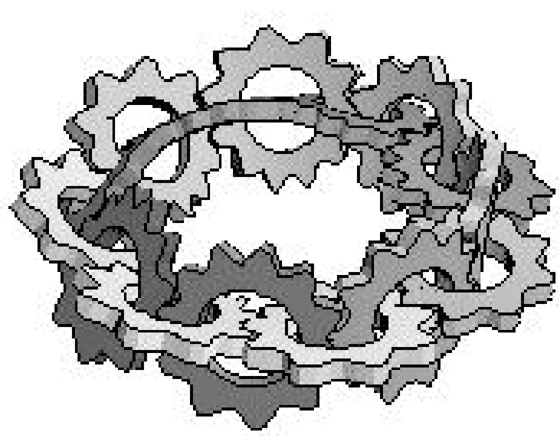
Paradox

Issue 2, 2008

THE MAGAZINE OF THE MELBOURNE UNIVERSITY MATHEMATICS AND STATISTICS SOCIETY



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Paradox

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PRINTED:	August, 2008

Words from the Editor

Welcome to this issue of Paradox, the magazine of the maths and stats society. As usual, in this issue we have amusing stories, deep quotes, and maths jokes that are worse than ever.

If you find the usual Paradox problems too easy, we have a list of unsolved, yet easy to understand, problems for you to try (in fact we included one such problem in the last edition, and to everyone's disappointment, no one was able to give a solution). We are very grateful to Dr Chris Ormerod for writing an interesting article on cellular automata. And finally we have devised our own lecture bingo, with the hopeful aim of brightening up your classes.

This issue, being only the second of the year, comes out rather late. One reason for this is the difficulty to obtain articles. So I once again urge you, the readers, to submit any material suitable for Paradox so this magazine may continue to thrive.

— James Wan

Words from the President

Let me ask you a question: what do committees, couches and T-shirts have in common?

Let me follow that up with more questions: are you passionate about proactive procrastination? Do you marvel at mathematics? Is not knowing the answer to the first question simply eating you up inside? If you answered yes to any of the above, then pay close attention.

STOP.

Now backtrack to where you picked up this issue of the Paradox.[1] Look past the metallic wire rack bursting full of mathemagical goodness, and you see the open door just beyond. Walk towards the light, and step into the MUMS room. This is where the magic happens, this is where a thousand monkeys typing random words spin out coherent seminars and Paradox articles; this is where a community of students miraculously put together the MUMS Puzzle Hunt, the Maths Olympics, regular seminars,

[1]Ignore this if you found it online.

all the while not seeming to do any work; and this is the only place where you'll find out the answer to the question at the start.

But enough about MUMS, you started reading this section for words, and words you shall have: Swifter, Higher and Stronger. Each year, from the venerated heights of the Michell Theatre seats, men and women from all walks of life join as one to celebrate (and cerebrate) the pinnacle of human achievement: the University Maths Olympics. Neither rain nor hail shall stop this hallowed indoor event, as mathletes young and old race to be the team to solve the most questions. So, come September, sign up a team of five for the UMO and experience it first hand.

— Yi Huang

Puzzle: what is wrong with this argument?

$$x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

$$1 + \frac{1}{x} + \frac{1}{x^2} + \cdots = \frac{x}{x-1}$$

Adding these up, we get

$$\cdots + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + x^3 + \cdots = 0.$$

Unsolved Problems

Problem 8 on the last edition of Paradox reads,

> Find 3 ways to write 3 as the sum of 3 3rd powers of integers.

Two of the ways are obvious, namely $1^3 + 1^3 + 1^3$ and $4^3 + 4^3 + (-5)^3$, while the third is more elusive. In fact, it is so elusive that by finding it, your fame and fortune would increase slightly beyond the $5 offered, for at the time of writing, this problem is unsolved.

The allure of problem solving is such that deceptively easy to understand problems can withstand the brightest minds and most concerted efforts. Below is a loose collection of problems that require almost no knowledge to understand (although a kindergarten, and sometimes even high school, education is preferred), but have remained unconquered. Who knows, maybe one of them may later become associated with your name. The author thanks Norm Do for compiling much of the list.

More Cubes. We know quite a bit about writing integers as sums of powers of other integers. For example, every positive integer is the sum of 4 squares. In fact, this generalises, and we know that every positive integer is the sum of at most 9 positive cubes (the number 23 actually requires 9 cubes). We also know that if we use signed cubes, then we need a sum of at most 5 terms. The open problem here is whether this can be improved to 4.

The easiest case is when the number is a multiple of 6, for then we can write it as a sum of 4 cubes: $6x = (x+1)^3 + (x-1)^3 - x^3 - x^3$. The other numbers are not so easy, for instance we don't even know if 33 can be written as the sum of 4 cubes.

Incidentally, it is not known if every integer is a sum of at most 3 signed squares.

A perfect cuboid is defined to be a box with all sides, diagonals and long diagonals integers. Of course, there is a problem with this, because we don't know if such a thing exists.

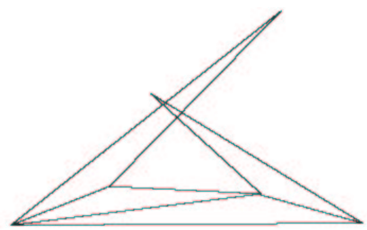
If we drop the long diagonal condition, then the shape is known as an Euler brick. It can be checked that there are infinitely many Euler bricks, for we take (a,b,c) to be a Pythagorean triple, and then $(a(4b^2-c^2), b(4a^2-c^2), 4abc)$ gives you the side lengths of an Euler brick.

Magic Squares. A magic square of order n is an $n \times n$ array of numbers such that the rows, columns and diagonals add up to the same number. Often we restrict the numbers to $1, 2, \ldots, n^2$. It is an unsolved problem to determine the number of distinct (excluding those obtained by rotation and reflection) magic squares of an arbitrary order. It is clear that there is 1 magic square of order 1, and that there is no magic square of order 2. A bit of thought reveals that there is only 1 distinct 3×3 magic square, once you've figured out to put the 5 in the middle and 1 in an adjacent square.

Even the number of 6×6 squares is not known, and is estimated to be 1.8×10^{19} using Monte Carlo simulation and methods from statistical mechanics.

Polygonal Nets. Shephard's conjecture states that all convex polyhedra have nets, that is, a self non-overlapping unfolding. Amazingly, this has not been proven.

Interestingly, there are convex shapes that, if you are not careful in the unfolding, yield overlapping ‘nets’. An example is given below.



The result is not true for non-convex polyhedra, for example, consider a shape made by gluing a small cube onto a large cube.

Irrationality. Although both π and e are irrational (not expressible as a fraction) and actually transcendental, it is not know if $\pi+e$ or $\frac{\pi}{e}$ is irrational. It is also unknown whether π and e are algebraically independent, i.e. whether there is a polynomial relation between them with rational coefficients. The Euler-Mascheroni constant, γ, is defined to be the limit of $1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n$ as $n\to\infty$. Its numerical value is $0.577\ldots$, incidentally very close to $\frac{1}{\sqrt{3}}$. It is not known if this number, which appears ubiquitously in maths, is irrational. Hilbert mentioned this problem as “unapproachable” and in front of which mathematicians stand helpless. G. H. Hardy is alleged to have offered to give up his chair at Oxford to anyone who can prove its irrationality.

