FH and CH are all diagonals across a cube face, and have the same length. Thus $\triangle$CFH is an equilateral triangle, and $\angle CFH = 60°$

Question 10 **CHANGE RUNNER NOW** **10 marks**

What is the largest number such that if you add 36 to the number written backwards, you get the original number back?

Answer: 95

By using some number theory, it is possible to show this cannot hold for three digits or more (proof omitted for conciseness).

So let $10a+b$ be a 2-digit number satisfying this. Then $10b+a+36=10a+b \Rightarrow b+4=a$. We want to maximise a, so here we let $a=9$ and $b=5$.

Question 11 **20 marks**

For not doing enough math problems, Han's teacher punishes him by making him write the Roman numerals up to 100: I II III IV V VI VII VIII IX X ... XCVII XCVIII XCIX C

How many times does he write the letter X?

Answer: 150

It is probably easier to break it down into sets of 10, i.e.:
1 to 9: 1 X
10 to 19: 11 X's
20 to 29: 21 X's
30 to 39: 31 X's
40 to 49: 11 X's
50 to 59: 1 X
60 to 69: 11 X's
70 to 79: 21 X's
80 to 89: 31 X's
90 to 99: 11 X's
Total: 150 X's

Question 12 **20 marks**

There exists two 2-digit numbers such that the difference between their sum and their product is 2009. What is the larger 2-digit number?

Answer: 68

We want to find integers a, b such that $ab - a - b = 2009$.
Then $2010 = ab - a - b + 1 = (a - 1)(b - 1) = 67 \times 30$.
So the two 2-digit numbers are 68 and 31.

Question 13 **20 marks**

People from a small island with 3000 inhabitants all get married exactly once in their life, and all to another inhabitant of the island. If each wedding has exactly 68 guests at it, in how many weddings will the average inhabitant participate in their lifetime?

Answer: 35

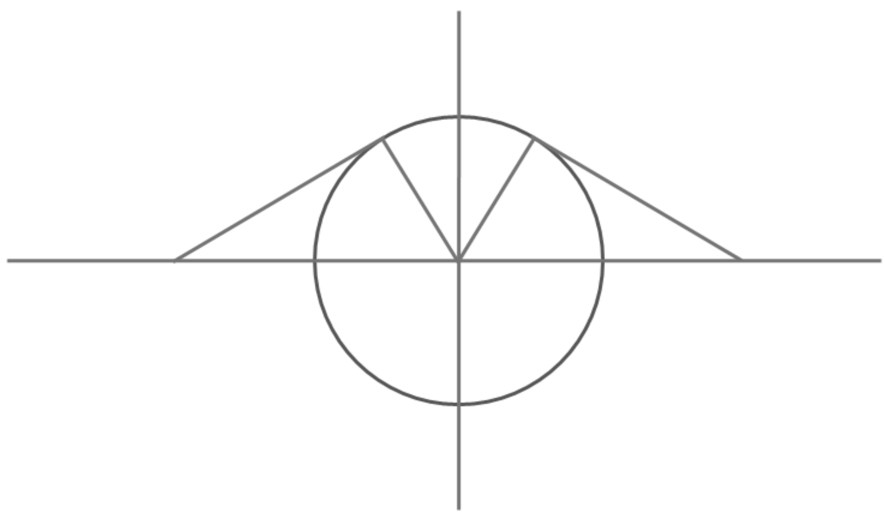
There are 1500 weddings. Total number of guests will then be 68×1500. Thus, on average each inhabitant will appear $\frac{68 \times 1500}{3000} = 34$ times as a guest. Add one more to include the wedding in which he/she got married brings it to 35.

Question 14 **20 marks**

Han stands at $(-2, 0)$ on a plane. Han's cake sits at $(2, 0)$ on the plane. Centred at $(0, 0)$ there is an impenetrable circle of radius 1. If Han is starving and infinitesimally thin, how far must he walk to reach his cake?

Answer: $2\sqrt{3} + \frac{\pi}{3}$

The trick is to minimize the distance. This means lines must come in tangent to the circle as shown in the diagram. The diagonal length can be calculated using Pythagoras to give $\sqrt{3}$. Furthermore, each of the triangles is actually half an equilateral triangle, so the angle subtended by the middle arc is $\frac{\pi}{3}$. Thus the total distance is $2\sqrt{3} + \frac{\pi}{3}$.



