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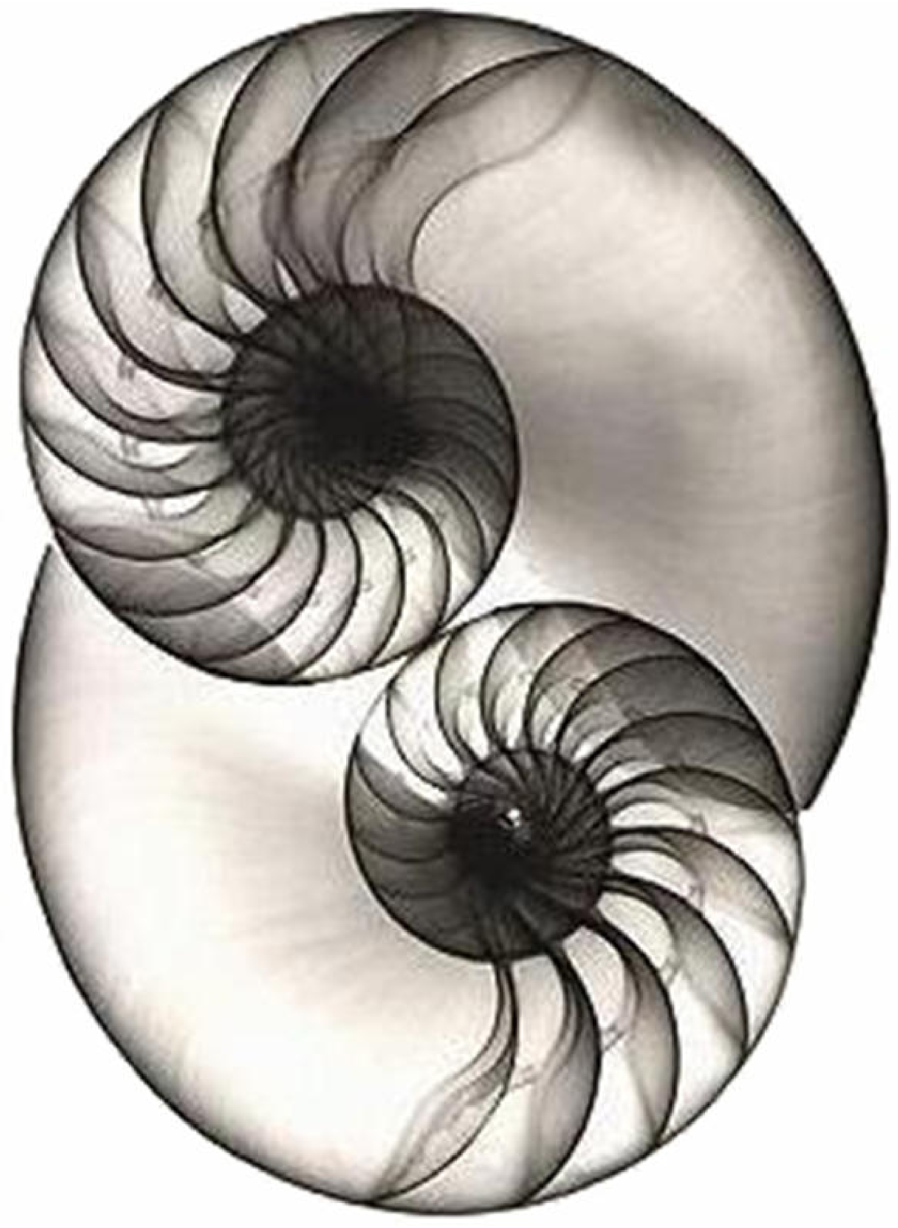
Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society



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Paradox

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Editor's Words

Welcome once again to the mid-year Paradox, the magazine of the Melbourne University Mathematics and Statistics Society. This edition contains an article about the Josephus problem (including a fascinating cultural history in parallel to the mathematical history). We have a report on the recent Puzzle Hunt, solutions to last edition's problems (and more to try out!), as well as a fun-packed section that shows you can't always tell a mathematician by his or her covering space. We also have articles from Perth (on the mystical origins of mathematical proof among the Pythagoreans) and Oxford (explaining the incredible world of surreal numbers). These articles mark the start of what hopefully will be a mutually beneficial sharing of articles between Paradox and the student maths magazines of these universities. Quotations and other curiosities are sprinkled throughout, so as to avoid unsightly white spaces.