Primes. Apart from the well known Goldbach Conjecture (that every even number >2 is the sum of 2 primes) and the Twin Prime Conjecture (that there are infinitely many pairs of primes 2 apart), there is a whole range of unsolved problems regarding prime numbers.

A prime p is said to be a Sophie Germain prime if both p and $2p+1$ are prime. Historically these numbers are important in proving cases of Fermat’s Last Theorem (alas, this has been solved). It is not known if there are infinitely many Sophie Germain primes.

Let n be a natural number. Are there infinitely many primes of the form n^2+1? I don’t know, and you probably don’t either. Also, we don’t know if there is always a prime between n^2 and $(n+1)^2$, a conundrum known as Legendre’s conjecture. We do know that there is always a prime between successive 4th powers.

Are there infinitely many palindromic primes? (Clearly all such primes must have an odd number of digits, otherwise 11 divides it.)

Are there infinitely many Fibonacci numbers that are also primes?

A balanced prime is a prime number that is equal to the arithmetic mean of the nearest primes above and below. ($5) Are there infinitely many?

Is there any value of n other than 1, 2, and 4, such that $n^n + 1$ is a prime?

Are there infinitely many primes of the form $n! - 1$?

$n^2 + n + 41$ is prime for integers $0 \leq n \leq 39$. Are there infinitely many primes of this form?

If p is a prime, is $2^p - 1$ always square free?

A repunit prime is a prime number that consists entirely of 1's in its base 10 representation. The smallest example is 11, and the next smallest is a string of 13 1's. It is easy to see that the length of any repunit prime must be a prime, for if it had a proper factor, we can take a repunit of the length of a factor, which will divide the "prime". Currently the largest repunit prime has 1031 1's. We don't know how many such primes there are.

Gilbreath's conjecture: we list all $\aleph_0$ primes, take the absolute difference of each pair of adjacent terms, which form a new sequence. Repeat. Do all new terms start with a 1?

Perfect Numbers. A perfect number, such as 28, has the property that the sum of all of its divisors is twice the number. We don't know if any odd perfect numbers exist. We also don't know if there are infinitely many perfect numbers, though this is related to the number of Mersenne primes (primes of the form $2^n - 1$).

A quasiperfect number n has sum of divisors $2n + 1$. No such number is known.

Zeta Function. The Riemann Zeta Hypothesis is too difficult to state here; remember we are aiming at the kindergarten level. However, the zeta function restricted to the positive integers is defined to be $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. For s even, we can evaluate the right hand side exactly (using complex integrals or whatever), and get rational multiples of powers of π. For instance, $\zeta(2) = \frac{\pi^2}{6}$. However, all methods fail when s is odd, and we hardly know anything about the exact forms of these values. It has been proven that $\zeta(3)$ is irrational, though we don't know if it is transcendental, or if any other odd zeta values are irrational.

Geometry. Here are a list of deceptively simple problems, vaguely all having some geometric meanings.

Is there a point in the plane that is a rational distance from each of the four corners of a unit square?

Is there a triangle with integer sides, medians, and area?

How many points can you find on the (half) parabola $y = x^2, x > 0$, so that the distance between any pair of them is rational?

Kabon triangle problem: How many disjoint triangles can be created with n lines?

It is not known if a single aperiodic (i.e. can only tile the plane non-periodically) tile can cover the plane.

Can any set of disks of total area 1 can be packed into a disk of area 2?

Is there a set S in the plane such that every set congruent to S contains exactly one lattice point (i.e. point with integer coefficients)?

Is there a constant A such that any set in the plane of area A must contain the vertices of a triangle with area 1?

It is not known if there are any necessary and sufficient conditions for a Hamiltonian circuit to exist; a Hamiltonian circuit on a graph is a loop that visits every vertex exactly once, traveling along edges. In contrast, there is a simple necessary condition for the existence of an Eulerian circuit, which visits every edge.

Does every simple (no self intersection) closed curve in the plane contain four points forming the vertices of a square?

It is easy to see that a semi-circle with radius 1 can contain any curve of length 1 (if you're allowed to rotate and move the curve around) – but what is the area of the smallest simply connected (no holes) set that can contain every curve of length 1?

Will all the $\frac{1}{k} \times \frac{1}{k+1}$ rectangles, for integer $k > 0$, fit together inside a 1×1 square? (It is an easy telescoping exercise to see that the areas agree.)

Egyptian Fractions. The Egyptian algorithm converts a rational number to a sum of reciprocals of positive integers. It is so named as the ancient Egyptians wrote fractions that way. Incidentally, they had a special symbol for $\frac{1}{2}$, and also used $\frac{2}{3}$ and $\frac{3}{4}$, although they are not reciprocals.

The algorithm takes a rational number, looks at the fractional part f, and finds the biggest reciprocal $\leq f$ (take the integer part of $\frac{1}{f}$), and calculates the difference. Now repeat. For instance, $1\frac{11}{28} = 1 + \frac{1}{3} + \frac{1}{17} + \frac{1}{1428}$. This method can produce huge (hundreds of digits) numbers in the denominator for innocent looking fractions. It is an interesting exercise to see that the algorithm always terminates.

Now, does the Egyptian algorithm always succeed in expressing a fraction with odd denominator as a sum of unit fractions with odd denominator?

Another open problem is the Erdös-Straus Conjecture: Is it true that the fraction $\frac{4}{n}$ can be written as a sum of three or fewer distinct unit fractions for $n > 2$?

Games. The minimum number of givens to render a unique solution to the standard sudoku is unknown; it is probably 17.

The minimum number of turns required to solve the Rubik's cube for an arbitrary starting position is not known, although it is bounded from above by 29.

The minimal number of moves required to order n disks on four rods in the Tower of Hanoi is unknown for general n, thought it is easy to verify that the sequence starts with $1, 3, 5, 9, 13, 17, 25, \ldots$

Diophantine Equations. The following are all unsolved problems asking for integer solutions.

Brown numbers are pairs of integers such that $n! + 1 = m^2$. Only three such numbers are known: $(4,5), (5,11), (7,71)$. It is conjectured that these are the only three such pairs.

A Diophantine n-tuple is a set of n positive integers such that the product of any two is one less than a square integer. Does there exist a Diophantine 5-tuple?

Can you find x, y, z, such that $(x+y+z)^3 = xyz$?

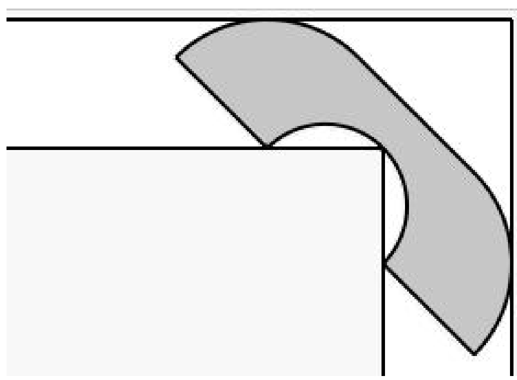
Miscellaneous. Take any positive integer of two digits or more, reverse the digits, and add to the original number. Now repeat the procedure with the sum. This procedure quickly produces palindromic numbers for most integers. The palindromic value for 89 is especially large, being 8813200023188. The numbers not known to produce palindromes are called Lychrel numbers. The first is 196, for which millions of digits have been computed.

