1. 1 chocolate bar bought by Ben can be passed around in the order Ben, David, Emily, Julia, Patrick, Ben, Emily, Patrick, David, Julia, Ben settling all 10 of the debts. Therefore the answer is **1**.

2. 100 (we wouldn't want to make question 2 too hard.)

3. the set can have at most one integer from each of the pairs $(0,1), (2,3), \ldots, (99,100)$ so the answer is atmost 50. the set $\{2, 4, 6, \ldots, 100\}$ satisfies the conditions, so the answer is **50**.

4. by repeated applications of pythogoras' theorem,

   $AC^2 = AB^2 + BC^2 = 2$

   $AD^2 = AC^2 + CD^2 = 3$

   $AE^2 = AD^2 + DE^2 = 4$

   $AF^2 = AE^2 + EF^2 = 5$

   $AG^2 = AF^2 + FG^2 = 6$

   So $AG = \sqrt{\mathbf{6}}$

5. Let $p$ be the proportion of the cake eaten by Andrew. Then Sam ate $2p$ of the cake and Han ate $4p$. so $1 = p + 2p + 4p = 7p$. $p = \frac{1}{7}$.

   So Andrew ate $\mathbf{\frac{1}{7}}$ of the cake.

6. The product of the numbers in the middle column is equal to the product of the numbers in the bottom row. So $9 \times 6 = 3\times ?$ so $? = \mathbf{18}$

7. By pythagoras' theorem, $30^2 = x^2 + y^2$. We also know that $y = 3x$. So $900 = 30^2 = 10x^2$ so $x = \sqrt{90} = 3\sqrt{10}$. So $x + y = 3\sqrt{10} + 3 \times 3\sqrt{10} = \mathbf{12\sqrt{10}}$ metres.

8. If you look at it sideways, $BX$ is the baseof triangle $BXO$ and the height is 5. The area of the whole square is 100. So $\frac{1}{2}5BX = \frac{1}{5}100 = 20$. So $BX = \frac{40}{5} = \mathbf{8}$

9. The inequalities can be rewritten as: $a^2 < b$, $b^2 < d$ and $c^2 < d$.

   Now, $a \geq 1$ so $b \geq a^2 + 1 \geq 2$ so $c \geq b^2 + 1 \geq 5$ so $d \geq c^2 + 1 \geq 26$. This can be achieved when $a = 1$, $b = 2$, $c = 5$ and $d = 26$. So the answer is **26**.

10. The turtle has 4 choices about which way to start, two choices of turning directions when he reaches the circle and 3 choices of lines to go back to the centre along. So the answer is $4 \times 2 \times 3 = 24$

11. every set of three of the seven lines defines exactly one triangle so the answer is $\binom{7}{3} = 35$.

12. You can do each colour separately. For each colour there are 4 ways of placing the three balls: $(3,0)$, $(2,1)$, $(1,2)$ and $(0,3)$. So the total number of ways is $4^3 = \mathbf{64}$

13. Let the sidelength of the square be $3c$. then the area of each of the small triangles is $\frac{1}{2} \times 2c \times c = c^2$ and the area of the whole square is $9c^2$. So the area of the inner square is $5c^2 = 1$. So $c^2 = \frac{1}{5}$. So the area of the original square is $9c^2 = \frac{9}{5}$.

14. Let $a$ be the number of blocks Aaron has. Then $3|a-2$, $5|a-3$, $7|a-4$ and $9|a-5$. So $3|2a-4$, $5|2a-6$, $7|2a-8$ and $9|2a-10$. So $3|2a-1$, $5|2a-1$, $7|2a-1$ and $9|2a-1$. Since $2a-1$ is divisible by 5, 7 and 9 it is also divisible by $5 \times 7 \times 9 = 315$. Also $1 \leq 2a-1 < 2 \times 300 = 600$. So $2a-1 = 315$. So $a = \mathbf{158}$

15. The answer is $\frac{2012}{4} + 1 = 504$ as the largest such set is $\{0, 4, 8, \ldots, 2012\}$.

    Proof: We will first show that at most two of any 8 consecutive numbers are in the set.