We hope you enjoy reading this magazine. Remember that we are very keen to publish articles written by students. You can email jokes, articles, quotes, problems and anything else that seems relevant to us at `paradox@ms.unimelb.edu.au`.

— Nick Sheridan

President's Words

After a much needed break, the Maths and Stats Society is raring to go. During the holidays, the new MUMS committee has been pursuing ways in which we can improve the society for all maths and statistics students. Several changes have been made to the website to provide more informative and relevant content for students. In particular, there is now a guide to free software which students may find useful in their studies and personal pursuits. Further improvements will appear later throughout the year. We would love to hear any comments or suggestions you may have, whether it be via email or in person. In fact, if you're walking through the building, feel free to pop in for a chat and see what's going on in the MUMS room.

Semester one saw excellent attendances at our seminars. Russell Love Theatre was packed with students who listened to talks including 'The Beauty of Mathematics' and 'Mathematics in the Movies'. Certainly the highlight of the semester proved to be the Puzzle Hunt which kept competitiors intrigued and challenged for over a week. While this year's competition proved to be more difficult than the inaugural event last year, more than 100 teams took up the challenge with Team 187 becoming the eventual victor.

Semester two promises to be just as action packed. We have another set of great seminars lined up. I strongly encourage you to attend these as they offer students a chance to discover interesting aspects of mathematics in an informal and friendly environment. Another highlight should be the University Maths Olympics (UMO). As the name suggests, this competition combines problem solving and physical prowess. Teams compete in an exciting and fast-moving atmosphere, so even if you choose not to compete, come along as a spectator and barrack for your favourite team. For a preview of how the UMO works, MUMS will be running its sister event, the School Maths Olympics, on Discovery Day, Sunday 21st August at 12pm. Both these competitions are held in an almost packed Theatre A and look set to be fantastic events for all.

— Andrew Kwok

"All mathematics is divided into three parts: cryptography (paid for by the CIA, KGB and the like), hydrodynamics (supported by manufacturers of atomic submarines) and celestial mechanics (financed by military and other institutions dealing with missiles, such as NASA)." – V. Arnold

A boy goes and buys a one-piece fishing pole that is $6'3''$ long. As he goes to get on the bus, the bus driver tells him that he can't take anything on the bus longer than $6'$. The boy goes back to town, buys one more thing, and the bus driver allows him on the bus. He did not damage the fishing pole in any way. What did he buy, and what did he do with it? (Solution on page 23)

Mathematics of the Beast

$$\sin(666^\circ) = \cos(6 \cdot 6 \cdot 6^\circ) = -\frac{\phi}{2}$$

$$\phi(666) = 6 \cdot 6 \cdot 6$$

(where ϕ is the golden ratio in the first infernal identity and the Euler-phi or totient function in the second)

The Josephus Problem

100 people are playing a game by stand in a circle, and from an arbitrary person they are numbered $1, 2, 3, \ldots 100$ in order. Now suppose some evil tormentor decides to spoil the fun and kill every third person (i.e. kill 3, 6, 9, etc) until there is only 1 left, what would be his or her number?

This category of problems, using m objects and getting rid of every nth, is called the Josephus problem, after Josephus Flavius (c. 37 – 101 AD). Josephus was a Jewish historian known for his account of the fall of the Temple. Legends have it that he was the commander of the Jewish rebels, the Zealots (fanatics who resisted Roman rule), and courageously defended the fortress of Jotapata for months until defeat was imminent. His 39 comrades decided to commit suicide, and in the typically mysterious ways of the Zealots, do so by standing in a circle and killing every 7th person. It just happened that Josephus was an accomplished mathematician, and by standing in the 24th spot, was able to wait until everyone else died, and surrender to the Roman forces.[1]

As he was about to be sent to emperor Nero and most likely die a gruesome death, Josephus prophesied that his captor, general Vespasian, would become emperor himself. For that he was freed, and adopted Vespasian's family name. Two years later, the general founded the Flavian dynasty.[2]

The general solution of the Josephus Problem eluded mathematicians for hundreds of years, until none other than Euler wrote the paper, "Observations on a new and singular type of progression", bringing the problem into the framework of analysis of recursively defined sequences. Meanwhile, many versions of the problem have popped up during the millennia and many of them are rather sadistic in nature, either to the characters that appear in the problem, or to the person attempting it. I recall the AMC paper when I was an ignorant year-7 contained a version of the problem, and it took me many long afternoons (after the competition, of course) writing down 1 to 100 and crossing numbers out, while never reaching the same answer more than twice. I was sadly unacquainted with the analysis of recursively defined sequences, no doubt.