Question 15 **CHANGE RUNNER NOW** **20 marks**

A restaurant offers 10% off the price of a bill that costs at least 10 dollars, 20% off the price of a bill that costs at least 20 dollars, and so on. Everything on the menu costs an integer number of dollars. What is the most you will pay?

Answer: \$29.50

The range of choices is limited since the original price must end with a 9 to maximize the amount you will pay. It is then possible to list out every possibility:
100% of \$9 = \$9
90% of \$19 = \$17.10
80% of \$29 = \$23.20
70% of \$39 = \$27.30
60% of \$49 = \$29.40
50% of \$59 = \$29.50
40% of \$69 = \$27.60
30% of \$79 = \$23.70
20% of \$89 = \$17.80
10% of \$99 = \$9.90
So the most you will pay is \$29.50 (i.e. when the bill is really \$50).

Question 16 **20 marks**

Imagine the digit 7 has been banned from use. So the "seventh number" is written using the symbol 8. How would the "two-thousand-and-ninth number" be written?

Answer: 2682

This is simply working in base 9, where the digit-set does not have 7, i.e. $\{0,1,2,3,4,5,6,8,9\}$. Converting 2009 from base 10 to base 9 normally gives 2672. Then since the digit 7 is banned, it has to be written as 2682 as per the digit-set.

Question 17 **20 marks**

$x = \pm 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 \pm 7 \pm 8 \pm 9 \pm 10$. How many possible values can x take?

Answer: 56

Notice that x is the sum of 5 odd and 5 even numbers and has to be odd. Furthermore, x must be at most 55 (by making all positive) and at least -55 (by making all negative). It's not too difficult to see that x can take all odd values between -55 and 55 inclusive, which is a total of 56 possible values.

Question 18 **20 marks**

A box contains 3 black balls, 3 red balls, and 3 yellow balls. Three of these are selected at random. What is the probability that they are all different colours?

Answer: $\frac{9}{28}$

There are 9 out of 9 ways of picking the first ball. There are 6 out of 8 ways of picking the second ball (since there only 8 balls left, of which 6 are of a different colour to the first). There are 3 out of 7 ways of picking the third ball (since there are only 7 balls left, of which 3 are of a different colour to the first two).

Thus the probability is just $\frac{9}{9} \times \frac{6}{8} \times \frac{3}{7} = \frac{9}{28}$.

Question 19 **20 marks**

An equilateral triangle has vertices at $(0, 0)$, $(a, 5)$ and $(b, 13)$. Find the area of the triangle.

Answer: $43\sqrt{3}$

Treat the points as points on an Argand diagram. Then cis $60°$ represents the rotation, so that: $(a + 5i)(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = (b + 13i) \Rightarrow 5 + a\sqrt{3} = 26 \Rightarrow a = 7\sqrt{3}$.

Then $L^2 = a^2 + 5^2 = 172$, where L is the length of the triangle's side.
Finally, area $= \frac{1}{2}L^2 \sin 60° = 43\sqrt{3}$.

Question 20 **CHANGE RUNNER NOW** **20 marks**

How many integers between 1 and 2009 inclusive can be written as the difference of two positive squares?

Answer: 1505

It's not too hard to check that 1, 2 and 4 cannot be written as the difference of two positive squares. Every odd number $(2n+1)$ can be written as $(n+1)^2 - n^2$, where $n \geq 1$. Every multiple of 4 greater than 4, i.e. $(4n+4)$ can be written as $(n+2)^2 - n^2$, where $n \geq 1$. The only other case is $4n+2$. Let $4n+2 = (a+b)(a-b)$, which is divisible by 2 but not 4. Then one of $(a+b), (a-b)$ must be odd and the other even. But then $(a+b)+(a-b) = 2a$ will be odd, contradiction. It follows that exactly 1505 numbers are expressible as the difference of two positive squares.

Question 21 **30 marks**

Julia asks Lu to think of a 3-digit number (abc) where a, b, and c represent digits in base 10 in the order indicated. Julia then asks Lu to form the numbers (acb), (bca), (bac), (cab), and (cba), to add these five numbers, and to reveal their sum, N. Lu tells Julia that $N = 2609$. Play the role of Julia and determine the original 3-digit number (abc).