    Suppose that 3 numbers $a < b < c$ with $c - a < 8$ are in the set. Then $c - a$ is either 1, 4 or 6 as are $b - a$ and $c - b$. But $(b-a) + (c-b) = (c-a)$ which can't happen if $(b-a)$, $(c-b)$, $(c-a)$ are each 1, 4 or 6. A contradiction.

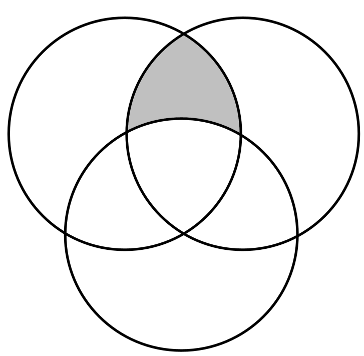
    So every set of 8 consectutive numbers contains at most 2 numbers in the set. So the set contains atmost two elements for each of

    $\{0, 2, 3, \ldots, 7\}, \{8, 9, 10, \ldots, 18\}, \ldots, \{2008, 2009, 2010, \ldots, 2015\}$

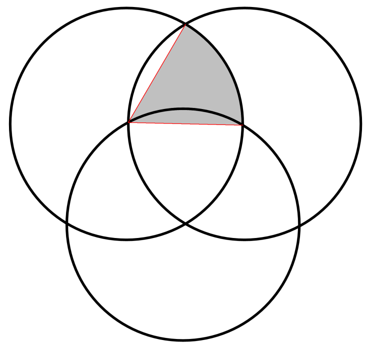
    making a total of $2 \times \frac{2016}{8} = \mathbf{504}$

16. The probability that $A$ is the closest vertex to $X$ is $\frac{1}{8}$ since each of the 8 vertices is equally likely to be the closest vertex to $X$. The next closest vertex has to be one of the vertices adjacent to $A$. So the probability that $B$ is the second closest vertex to $X$ (given that $A$ is the closest) is $\frac{1}{3}$. so the answer is $\frac{1}{8} \times \frac{1}{3} = \mathbf{\frac{1}{24}}$.

17. In the diagram, each circle has radius 30m and we need to find the area of the shaded region.

    We can cut out a piece of this region and put it back as shown in the new diagram. So the area in question is the area of the new shaded region.



This region is exactly one sixth of the size of a whole circle of radius 30m. So that answer is $\frac{1}{6}\pi \times 30^2 = 150\pi\text{m}^2$

18. there is one way with 1 right turn

    If the path has 2 or 3 right turns it is uniquely define by the position of the second right turn, and there are 6 possible positions for this, so this case yields 6 paths.

    If the path has 4 or 5 right turns it is uniquely defined by the positions of the second and fourth right turn, and there are four possible positions of each of these (and each pair defines a path). So this case yields 16 paths.

    If the path has 6 right turns it is uniquely defined by the position of the third right turn, And there are 4 possible positions of it. So this case yields 4 paths.

    If the path has 7 right turns it is uniquely defined by the height of the fourth straight section and there are two of these. So this case yields 2 paths.

    There is only one path with 8 right turns (an inward spiral) and none with more than that many turns.

    So the total number of paths is $1 + 6 + 16 + 4 + 2 + 1 = \mathbf{30}$

19. The number must be a multiple of 5, so the last digit is 0 or 5. Since the number is not 0, the product of its digits isn't 0, so the last digit must be 5.

    So the product of it's digits is a mutiple of 5, so it is a mutiple of 25. Therefore the second last digit is 2 or 7. If the second last digit was 2 the number would have to be even. So the second last digit is 7. Let the first digit be $a$. then the number is $100a + 75 = 5 \times a \times 7 \times 5 = 175a$. So $75a = 75$. So $a = 1$.

    Therefore the number is **175**

20. $1 \ominus b = 0 \ominus (0 \ominus b) = (b + 1) + 1 = b + 2$

    $2 \ominus b = 0 \ominus (0 \ominus b) = (b + 2) + 2 = b + 4$

    $3 \ominus b = 0 \ominus (0 \ominus b) = (b + 4) + 4 = b + 8$

    $4 \ominus b = 0 \ominus (0 \ominus b) = (b + 8) + 8 = b + 16$

    $5 \ominus b = 0 \ominus (0 \ominus b) = (b + 16) + 16 = b + 32$

$6 \ominus b = 0 \ominus (0 \ominus b) = (b + 32) + 32 = b + 64$

So $6 \ominus 7 = 64 + 7 = \mathbf{71}$

As you might be able to see, $a \ominus b = 2^a + b$ in general.