A medieval version of the problem states: a ship carrying 15 Turk and 15

[1] Another version claims that 41 people were in the circle and every third was killed. Josephus saved another man by instructing him where to stand, so he would be the second last to go.

[2] Out of interest, the Colosseum was commissioned by Vespasian and known as the Flavian Amphitheatre.

Christian passengers encountered a storm, and for survival, half of the passengers had to be thrown into the sea. Naturally, the passengers were placed in a circle, and every ninth man, reckoning from a certain point, was cast overboard. Now find an arrangement by which all the Christians could be saved. The solution is that we must arrange the men thus: *CCCCTTTTTCCTCCCTCTTCCTTTCTTCCT*, where C stands for a Christian and T for a Turk. Medieval scholars were so thrilled about the prospect of drowning Turks that they even invented a mnemonic for it: From numbers' aid and art, never will fame depart, where $a = 1, e = 2, i = 3, o = 4$, and $u = 5$. Noting only the vowels, the order is o Christians, u Turks, etc.

In the case of killing (or drowning, defenestrating, etc) every second man, a general solution is much simpler, and we shall consider the trivial case when the number of people, m, is a power of 2. So let $m = 2^k$. We kill every second person, so after one round of killing (i.e. until we reach the highest numbered person), only 2^{k-1} people remain. Now note the last person killed was the person with the highest number, and when the next round begins, we start counting from person number 1 again. But now the number of people is another power of 2, so the same process repeats, until we get down to say, 2 people. The second person gets killed, so number 1 remains. So for all $m = 2^k$, the first person survives.

Since we can write any number as $m = 2^k + x$, we choose the largest k such that x is non-negative. This ensures that $2^k > \frac{m}{2}$ (if not then use $k+1$ instead of k), and so $x < \frac{m}{2}$. Now if we kill x people, then we have 2^k people left, and the last killed person was numbered $2x$. So the problem is reduced to the 2^k case except instead of starting from 1, we are starting from $2x + 1$. Note that $2x$ and $2x + 1 \leq m$, so we don't have an overflow. As in the 2^k case, the starting person survives, so the survivor is numbered $2x + 1$.

This solution can be succinctly represented in binary. Express m in binary, take the leftmost 1 and append it at the right end, and convert to decimal: for it is number of the survivor. The proof is trivial, and I shall leave it to the astute readers.

A general recursive solution can thus be constructed based on the above idea. Let $s(m)$ denote the survivor of the original m people. Then to find $s(m+1)$, we first kill 1 person so the total number is now m. Hence the position of $s(m+1)$ is the position of $s(m)$, plus the first n people skipped: it is $s(m) + n$. But we may have an overflow over the limit $m + 1$, so $s(m + 1)$ is taken as the remainder of $s(m) + n$ after dividing by $m + 1$. Or,

$s(m+1) \equiv s(m) + n \pmod{m+1}$

So to figure out the problem when $m = 100$ and $n = 3$, just pick a "bloody" obvious number of people to "kill", say $m = 3$, and build on from that until you get to $m = 100$. In most iterations you'd be just adding 3 to the last result, so it is quite manageable. If you ever decide, as a purely masochistic test, to try the problem, check your solution against number 91.

Any smarter ways of doing it? I have no idea. But killing numbers is certainly a safer way to relieve stress than some other forms I could name.