Are there only finitely many perfect squares with just two different nonzero decimal digits?

A family of finite sets is said to be union-closed if the union of any two members of the family is also a member of the family. In a finite union-closed family of sets, must some element appear in at least half of the sets? (It is simple to check that if the set is the power set of $1,\ldots,n$, then every element appears in exactly half of the sets.)

Does every graph with minimum degree 3 have a cycle whose length is a power of 2?

The moving sofa problem asks for the rigid two-dimensional shape (a 'sofa') of largest area A that can be maneuvered through an L-shaped planar region with legs of unit width. As a semi-circle of unit radius can pass through the corner, a lower bound for the sofa constant $A = \frac{\pi}{2}$ is readily obtained. The true value for A is yet to be revealed.



A considerably better lower bound of $\frac{2}{\pi} + \frac{\pi}{2}$ is based on the shape above, consisting of two quarter unit circles on either side of a $1 \times \frac{4}{\pi}$ rectangle, from which a semicircle of radius $\frac{2}{\pi}$ has been removed. To see that the removed figure is a circle, note that the distance between the two inner contact points with the 'wall' is fixed, and the 'wall' always cuts out a right angle. To see that the width of the rectangle is $\frac{4}{\pi}$, we let a be its width and note the area of the rectangle minus the circle is $a - \frac{a^2\pi}{8}$, with a maximum at $a = \frac{4}{\pi}$.

The nested radical constant is $\sqrt{1+\sqrt{2+\sqrt{3+\cdots}}}$. It is easy to check that it converges, but no closed-form expression is known.

We close with a problem that the average person can contribute to. Sierpinski numbers are integers k such that $2^n k+1$ is composite for all n. It is surprising that such numbers exist at all, let along that in fact there are infinitely many of them. It is conjectured that 78557 is the smallest one. There were 17 candidates for the smallest number, and a computing effort known as "seventeen or bust" was carried out to eliminate them. At the time of writing the only remaining candidates are 10223, 21181, 22699, 24737, 55459, and 67607. One only needs to produce an n such that $2^n k+1$ is prime for one of the above k to eliminate it. Of course, such n's are likely to be huge.

Quotes 1

“I take space to be absolute.” – Newton

“I hold space to be something purely relative as time is.” – Leibniz

∞

“Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half.” – Leibniz

Newton called Leibniz a fraud.

∞

“In conclusion I wish to say that in working at the problem here dealt with I have had the loyal assistance of my friend and colleague M. Besso, and that I am indebted to him for several valuable suggestions.” – Albert Einstein

“You had, by the way, overestimated the meaningfulness of my observations again. . . what is certain is that I was not aware of this consequence of my comments and cannot grasp the argument even now.” – Michele Besso

∞

“No more fiction for us: we calculate; but that we may calculate, we had to make fiction first.” – Nietzsche

∞

“The advancement and perfection of mathematics are intimately connected with the prosperity of the state.” – Napoleon

∞

“I have yet to see any problem, however complicated which, when looked at the right way, did not become still more complicated.” – Paul Anderson

A question on the British matriculation, 1896: find the prime factors of 5679431432056743205685679432.

Cellular Automata

1 Introduction

Cellular automata (CA), broadly speaking, are dynamical systems that are discrete in time, space and state. Cellular automata feature ubiquitously in scientific modelling. There are CA models of cancer, tissue regrowth, HIV/AIDS, fire, traffic, stock markets, fluid mechanics, solitons, qubits, bees and many more obscure (and not so obscure) topics. The modelling aspect of cellular automata seems to have one common thread; a seemingly obligatory citation of the work of Wolfram [5]. The sheer simplicity of programming a computer to compute CA makes it a fantastic medium to do simulations of complex behavior. In the literature, CA models have been programmed in everything from machine code to Microsoft Excel.

Instead of the usual programming a computer to simulate CA, we intend to demonstrate how a CA can simulate a computer. We will split this up into 3 proceeding sections. §2 will introduce some background material, mainly some definitions. §3 will introduce Conway's Game of Life along with a few building blocks we shall use in §4, which will focus on constructing configurations that will compute three main logical operations: NOT, AND and OR.

2 Definitions and general boring stuff

Let us give 3 definitions of a cellular automata. The first shall be our most basic definition, then we shall increase the complexity gradually. Each successive definition shall be a subclass of the previously defined system. We shall also present some examples of interesting systems that all go under the guise of CA.

Definition 2.1. A cellular automaton is a dynamical system that is discrete in time, space and state.

The prototypical example is when one takes the integers, $\mathbb{Z}$, and assigns a value of 0 or 1 (i.e. an element of $\mathbb{Z}_2$) at each point on $\mathbb{Z}$. At a time t, we represent the systems state by the bi-infinite sequence $(x_n^t)_{n\in\mathbb{Z}}$. We define the state of the system in the next time step, $t+1$, by some rule. The prototypical example being where the rule is such that the new bi-infinite sequence,$(x_n^{t+1})_{n\in\mathbb{Z}}$, is defined by the equation $x_n^{t+1} = f(x_{n-1}^t, x_n^t, x_{n+1}^t)$

where $f : \mathbb{Z}_2^3 \to \mathbb{Z}_2$. Given an initial condition, that is an initial bi-infinite sequence, (x_n^0), x_n^t is determined for all time by f. For our particular case, there are $2^{2^3} = 256$ choices of function. For example, $f(x_{n-1}^t, x_n^t, x_{n+1}^t) = x_{n-1}^t + x_{n+1}^t \mod 2$ is a simple rule. This example is called rule 90 in Wolframs class[5]. The evolution of this system with a very simple initial condition can be seen in figure 1.

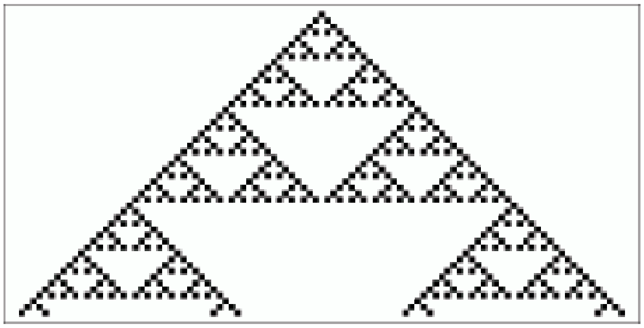


Figure 1: Rule 90 with an initial condition where $u_n^0 = \delta_{n0}$.

This example is classical in that this system is one that possesses the following additional properties:

1. The system is deterministic.
2. The number of states that any point can be in is finite.
3. The rule that defines the state of any point in the next time step depends only on the state of that point and the states of those points around it.

It is important to note that there are many examples of systems that are put in the class of CA that violate these properties. Some of these systems are very extensively studied in modelling. In particular, a class of CA called box-and-ball systems violate the last of these properties, locality, yet it is the focal point of a large body of research in mathematical physics. However, we will disregard such examples to give the following more formal definition of CA.

Definition 2.2. A cellular automata is sequence, $(C_t)_{t\in\mathbb{N}}$, defined a 5-tuple (L, Σ, U, f, C_0) where

1. L is a lattice.
2. $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$ is an alphabet.
3. $U = (u_1, \ldots, u_n)$ is some finite sequence of lattice elements.