21. Note that the number of actions is one more than the number of pairs of adjacent sheets of paper which are different colours.

    The total number of pairs of sheets of paper is $\binom{6}{2} = 15$, and the total number of pairs of sheets which are different colours is $3 \times 2 + 3 \times 1 + 2 \times 1 = 11$. For each pair of adjacent positions, the probability that the pieces of paper in them are different colours is $\frac{11}{15}$.

    Since there are 5 pairs of adjacent positions of the 6 sheets, the expected number of pairs of adjacent sheets of paper which are different colours is $5 \times \frac{11}{15} = \frac{11}{3}$. So the expected number of action is $\frac{11}{3} + 1 = \mathbf{\frac{14}{3}}$.

22. Let $A$ be the state where the players are equal and let $B$ be the state where the server is one game behind. Let $a$ and $b$ be the probabilies that the server wins from sates $A$ and $B$ respectively. So the answer is $a$.

    From state $A$ the probability that we move into state $B$ and change server is $\frac{2}{3}$ and the probability that we move into state $B$ without changing server is $\frac{1}{3}$. So $a = \frac{2}{3}(1 - b) + \frac{1}{3}b = \frac{2-b}{3}$. So $b = 2 - 3a$

    From state $B$ the probability that we move into state $A$ and change server is $\frac{2}{3}$ and the probability that the server loses the game (and hence the set) is $\frac{1}{3}$. So $b = \frac{2}{3}(1 - a)$.

    So $2a = 2 - 3b = 2 - 3(2 - 3a) = 9a - 4 \Rightarrow 7a = 4 \Rightarrow a = \mathbf{\frac{4}{7}}$

23. First we show that the optimal solution must be a straight path by showing that for any way of rowing across the river which is not straight there is a way of rowing straight which is better (of course, this was not required in the competition).

    Assume that Nepa rows in a way which is not straight. Place a balloon on the water (which travels at 2m/s, just like the water) on the opposite side of the river to Nepa so that he will crash into it as he reaches the other side of the river. If we consider a reference frame which moves with the river, Nepa always moves at (at most) 1m/s and the balloon is stationary. So the fastest way for Nepa to reach the balloon is to row straight towards it. This new method of reaching the balloon will get him to it faster than the old one, so it will descrease his final distance from the post. Therefore the optimal path is straight.

    Now, in the new reference frame, let Nepa's speed in the $x$ direction be $a$ and let his speed in the $y$ direction be $b$. Then $a^2 + b^2 = 1$. The time taken is $\frac{500}{b}$ seconds and the distance downstream is this time times $2 - a$. So the total distance downstream is $\frac{500}{b}(2 - a)$, so we need to minimise $\frac{2-a}{b}$.

    Now, if we consider the point $(a, b)$ in the plane, it must be on the circle of radius 1 around the origin. Now, $\frac{a-2}{b}$ is the gradient of the line which passes through the point $(a,b)$ and the point $(2,0)$. Clearly this gradient will be minimised when the line is tangent to the circle. So triangle with vertices at $(0,0)$, $(a,b)$ and $(2,0)$ is rightangled at $(a,b)$. So $(a,b)$ is on the circle of radius 1 around $(1,0)$. Therefore, since $(a,b)$ is in the positive quadrant, $a=\frac{1}{2}$ and $b=\frac{\sqrt{3}}{2}$.

So the answer is $\frac{500}{b}(2-a)=500\sqrt{3}$

24. The sequence which achieves the minimum value of $a_{2012}$ is given by $a_n=\frac{(n-4)(n-5)}{2}+1$, so the answer is $\frac{2008\times 2007}{2}+1=\mathbf{2015029}$. This way the sequence is given by $11,7,4,2,1,1,2,4,\ldots$ so that we always have $a_{n-1}+a_{n+1}=2a_n+1$ and it gets as small as possible while still being positive.

    Now we prove that this is minimal. First we will prove that $a_4\le a_5$.