— James Wan

Letter to the Editor of Tracks surfing magazine, January 2002: The Perfect Curve

The article in November *Tracks* on surboard shape and design was great. Having shaved a bit of foam myself as a backyarder, it was of special interest. Something got my attention though – the comment regarding a planshape being a "perfect parabola". It got me thinking... is it? Galileo was fooled by a similar curve, known as a "catenary" – the curve traced by a chain suspended between two points and acted on by gravity. It looks just like a parabola (in fact he believed it was), but is subtly different. I have a sneaky feeling that there are all sorts of different "perfect" curves in a surfboard, not just parabolas. Take a look at the fat tail of one of the new nugget-style trick boards – it looks like a perfect Kappa curve. Not even remotely parabolic. A more amazing one is the asymmetrical curve you see in a rail when you look at it along the line of a board. I suspect for "low" and "pinched" rails it's a shape identical with what's known as a "pursuit curve". The most scary one I've seen, that possibly describes a planshape in its entirety as a continuous function, is a gem know as the "Pearls de Sluze". For values of $x > 0$, the function generates a near-perfect shape for a classic roundtail shortboard outline! Spooky, huh??!! There's a really nice little website where you can check out all these amazing curves and the formulae that describe them. It's at: `http://www-groups.dcs.st-andrews.ac.uk/~history/Curves/Curves.html` [Editor's note: The graph would need to be stretched a bit in the x-direction.]

Perfect Numbers and the Order of Pythagoras

A number equal to the sum of all its factors except itself is known in the present day as a perfect number. The discovery of perfect numbers can be traced to the Pythagorean Order – a reclusive, dedicated brotherhood of mathematicians in Ancient Greece, led by the philosopher Pythagoras. For these radical scholars mathematics was virtually a religion, in which numbers themselves embodied the most fundamental and universal truths – and in this setting perfect numbers were seen almost as Gods, to be studied with the utmost reverence and awe. While some today might consider this attitude towards perfect numbers as being slightly extreme, it gives an interesting indication of the ideals and motivations that allowed mathematics, over two millennia ago, to emerge in Greece as an independent discipline – and it's entirely plausible that mathematics would be very different today had it not been for Pythagoras and his eccentric numerical passions.

Although we remember him today almost solely for his pioneering work on right-angled triangles, Pythagoras in his own time was certainly more than just a mathematician: he was a philosopher, a prophet, a mystic, a lyre player, an astronomer, a visionary, a vegetarian, and was viewed by many as a god-like figure. He was born on the Isle of Samos in about 560 BC, and led a life which is still largely shrouded in mystery. After spending his childhood in his birthplace, where he was well-educated in poetry and music, he travelled widely in Egypt and Babylon. Enthralled by the exotic religious and mystical practices he observed on these travels, he decided on his return to the Greek empire to form a brotherhood of free-thinkers, for the purpose of advancing Greek civilisation. He had chosen the perfect time and place to do this: a two-hundred year period of Greek colonisation throughout the Mediterranean was drawing to a close, and this had brought with it a rapid increase in wealth and prosperity, as well as a fervour of intellectual and artistic activity. Greek culture had been reborn into the Classical era, and the arts and sciences were flourishing with the vigour of a youthful empire. This was clearly an ideal setting for radical thinkers to be thinking radical thoughts, and Pythagoras chose the remote colony of Croton, in what is now Italy, as a base for his religious Order.

Ideologically, the newly-formed Pythagorean brotherhood reflected the changing intellectual climate of the Greek empire – particularly the shift away from mythology as a means of explaining universal phenomena. Its core beliefs focussed on the divine nature of the soul and its actualisation through knowledge, the importance of proportion and harmony in the universe, and also humanitarian issues such as ethics and virtue. In the tradition of a religious cult, Pythagoras and his followers maintained an austere lifestyle, observing strict codes of secrecy and abstinence (including an avoidance of meat, animal skins and beans); these bore a strong resemblance to doctrines of the Egyptian priesthood, and were probably absorbed by Pythagoras during travels in his earlier years.

How the Order came to be so entangled with mathematics, given its blatantly religious origins, is certainly an interesting question. It is known that Pythagoras developed a lasting fascination for the subject during his childhood, and that Greek secular culture also had a considerable influence. The Greeks had a high regard for beauty of form and proportion, which is evident in their architecture, in their sculpture, and even in their dramatic works. The mathematics behind these aesthetic qualities was widely appreciated – to the extent that after the fall of Athens to the Persian invasion in roughly 490 BC the city was rebuilt entirely to a geometric design. As a result, scholars tended to regard mathematical truths with reverential awe: to them they seemed the purest manifestation of art and the divine. The Pythagoreans, ever on the look-out for things to worship, found themselves drawn towards mathematics, and it soon became the centrepiece of their ideology – a fact expressed with notable eloquence by their dictum: 'everything is numbers'. Having found an ideological foundation-stone, Pythagoras and his followers set about formalising the discipline, and in doing so introduced some much-needed consistency and accountability into the study of mathematics via the notions of rigour and proof.