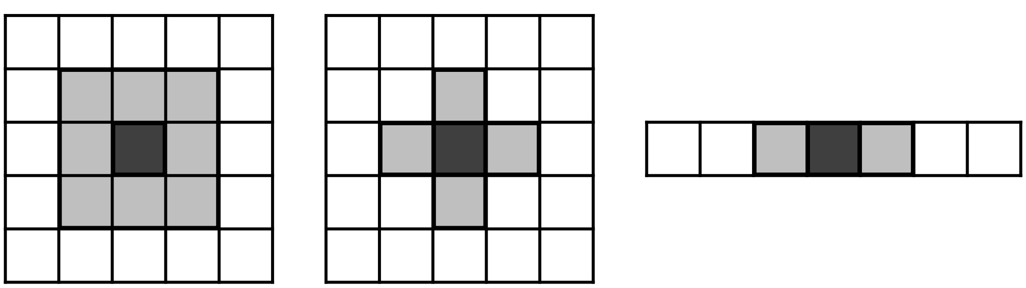
4. $f : \Sigma^U \to \Sigma$ is a function.

5. $C_0 : L \to \Sigma$ is some initial configuration.

where the sequence is specified so that for all $v \in L$, $C_{t+1}(v) = f(C_t(v+U))$.

If that seems a bit bewildering, let us take the previous example in this framework. Lattice in this context is just an additive group, like a grid. The lattice is the space on which this cellular automata exists, that is $L = \mathbb{Z}$. The alphabet is the states that each point in the space may be, that is $\Sigma = \{0,1\} = \mathbb{Z}_2$. The U defines the neighbourhood, that is what points are around another point, here $U = (-1,0,1)$. Now we can make the correspondence of notation by saying $x_n^t = C_t(n)$. We define the global transition that defines C_{t+1} by $x_n^{t+1} = C_{t+1}(n) = f(C_t(n-1), C_t(n), C_t(n+1)) = f(x_{n-1}^t, x_n^t, x_{n+1}^t)$.

Now we don't want to consider all lattices for a very good reason. Think about changing a single point's state, say $C_0(0)$, where 0 is the identity element of the lattice. Since the rule is acting locally, the only possible effect will be on the elements $C_1(U)$, then $C_2(U+U)$, and for arbitrary $t \in \mathbb{N}$, the effect of changing the state of $C_0(0)$ will only have an effect on $C_t(tU)$. Hence the effect of changing $C_0(0)$ will only ever affect the set generated by U. Hence, we will assume L is generated by U, hence, making it finitely generated. Before you start running for the hills, almost all examples you'll ever come across are where L is one of the groups with neighbourhoods appearing in figure 2.



(a) (b) (c)

Figure 2: The three most common groups with associated neighbourhoods: (a) $\mathbb{Z}^2$ with a Moore neighbourhood, (b)$\mathbb{Z}^2$ with a Neumann neighbourhood, (c) $\mathbb{Z}$ with a radial neighbourhood.

We finally add one more definition of cellular automata which is a minor extension of the definition above. Instead of a lattice, which is an abelian group, we now consider the case where it's just a group, G. We will now refer to the identity element as e instead of 0. Because the above argument works for any group, we may assume G

is a finitely generated group, moreover, generated by U. Without loss of generality, we may assume that $\{u_i\} = \{u_i^{-1}\}$. We can assume this because we can simply add elements to the end of U and extend the old transition function $f : \Sigma^U \to \Sigma$ so that it has no real dependence on any added values. The Cayley graph of a finitely generated group, G, associated with a generating set, $\{u_i\}$, is the graph, Γ, in which each element of G is a vertex and two vertices, $g_1, g_2 \in G$, have an edge joining them if $g_1 = g_2 u_i$ for some i. Because $\{u_i\} = \{u_i^{-1}\}$, we treat this as an undirected graph. We may endow G with a metric that comes from the Cayley graph, Γ [2]. Now we define the distance between g_1 and g_2, $d(g_1, g_2)$, to be the minimum path distance from vertices g_1 to g_2 on Γ. That is, two elements, $g_1, g_2 \in G$, are far away if you need to multiply g_2 by many elements of U before you equal g_1.

Now we define a metric on Σ^G by

$$d(C, C') = \exp\left(-\inf\{d(e,g) | C(g) \neq C'(g) : g \in G\}\right) \tag{2.1}$$

or 0 if $C = C'$, which in simple terms says that two configurations are close if they only disagree far away from e.

Definition 2.3. A cellular automaton is a sequence $(C_t)_{t\in\mathbb{N}}$ satisfying $C_{t+1} = F(C_t)$ where G is a finitely generated group, Σ is a finite alphabet and F is a continuous endomorphism of Σ^G that commutes with the group action.

Here, the continuity is interpreted in terms of the topology induced by (2.1). This setting is more natural from a topological dynamical point of view. This generalization was considered in the famous result of Machi and Mignosi who prove what's known as a Garden of Eden theorem. That is, a mapping is surjective if and only if it is pre-injective.

3 Conway's Game of Life and the Zoo

Now that we have defined what a cellular automaton is, we may introduce an important example, namely Conway's. The basic setting is where $\Sigma = \{0, 1\}$ and the space and neighbourhood coincides with (a) in figure 2. That is $G = \mathbb{Z}^2$ and U is the sequence of elements $u_i = (u_1^{(i)}, u_2^{(2)})$ such that $|u_1^{(i)}|^2 + |u_2^{(i)}|^2 \leq 2$ in any order. The function, simply put is

$$X^t_{(i,j)} = \begin{cases} 1 & \text{if } X^t_{(i,j)} \text{ and } \sum_{u_1 u_2 = 1} X_{(i,j)+u} = 2 \text{ or } \sum_{u_1 u_2 = 1} X_{(i,j)+u} = 3 \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

The reason given for an alive cell dying when it has more than 3 live neighbours is overcrowding, while the reason for a cell dying when it has fewer than 2 live cells is loneliness. A dead cell with 3 neighbours comes to life as if supported by its surrounding cells. We demonstrate the first few evolutions of a moderately random set of initial conditions in figure 3.

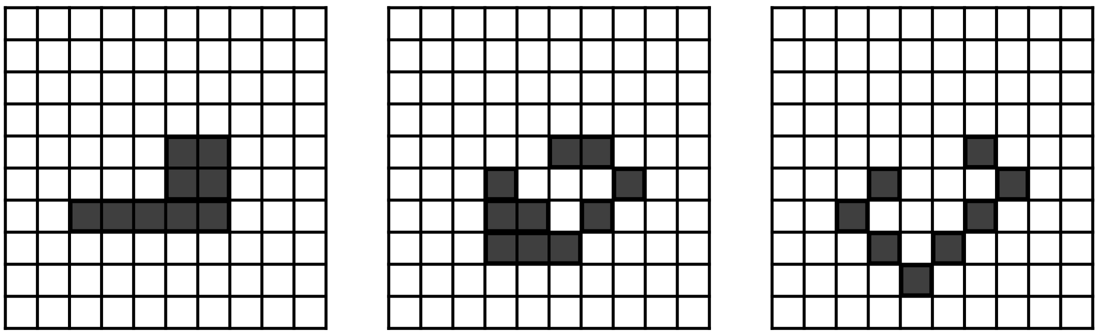


Figure 3: These are the first few states of a random set of initial conditions.