    The inequality in the question can be rewritten as $a_{n+1}-a_n>a_n-a_{n-1}$. So $a_5-a_4>a_4-a_3>a_3-a_2>a_2-a_1>a_1-a_0$. So

    $a_4-a_3\le a_5-a_4-1$

    $a_3-a_2\le a_5-a_4-2$

    $a_2-a_1\le a_5-a_4-3$

    $a_1-a_0\le a_5-a_4-4$

    So $a_4-a_0=(a_4-a_3)+(a_3-a_2)+(a_2-a_1)+(a_1-a_0)\le 4(a_5-a_4)-10$

    Also, $a_4-a_0=a_4-11\ge 1-11=-10$. So $4(a_5-a_4)-10\ge -10$ hence $a_4\le a_5$.

    Now, for $n>4$, $a_{n+1}-a_n>a_n-a_{n-1}>\ldots>a_5-a_4\ge 0$. So $a_{n+1}-a_n\ge a_n-a_{n-1}+1\ge\ldots\ge a_5-a_4+n-4\ge n-4$

    So $a_{2012}=(a_{2012}-a_{2011})+(a_{2011}-a_{2010})+\ldots+(a_6-a_5)+a_5\ge 2007+2006+\ldots+2+1+1=\frac{2007\times 2008}{2}+1=2015029$ as required.

25. This is the optimal strategy:

| Direction | Travellers | Time for crossing(min) | Time so far(min) | people across the bridge |
|---|---|---|---|---|
| → | A&B | 2 | 2 | A,B |
| ← | A | 1 | 3 | B |
| → | C&D | 4 | 7 | B,C,D |
| ← | B | 2 | 9 | C,D |
| → | A&B | 2 | 11 | A,B,C,D |
| ← | A | 1 | 12 | B,C,D |
| → | E&F | 6 | 18 | B,C,D,E,F |
| ← | B | 2 | 20 | C,D,E,F |
| → | A&B | 2 | 22 | A,B,C,D,E,F |
| ← | A | 1 | 23 | B,C,D,E,F |
| → | G&H | 8 | 31 | B,C,D,E,F,G,H |
| ← | B | 2 | 33 | C,D,E,F,G,H |
| → | A&B | 2 | 35 | A,B,C,D,E,F,G,H |
| ← | A | 1 | 36 | B,C,D,E,F,G,H |
| → | I&J | 10 | 46 | B,C,D,E,F,G,H,I,J |
| ← | B | 2 | 48 | C,D,E,F,G,H,I,J |
| → | A&B | 2 | 50 | A,B,C,D,E,F,G,H,I,J |
| ← | A | 1 | 51 | B,C,D,E,F,G,H,I,J |
| → | K&L | 12 | 63 | B,C,D,E,F,G,H,I,J,K,L |
| ← | B | 2 | 65 | C,D,E,F,G,H,I,J,K,L |
| → | A&B | 2 | 67 | A,B,C,D,E,F,G,H,I,J,K,L |

So the minimum possible amount of time is **67 minutes**

Proof:

Each time people crossover the bridge forwards, one person has to go back, so there are 11 forwards crossings and 10 back crossings. Let $n$ be the number of times Billie crossed the bridge backwards. Then (s)he must have crossed it $n+1$ times forwards.

The minimum possible amount of time taken during backwards crossings occurs if only Alex and Billie crossed backwards. In this case, since Billie crossed backwards $n$ times, Alex crossed backwards $10-n$ times, so the number of minutes taken in backwards crossings is atleast $2 \times n + 1 \times (10-n) = n + 10$.

Clearly the first, second, thord fourth and fifth longest forwards crossings take ateast 12, 10, 8, 6 and 4 minutes respectively (to get the 10 slowest people across the bridge).

the other 6 forwards crossings each take atleast 2 minutes (though atmost $n+1$ take exactly 2 minutes). So atleast $5-n$ of the remaining 6 take more than 2 minutes. So the number of minutes taken for forwards crossings is at least $12+10+8+6+4+2\times 6+(5-n) = 40+12+5-n = 57-n$

So the total amount of time taken is atleast $(n+10)+(57-n) = 67$ minutes as required.