The Pythagoreans doctrine 'everything is numbers' refers implicitly to the natural numbers – the positive integers – as opposed to all real numbers. Pythagoras had based this radical concept of the universe on a few observations of whole number relationships in the natural world – for instance, in the ratios of lengths of strings vibrating with harmonious musical intervals, or in the orbits of the planets. The Order held their faith in natural numbers with passionate emotion; so much so that when a member had the audacity to prove that the diagonal of a unit square is irrational (and so cannot be expressed as a ratio of natural numbers) Pythagoras was distraught – gutted, like a right-triangle stripped of its hypotenuse – and in a state of desperation ordered his brotherhood to divide into several sects, each with the task of trying to resolve the issue in the light of his convictions. Of course it didn't help that the proof of this unstomachable truth was one of the most remarkably simple and – some might say – 'aesthetically appealing' of its kind:

Theorem The length of the diagonal of a unit square (equal to the square root of 2) is irrational.

Proof. Suppose that $\sqrt{2} = \frac{a}{b}$ for some integers a and b, whose greatest common divisor is 1.

Then $(\frac{a}{b})^2 = 2$, and hence $a^2 = 2b^2$. Since b^2 is an integer, a^2 is even, and hence a is even. Since 2 divides a, 2 cannot divide b, and so b is therefore odd.

Now, since a is even, there exists an integer c such that $a = 2c$. So $a^2 = 4c^2 = 2b^2$. Thus $b^2 = 2c^2$. Since c^2 is an integer, b^2 is even, and hence b is even. This contradicts the earlier conclusion that b is odd. So the square root of 2 is not rational, and is therefore irrational. QED

As well as being deeply religious and passionate, the Pythagoreans' attitude to numbers showed clear signs of the Greek mythological traditions of drama and symbolism. This was most evident in their habit of attributing 'personalities' to individual numbers. Records of these survive for all numbers from one to nine: one was representative of peace and tranquillity, having no lesser parts and being the symbol of identity, unity and existence. Two, however, was the origin of contrast, or disunity, and accordingly was an evil number – and so on. Other symbolism had a more 'number-theoretical' basis – all odd numbers, for example, were thought to be male, and even numbers female; prime numbers were seen as virile, whereas composites were effeminate. Inevitably, of course, Pythagoras was driven by his unstoppable idealism towards the notion of a 'perfect' number.

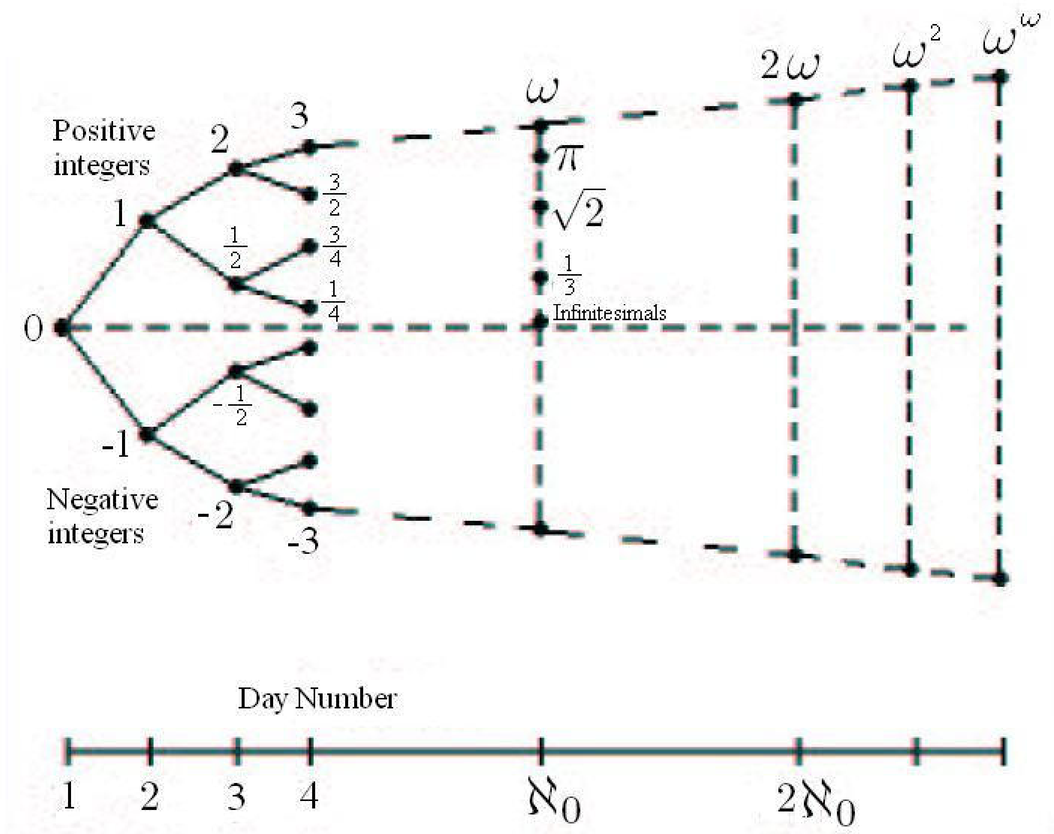
The members of the Pythagorean Order put forward a number of candidates for this auspicious title. Among those suggested were ten, on the basis that it is the sum of the first four 'geometrical numbers' (1,2,3 and 4), and also four, the first mathematical power, and therefore apparently "the most perfect of numbers; . . . which gives the human soul its eternal nature". One especially enlightened individual nominated three, arguing that it was the first 'real' number (one, after all, was indivisible, and two, being even, was female). However, if appellatory adhesion is anything to judge by, it is the numbers equal to the sum of their proper factors which truly deserved the title.