Now that we have a system, we wish to construct a configuration that will be able to compute things. To do this we are required to construct tools that we may put into our system. That is, we will find configurations that have desirable properties that we may exploit to do our calculations. We will split these up into a few categories:

Gliders: The basic object that we will use to transmit information from one point to another will be the glider. It is an object that undergoes some evolution, then returns to the same configuration with some sort of shift in position. The glider we shall use for our construction and the evolution of the glider is shown in figure 4.

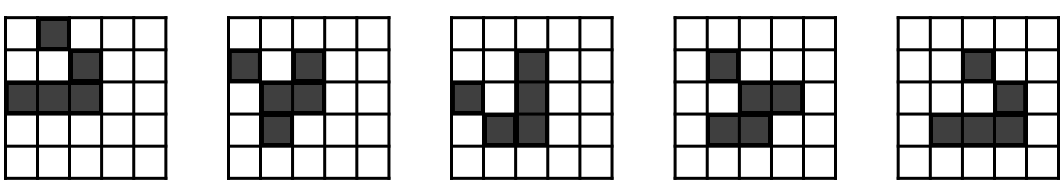


Figure 4: This little guy is the basic unit for our construction.

Guns: A gun is a object that shoots out a glider. That is, the configuration undergoes some evolution and produces a glider that shoots off in some direction. Figure 5 shows two time states, the initial time, and 30 time steps later in which the configuration is the same, however, there is a little glider that will shoot off diagonally down and to the left. Needless to say, the resulting glider does not interfere with other gliders or the original gun, hence, this gun produces a steady stream of gliders.

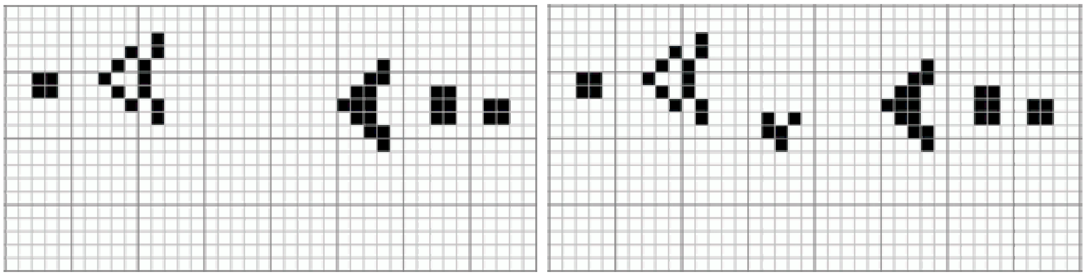


Figure 5: This is the P30 gun, at time t and $t+30$ respectively.

It is important to note that we may point two guns at each other, that is one down and to the left, and the other down and to the right, then the resulting gliders will destroy each other.

Blocker: A Blocker is an object that is able to withstand attacks from a glider gun. That is, the glider get shot towards a blocker, and then undergoes some evolution, then returns to the configuration of the blocker. Essentially, it destroys a glider. Figure 6 shows a basic blockers interaction with our glider. We will also associate a variable with a blocker and a Gun. The truth of this block is indicated in figure 6 by the light grey cell. If the cell is alive, then that variable is true and the blocker dies leaving the gun to emit a stream of gliders, when the cell is dead, the blocker stays alive to destroy the steam of gliders.

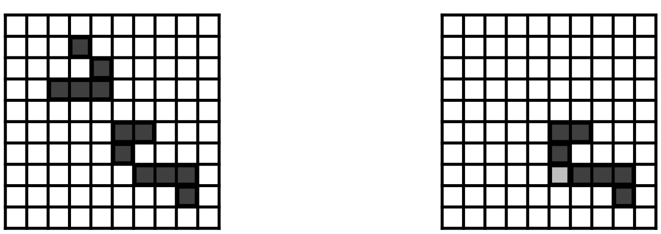


Figure 6: This is step t and $t+5$ of the interaction between a blocker and a glider. At time $t+5$ we have marked the single square we will use as input.

Accepter: The Accepter basically acts like a blocker, however, once it is hit by a glider, it undergoes some evolution and ends up in an altered state. We shall use a simple 2×2 box which will get destroyed by a glider (when hit in a certain way) along with a blocker. The output can be interpreted as true if the box is destroyed or not. To concatenate logical expressions, we shall remove the accepter allowing the glider stream to pass through.

It should be mentioned that there are whole classes of interesting objects we do not require for our calculations and will not give examples of, but are worth mentioning nonetheless because they are fun:

Puffers: These objects act like gliders in that they reproduce their original configuration with some shift, however, they leave a residue of periodic or stable objects. There are some massive puffers available to download.

Spaceships: Well, technically, there is no difference between a spaceship an a glider, however, when gliders get massive, it seems somewhat more apt to call such objects spaceships.

Breeders: Sort of like the puffers, however, they leave a residue of guns.

There are a few people who collect configurations[6]. Various people engineer objects in life to compete for various titles, such as slowest ship, fastest glider, smallest Garden of Eden configuration. I believe the largest constructed object is Gabriel Nivasch's 4,195 by 330,721 cell ship with an initial population of 11,967,399 alive cells, and is the only known object that travels at 0.378 times the speed of light (light here moves one square per time unit).

Other interesting objects include Dean Hickerson's massive (approximately) 1500 cell by 1000 cell prime calculator which is essentially a gun that produces a glider at time PN, for some period P, if and only if N is prime. Also worth mentioning is Paul Rendell's (approximately) 1600 cell by 1600 cell Turing machine. Many of these configurations will run on Life32, which is a nifty little program by Johan Bontes[1]. The site also has links and a few megs of engineered configurations to play around with.

4 Constructing logical gates

We will now give some details as to how to construct three main logical gates: AND, OR and NOT. These are the basic gates required to do any numerical calculation. One can concatenate these objects to build larger objects and do any single calculation. However, we must say, this will only do a set of binary calculations and the resulting construction does not achieve all that is necessary to be universal (that is, to be able to simulate a Turing machine).

Instead of drawing all the states of every cell, we shall use the following notation:

1. Gun and Glider stream:

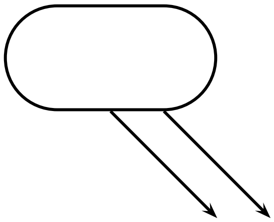


Figure 7: Diagramatic representation of a glider gun and stream of gliders.

2. Blocker associated with a variable *A*:

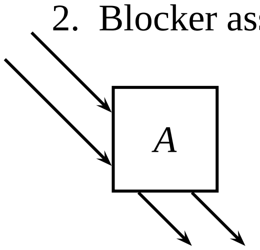


Figure 8: Diagramatic representation a blocker associated with a variable *A*. A generic blocker associated with no variable will have nothing coming out.

3. An accepter:

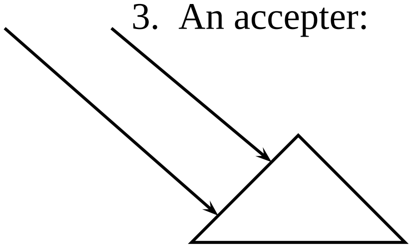


Figure 9: Diagramatic representation an accepter. An accepter output is “True” if the stream reaches the accepter.