Answer: 943

Let m be the original 3-digit number m $= (100a + 10b + c)$.
We are given that $N = 122a + 212b + 221c$. Then $N + m = 222(a + b + c) = 2609 + m$.
Then $2609 + m$ must be a multiple of 222, which gives us $m = 55, 277, 499, 721, 943$.
Checking all of those shows that 943 is the only one that satisfies $N = 2609$.

Question 22 **30 marks**

A security door keycode consists of 3 digits (0 to 9). The keys have no memory, so that pressing 4705 will unlock a door with code 705. However, you do know for certain that the three-digit code does not begin with a 5. What is the minimum number of keys you have to press in order to be assured of cracking the code?

Answer: 999

We assert that there must be at least 99 5's, and at least 100 each of every other digit. By explicit construction, we show that such a string of numbers does exist that satisfies the problem and has a length of 999, which is:

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81790913635783152117235345570553679541318573815761778539057505312959631307
33657167730395075503624546818023860766278089002500812409686302836021622308
45020508674041868528810261228584052005862904636807886526672803400700047482
70227460328696425446706650429822522296087864142094625660542432207224108236
46920496201666160092487202724658281464704412061001712397213961992375839383
71453567178977639169978753893387190351217347718911497325334333219585176739
27134119473703398337695301374715659305198113859529934995637921510935019361
33095749394951879765154355693163354579439945132971015985514311835907998999
01262342218461442320834883264580676289226346694287088988876403062128472689
66492825884388214085626784276841644782038483826908011161150979877527796537
81914759417561554793275727910373691925996751655929372577291587314197099125
11059743770777415323196975491701486188090249844906874265604300698663804524
43444568292601048506436683040744894401824760654805648168854029484940632421
0609800317459181352331071127358955255
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Question 23 **30 marks**

Yi, Han and Sam have a snowball fight, whereby anyone hit in the face is knocked out, and the last man standing wins. Yi targets Han if he's still standing, and otherwise targets Sam. Han targets Sam if he's still standing, and otherwise targets Yi. Sam targets Yi if he's still standing, and otherwise targets Han. They take turns launching snowballs in the order Yi, Han, Sam (skipping anyone knocked out). If each has 50% chance of a critical hit with any snowball, what's the probability that Yi wins?

Answer: $\frac{8}{21}$

If there are two players, then $p_1 = \frac{1}{2} + \frac{1}{2}(1 - p_1)$, so $p_1 = \frac{2}{3}, p_2 = \frac{1}{3}$.

Since there are three players, $p_1 = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2}p_3$, and $p_2 = \frac{1}{2}p_1$.

Now, $1 = p_1 + p_2 + p_3 = \frac{3}{2}(\frac{1}{2} \times \frac{1}{3} + \frac{1}{2}p_3) + p_3 = \frac{1}{4} + \frac{7}{4}p_3 \Rightarrow p_3 = \frac{3}{7}$.

Finally, $p_1 = \frac{1}{6} + \frac{3}{14} = \frac{8}{21}$ as required.

Question 24 **30 marks**

Chris has a triangle with side lengths 26, 28, 30. He draws lines connecting the midpoints of each side. If he then folds these lines and connects the vertices to form a tetrahedron, what is the volume of this tetrahedron?

Answer: $42\sqrt{55}$

The base is a triangle of side lengths 13, 14, 15. Let the three points lie on the xy-plane, i.e. at coordinates $(0,0), (5,12), (14,0)$, so that area $A = 84$. Some algebraic manipulation shows that the height of the tetrahedron will then be $h = \frac{3\sqrt{55}}{2}$.

Thus, volume $= \frac{1}{3}Ah = \frac{1}{3} \times 84 \times \frac{3\sqrt{55}}{2} = 42\sqrt{55}$.

Question 25 **30 marks**

Find a 9-digit number, made up of only the digits 2 and 9, that leaves a remainder of 29 when divided by 2^9.

Answer: 229299229

Assume the last 2 digits are 29. Then we only need to find a 7-digit number divisible by $2^7 = 128$. Consider this 7-digit number. It has the following properties:
It must be divisible by 2, so its last digit must be divisible by 2. (i.e. 2).
It must be divisible by 4, so its last digit must be divisible by 4 (i.e. 92).
Repeating this process another 7 times, each time adding one more digit in front, gives the 7-digit number 2292992. Thus our 9-digit number is 229299229.