With so many hours of secrecy and abstinence to wile away, it's likely that the Pythagoreans found themselves, perhaps under the shade of an olive grove on a dusty Saturday afternoon, killing time by adding up the proper factors of their favourite natural numbers. They'd have found that some numbers yielded a smaller total; while others (blatantly flouting Pythagoras' belief in frugality and restraint) gave a total larger than the original. The shining pinnacle of virtue in-between, occupied by those rare numbers whose factors gave a total exactly equal to the original, would have been only too clear, and seeing these numbers in such blazing relief against the wretched 'deficient' numbers and the over-indulgent 'superabundant' numbers, the Pythagoreans would surely have thought them the most perfect numbers of all. Whether olive trees and dusty afternoons were involved or not, the discovery of perfect numbers certainly captivated the attention of the brotherhood, and they quickly became engrossed in trying to discover a method for generating all of them. Although they failed at this (in fact, there is still no such method known today) they did manage to find the first four (which are 6, 28, 496 and 8128). Discovering these awakened an interest in related questions – of particular and urgent interest was the question of whether odd perfect numbers could exist (as mentioned earlier, odd numbers were thought to be male – and suffice it to say that the Pythagoreans were pioneering mathematicians and not pioneering feminists, so an odd perfect number would have been the perfectest of the perfect). They also explored the amicable numbers – pairs of numbers in which the sum of the factors of one equals the other, such as 220 and 284 – which (not surprisingly) they came to view as symbols of friendship.

Thus, perfect numbers were so named as a result of a mystical, superstitious, and deeply religious view of numbers. As it turned out, Pythagoras' driving conviction, that all universal truths were embodied by the natural numbers, was fatally flawed; in addition, there are probably few mathematicians today who would consider the assignment of genders to numbers to be of paramount importance. However, the legacy of the Pythagoreans should not be underestimated. Aside from having made elemental discoveries in geometry and number theory, the Pythagoreans were the first to appreciate the necessity for mathematical rigour and proof. What's more, it was their passion for numbers themselves – in an abstract sense – that led to a formalised study of mathematics for its own sake. Perhaps it's therefore not unrealistic to think of their bizarre-seeming mathematical world – in which numbers equal to the sum of their factors were worshipped like Gods – as representing the infancy of modern pure mathematics.

— Geoffrey Pearce

The Wonderful World of Surreal Numbers



We've all heard of integers, rationals, reals, even complex numbers, but what on earth are surreal numbers? They are a beautiful way of defining a class of numbers which includes all reals, but also ordinal numbers; i.e. all the different infinities and even infinitesimal numbers. Not only this but we get a full system of arithmetic for all these numbers. Ever wondered what $(\infty - 1)$ is, or $\sqrt{\infty}$? Before we get stuck into that, let's learn some history.