Using these tools, we have the following diagrams that represent each logical gate:

NOT *A*: To build NOT *A*, we simply have a stream that is going to be blocked only if *A* is true. So if *A* is true in figure 10, then the stream of gliders passes through *A* stopping the other stream of gliders from reaching the acceptor, otherwise if *A* is not true the stream on the left will reach the accepter.

A AND B: Depicted in figure 11 is the stream from the left gun passing through to the accepter if *A* is true, and not being blocked by the stream on the far right if *B* is

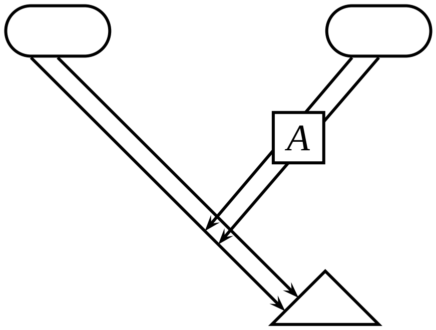


Figure 10: The logical gate representing NOT *A*

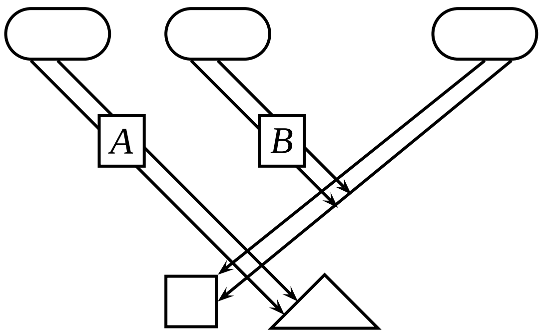


Figure 11: The logical gate representing *A* AND *B*.

true. If *B* is not true, the stream of gliders from the gun on the far right stops the gliders from the far left or hits a blocker if *A* is not true.

A OR B: The OR gate is the more complicated and requires 4 guns. The right 3 guns in figure 12 and their streams essentially represent NOT (*A* OR *B*), and then we are really concatenating this with the NOT operation. The stream from gun on the far right is only blocked if either *A* or *B* is true. Hence, the stream on the far left passes to the acceptor if either *A* or *B* are true.

Hence we have constructed three basic logical gates and have shown that the Game of Life can at least do any single binary calculation using a series of logical gates.

5 Some additional food for thought

We have shown that you can do basic calculations, however, what is required in order for something to be universal is a fair bit more technical, but it can and has been done[3] and available to download for the Life32 program [1]. This is not the only

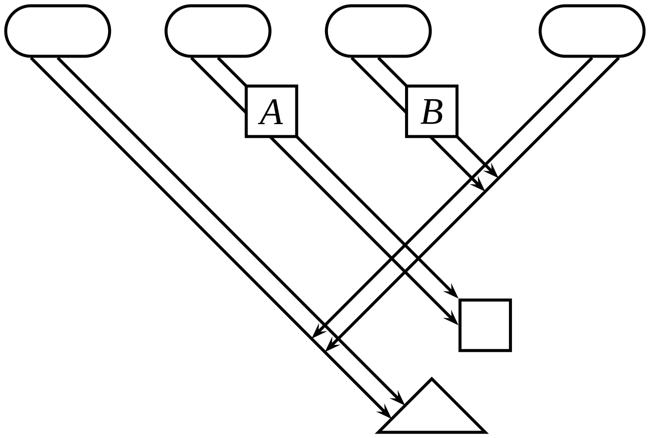


Figure 12: The logical gate representing *A* OR *B*.

cellular automata that is capable of being programmed to do calculations, but it is the most popular. With a much larger alphabet it is even possible to get a CA to simulate a Turing machine in a way that each time step of the CA is in correspondence with a single time step of a Turing machine.

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[1] Bontes, J. Life32, http://www.xs4all.nl/ jbontes/

[2] Machi, A. and Mignosi, F. Garden of Eden configurations for cellular automata on Cayley graphs of groups. *SIAM J. Discrete Math.* **6** (1), 1993 , 44–56.

[3] Rendell, P. Turing universality of the game of life, in: A. Adamatzky (Ed.), Collision-based Computation, Springer, 2002, p. 513.

[4] Rennard, J-P. Implementation of logical functions in the game of life. In Andrew Adamatzky, editor, Collision-Based Computing, pages 491512. *Springer*, London, 2002. URL http://www.rennard.org/alife/CollisionBasedRennard.pdf.

[5] Wolfram, Stephen. A new kind of science. *Wolfram Media, Inc.*, Champaign, IL, 2002.

[6] Game of Life News: http://pentadecathlon.com/lifeNews/

— Chris Ormerod

Quotes 2

“To deprive a mathematician of existence proofs would be like depriving a boxer of his gloves.” – David Hilbert

“For a mathematician he did not have enough imagination, but he has become a poet and now he is doing fine.” – David Hilbert, on a former student

“Medicine makes people ill, mathematics make them sad and theology makes them sinful.” – Martin Luther

“If your experiment needs statistics, you ought to have done a better experiment.” – Ernest Rutherford

“Mathematics contains much that will neither hurt one if one does not know it nor help one if one does know it.” – J. B. Mencken (journalist and essayist)

“There are only two kinds of math books. Those you cannot read beyond he first sentence, and those you cannot read beyond the first page.” – C. N. Yang (Nobel laureate in physics)

“Education is an admirable thing, but it is well to remember from time to time that nothing worth knowing can be taught.” – Oscar Wilde

“The perplexity of life arises from there being too many interesting things in it for us to be interested properly in any of them.” – G. K. Chesterton (writer)

“The intellectuals’ chief cause of anguish are one another’s works.” – Jacques Barzun (historian)

“As an adolescent I aspired to lasting fame, I craved factual certainty, and I thirsted for a meaningful vision of human life – so I became a scientist. This is like becoming an archbishop so you can meet girls.” – M. Cartmill

“Our Father who art in Heaven, give me bread and beer. Listen for once.” – From Abel’s notebook

Solution to puzzle: The first equation converges if $|x| < 1$, while the second converges if $|x| > 1$.

Maths Jokes

Mathematic puns are the first sine of madness.

∞

(Another twist on the perennial primes joke.)

Chemist: 3 is a prime, 5 is a prime... hey let's publish!

Physicist: 3 is a prime, 5 is a prime, 7 is a prime, $\frac{9}{3}$ is a prime with renormalisation, ...

Quantum physicist: all numbers are both prime and non-prime until observed.

Computer scientist: 11 is a prime, 101 is a prime, 111 is a prime...

Programmer: 3 is a prime, 5 is a prime, 7 is a prime, 9 will be fixed in the next release...

Unix programmer: 3 is a prime, 5 is a prime, 7 is a prime, segmentation fault, core dumped.

Minesweeper addict: 3 is red, 5 is maroon, 7 is black, 9 is impossible.

Politician: what's a number?

Philosopher: what is ?

Economist: 3 is a prime, 5 is a prime, 7 is a prime, 11 is a prime... the prime rates are dropping.

Professor: 3 is a prime. That's an interesting statement. I'll get my research students to look into it.

Poker player: 3 is a prime, 5 is a prime, 7 is a prime, J is a prime, K is a prime.

∞

Classification of mathematical problems as linear and nonlinear is like classification of the Universe as bananas and non-bananas.

∞

One evening René Descartes went to relax at a local tavern. The tender approached and said, “Ah, good evening Monsieur Descartes! Shall I serve you the usual drink?” Descartes replied, “I think not”, and promptly vanished.

(What actually happened: Descartes maintained a custom of remaining in bed until 11 a.m. However, after being employed by the Queen of Sweden, he had to walk to the palace at 5 a.m. every morning in the cold for 4 months, and died of pneumonia.)

∞

Q: How is a PhD student in theology like the Laplacian operator?

A: Div grad.

∞

Q: Why is this obvious?

A: That depends on what your definition of ‘is’ is.

∞

Q: How many problems will there be on the exam?

A: I think you will have lots of problems on the exam.

∞

Stats jokes:

Are statisticians normal?

“Smoking is a leading cause of statistics.” – Fletcher Knebel

“There are three kinds of lies: lies, damned lies, and statistics.” – Attributed by Mark Twain to Benjamin Disraeli.

“I could prove God statistically.” – George Gallup

“If I had only one day left to live, I would live it in my statistics class: it would seem so much longer.” – Allegedly found in the inside cover of a statistics textbook.

“The Bureau of Incomplete Statistics reports that one out of three.”

∞

A question was to find $\int 1 dx$. One student, having obviously copied someone, answered "$x + \cos n \tan t$".

∞

Try this if you are asked to prove any mathematical theorem: "If this was not true, you would not ask me to prove it."

True Stories

Einstein one day took a wrong turn on the way home and became confused and lost. He wandered down the street and found a police station, and asked whether he could be directed to Professor Einstein's house.

∞

One day Hilbert and his wife were entertaining some people for dinner. Mrs. Hilbert told the professor to go change his clothes. He went upstairs, took off his jacket, shoes, etc., but being a creature of habit, he then brushed his teeth, got into bed, and went to sleep.

On another occasion, Hilbert forgot to wear a necktie to someone's dinner. He then mailed a tie to the hosts, and instructed them to stare at it for three hours.

∞

Erdös said there are three steps in the mental degradation of a mathematician:

1. First you forget your theorems.
2. Next you forget to zip up.
3. Last you forget to zip down.

Erdös also defines a trivial being as someone who doesn't do maths.

∞

Littlewood was once asked what God was doing before the Creation. He said, "Millions of words must have been written on this; but he was doing Pure Mathematics and thought it would be a pleasant change to do some Applied."

Littlewood also made this comment on Hardy: “all individuals are unique, but some are uniquer than others.”

∞

Norbert Wiener needed constant reassurance of his creativity and place in the history of mathematics. A colleague’s duty was to assure him that he was a good mathematician.

It was Wiener’s custom to stick his finger in the grooves of the MIT corridors, close his eyes, lower his head in thought and walk down a corridor. Professors were told to close their classroom doors or Wiener would follow the corridor wainscoting to the classroom and follow it around the room until it led him back to the corridor.

During World War II, Wiener was working to perfect the antiaircraft gun’s fire control system. Field performance data were fed back to him and he in turn kept refining his equations to be sent to all American ships. Japan, in the meantime, was perfecting their Kamikaze tactics: they replaced single plane attacks by coordinated mass attacks. A few days before Japan surrendered, an American destroyer was the target of such an attack. All hands on the American ship, having just received Wiener’s latest equations, managed to shoot down every one of the attacking planes. After the war, the entire crew paid a visit to him in grateful tribute.

∞

Nobel laureate Richard Feynman, a passionate drummer, was asked by a Swedish encyclopedia publisher to supply a photograph of himself “beating the drum to give a human approach to a presentation of the difficult matter that theoretical physics represents.” Feynman replied:

“The fact that I beat a drum has nothing to do with the fact that I do theoretical physics. Theoretical physics is a human endeavor, one of the higher developments of human beings, and the perpetual desire to prove that people who do it are human by showing that they do other things that a few other humans do (like playing bongo drums) is insulting to me. I am human enough to tell you to go to hell.”

∞

Feynman was once celebrated as the “smartest man in the world.” His mother’s response? “If that’s the world’s smartest man, God help us!”

∞

Feynman's van had Feynman diagrams painted all over it and a license plate that said 'Quantum'. When asked if anyone ever recognised the diagrams, he said, "Yes. Once we were driving in the Midwest and we pulled into a McDonald's. Someone came up to me and asked me why I have Feynman diagrams all over my van. I replied, 'Because I AM Feynman!' The young man went, 'Ahhhhh...'"

∞

One day the Indian writer R. K. Laxman met the mathematician Bertrand Russell. "You Indians have invented nothing," Russell declared. "Absolutely nothing!" Laxman was understandably stunned – until Russell explained that he was speaking about the number zero.

∞

The Russian physicist Peter Kapitza once gave Paul Dirac an English translation of Dostoevsky's Crime and Punishment. When asked how he liked it, Dirac replied, "it is nice, but in one of the chapters the author made a mistake. He describes the Sun rising twice on the same day."

∞

As a professor at Cambridge, J. J. Thomson introduced laboratory work to the curriculum. "We found many cases," he recalled, "where men could solve the most complicated problems about lenses, yet when given a lens and asked to find the image of a candle flame, would not know on which side of the lens to look for the image." "Perhaps the most interesting point," Thomson also observed, "was their intense surprise when any mathematical formula gave the right result."

A Mathematical Model for Procrastination

1.

Solutions to Problems from Last Edition

We had a number of correct solutions to the problems from last issue. Below are the prize winners. The prize money may be collected from the MUMS room (G24) in the Richard Berry Building.

Narthana Epa may collect \$3 for solving question 3.

James Zhao may collect \$5 for solving questions 1 and 2.

Jie Zhou may collect \$5 for solving questions 1 and 2.

Wonki Noh may collect \$8 for solving questions 3 and 5.

Fen Niu Lim may collect \$12 for solving questions 3, 5 and 7 (partial).

Kate Mulcahy may collect \$12 for solving questions 3, 5 and 7 (partial).

1. A die is thrown until a 6 is obtained. What is the probability that a 5 is not obtained before that?

   Solution: either the 6 comes before the 5, or the 5 comes before the 6; the two events have equal probability due to symmetry. Hence the answer is $\frac{1}{2}$.

2. A $3 \times 3 \times 3$ cube can be cut up into 27 unit cubes. What is the minimum number of straight cuts required to do this, if you are allowed to move the pieces around in between cuts?

   Solution: we must separate the middle cube from the rest; this requires at least 6 cuts. In fact, the obvious 6 cuts suffice as they separate all small cubes. So 6 is the answer.

3. Show that the medians of a triangle can also form a triangle, with area $\frac{3}{4}$ that of the original one.

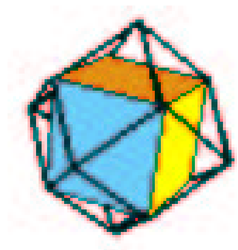
   Solution: let the triangle be denoted ABC, and let A', B', C' be the midpoints of BC, CA, AB respectively. Construct D' so $AD' = BB'$. It is now easy to check that $AD = CC'$ and $AA'D$ is the triangle formed by the medians.

   Let AC and $A'D$ intersect at E. As $A'B'DC$ is a parallelogram, E bisects $B'C$ and $A'D$, so $AE/AC = 3/4$. But the ratio of area of $AA'D$ and area of ABC = ratio of $AA'E$ and $AA'C$ = ratio of AE and AC, and so is $3/4$.

4. Find the volume enclosed by the graphs of $|x| + |y| = 1, |y| + |z| = 1, |z| + |x| = 1$.

Solution: it is easy to see that the region bounded by the first 2 graphs is an octahedron; now the third prism cuts the octahedron from the top, creating a rhombic dodecahedron (see picture), which has many fascinating properties (it is found in the structure of honeycombs and garnet).

To find the area, we note that the polyhedron is the convex hull of $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ and all permutations of $(0,0,\pm 1)$ – that is, a square of side length 1 with 6 pyramids of height $\frac{1}{2}$ on each face. Hence the total area is $1+6\cdot\frac{1}{3}\cdot 1\cdot\frac{1}{2} = 2$.



5. In an equilateral triangle ABC, point Q is on BC, and AQ meets the circumcircle of the triangle at P. Prove that $1/PB + 1/PC = 1/PQ$.

   Solution: observe that $\angle BPQ = \angle CPQ = 60°$. By equating triangle areas, we have $\frac{1}{2}PB\cdot PQ\sin 60° + \frac{1}{2}PC\cdot PQ\sin 60° = \frac{1}{2}PB\cdot PC\sin 120°$. This simplifies to give the result.

6. Starting with one amoeba, every second it splits into either 0, 1, 2 or 3 amoebae with equal probability. What is the probability that the population eventually dies out? What if it can only split into 0, 1 or 2 amoebae with equal probability?

   Solution: let the probability be p, then after a splitting, we observe that $p = \frac{1}{4}(1+p+p^2+p^3)$. Solving for p, we get either 1 or $\sqrt{2}-1$. Now there are several methods to show that $\sqrt{2}-1$ is the only stable equilibrium of the recursive process (starting at $x = \frac{1}{4}$ and apply $f(x) = \frac{1}{4}(1+x+x^2+x^3)$ repeatedly), and so is the answer.

   When the amoeba only splits into 0, 1 or 2, we get $p = \frac{1}{3}(1+p+p^2)$, and so $p = 1$, and so the population will eventually die out.

7. It is well known that no four distinct integer squares can be in arithmetic progression; it is obvious that three can. Find a way to generate all of them.

   Solution: let x^2, y^2, z^2 be in arithmetic progression, then $(x/y)^2 + (z/y)^2 = 2$, and we seek all rational solutions of this equation. But there is a bijection between all these solutions and points of intersection between the graphs of $X^2+Y^2 = 2$ and $Y = p/q(X+1)+1$. Hence, all answers (x,y,z) are given by $(q^2-2pq-p^2, p^2+q^2, q^2+2pq-p^2)$ and their integer multiples. Note that this is the same as $(a-b,c,a+b)$ where (a,b,c) form a (not necessarily primitive) Pythagorean triple.

8. Find 3 ways to write 3 as the sum of 3 3rd powers of integers.

   Solution: this was a prank, for the problem is unsolved. See our article on Unsolved Problems. Of course, if you do happen to find a solution, you are welcome to send it in.

9. Find an explicit formula for $\cos 1^\circ$.

   Solution: we didn't receive any solutions for this problem. A relatively simple expression, found by computing $\cos(\frac{30}{3}-\frac{36}{4})^\circ$, is

$$\frac{\sqrt[6]{2}}{8}\left((\sqrt[3]{\sqrt{3}-i}+\sqrt[3]{\sqrt{3}+i})\sqrt{\sqrt{10+2\sqrt{5}}+4}+(\sqrt[3]{\sqrt{3}-i}-\sqrt[3]{\sqrt{3}+i})\sqrt{\sqrt{10+2\sqrt{5}}-4}\right).$$

Paradox **Problems**

Below are some puzzles and problems for which cash prizes are awarded. Bear in mind that anyone who submits a clear and elegant solution may claim the indicated amount (unless two solutions are the same, in which case only the first submission will be rewarded). Either email the solution to the editor (see inside front cover for address) or drop a hard copy into the MUMS room (G24) in the Richard Berry Building; please include your name.

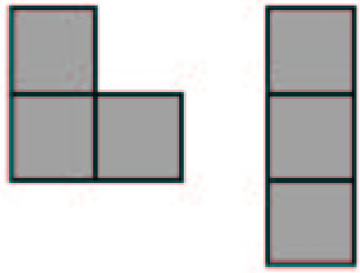
If you enjoy Paradox Problems, then you should also look at the Puzzle Corner column which appears in each issue of the Gazette of the Australian Mathematical Society. There are fun problems to be solved, with the best submission receiving a book voucher worth **$50**.

The Puzzle Corner can be found online at http://www.austms.org.au/Gazette.

1. ($2) Find a quintic polynomial $P(x)$ if $x^3|P(x)$ and $(x-1)^3|P(x)-1$.

2. ($2) Show that n^4+4^n is a prime if and only if $n=1$.

3. ($2) What shape can be made with either 2 of the first figure and 1 of the second figure, or 1 of the first figure and 2 of the second figure?



4. ($4) Show that for an integer n, $n+3$ and n^2+3 can't both be perfect cubes.

5. ($4) Given 3 parallel lines, construct (using ruler and compass) an equilateral triangle such that each vertex lies on a line.

6. ($4) Find $\sum_{n=0}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$.

7. ($5) For positive integers m and n, if $\frac{m}{n} < \sqrt{2}$, then $\frac{m}{n} < \sqrt{2}(1 - \frac{1}{4n^2})$.

8. ($7) Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(m + f(n)) = f(m) + n$ for all m, n.

Paradox would like to thank Adib Surani, Yi Huang, Sam Chow, Julia Wang, Tharatorn Supasiti, Norm Do, Kate Mulcahy and Joanna Cheng for their contributions to this issue.

Maths Lecture Bingo

To play, simply take out the bingo sheet overpage and attend the lecture of your choice. Mark over the square that corresponds to an event that has occurred during the lecture. The first one to form a row/column/main diagonal or mark all four corners yells “Bingo!” to win.

Note: all of the above events have actually occurred at one time or another.

a student's phone rang	"just believe me" or "too hard for you"	the equation on the board looked like gibberish	lecturer wasted 15 minutes complaining not having enough time	lecturer referred to his work
went through about 20 theorems/ lemmas in a single class	lecturer proved the wrong theorem	someone snored	"QED" or "□"	lecturer's phone rang
you could hear a pin drop	you fell asleep	you are about to (legitimately) call out "bingo"	"Brouwer's fixed point theorem"	"real life application" that is not real (only in pure maths)
lecturer was missing	"obvious" or "clear" or "trivial"	"assignment"	lecturer misspelt	lecturer had a hangover (probably an engineering lecturer)
"refer to the book"	no one sat in the front row	"left as an exercise"	lecturer turned up with coffee	the lecturer attempted a maths pun (and no one laughed)