It all started a long, long time ago in a galaxy far, far away (Cambridge in the 1970s). A man by the name of John H. Conway was playing Go, an ancient Chinese game that's very elegant in itself. After much thought, he realised that the later stages of the game could be thought of as the sum of many smaller games. Conway then applied his ideas to other games like Checkers and Dominoes. It seemed that these games were behaving as if they were numbers.

Conway's ideas led him to define a new family of numbers from sets, which were constructed essentially by a series of binary choices. It turned out that this wonderfully simple new system included all the real numbers and more.

In 1972, Conway described his system of numbers to computer scientist Donald Knuth at Stanford. Knuth (creator of the TEX typesetting system) then went away and wrote a short novelette introducing these numbers. It was the first time for a major mathematical discovery to be published in a work of fiction first. Knuth coined the term *surreal numbers*; taking "sur" from the French for "above". The surreal numbers satisfy the axioms for a field (but the question of whether or not they constitute a field is complicated by the fact that, collectively, they are too large to form a set).

Only two axioms are needed to give you all the surreal numbers. Three more and you get additon and subtraction. One further axiom gives you multiplication. So not only can you get the full structure of the reals from six axioms, but we also get the full power of this arithmetic for the rest of the surreal numbers! So without further ado let's jump into this beautiful landscape.

Rule 1 – Defining what a number is

A number x is an ordered pair of sets of numbers created previously

$$x = (X_L, X_R), \text{ where } X_L \not\geq X_R$$

I'm using notation where $A \not\geq B$ means that, for any a in A and b in B,

$$a \not\geq b$$

"But what is $\not\geq$?" I hear you say. Well that's the beauty of it – we don't have to define it! On the 1st day there was nothing and so the guy in charge said "Let there be 0". We haven't created any numbers yet, so the left and right sets of zero have to be empty! Yes that's it!

$$0 = (\emptyset, \emptyset), \text{ where } \emptyset \text{ is the empty set}$$

We have to check that 0 is a number. To do this we must check that each element of the left set of zero is $\not\geq$ each element of the right set. This is trivial because neither set has any elements!

So zero was created on the first day. Now on the second day we can use 0 as an element in the left and right sets. So we get two new numbers,

$$1 = (\{0\}, \emptyset)$$
$$-1 = (\emptyset, \{0\})$$

It is easy to check that these are both numbers also, because of the empty set again. Now you ask me why the first one is called 1 and the second -1. Here's where we need our second rule.

Rule 2 – Defining "x is less than or like y"

$$x \leq y \quad \text{means} \quad X_L \not\geq y \text{ and } x \not\geq Y_R$$

The first thing to prove is that 0 is less than or like 0. This also falls through because of the empty set. I bet you've never proved that $0 \leq 0$ before!

We can go on and prove that $-1 \leq 0$ in the same way. If we try and prove that $-1 = 0$ by then showing $0 \leq -1$ we hit a brick wall (thankfully). As it is equivalent to $0 \not\geq 0$ which we already know is false:

$$0 \leq -1 \text{ by definition, means } \emptyset \not\geq -1 \text{ and } 0 \not\geq 0$$

Thus -1 is strictly less than 0. Well, it all makes good sense so far. We can carry on this way and on the 3rd day it turns out we get 17 new numbers, although most of them are *like* each other and we are left with only 4 new ones: $2, -2, \frac{1}{2}$ and $-\frac{1}{2}$.

Another thing we need – and which is normally *assumed* when dealing with the reals – is the transitive law. We can prove (yes, I said *prove*) the transitive law.

To do this we assume the opposite; that we have 3 numbers satisfying the following:

$$x \leq y, \;\; y \leq z \;\; \text{and} \;\; x \not\leq z$$

We will call such a triple of numbers, (x, y, z) a *naughty* triple of numbers. The first two of these mean,

$$X_L \not\geq y, x \not\geq Y_R$$
$$Y_L \not\geq z, y \not\geq Z_R$$

The last condition for (x, y, z) to be *naughty* means that at least one of the following is true.

$$\text{There is some } x_L \in X_L \text{ such that } x_L \geq z$$
$$\text{There is some } z_R \in Z_R \text{ such that } x \geq z_R$$

Case 1: $(x_L \geq z)$ This means that (y, z, x_L) are *naughty* numbers. This is because we have, $(y \leq z)$ and $(z \leq x_L)$. Then using the initial assumptions about x, y and z we also have $(X_L \not\geq y)$. Therefore $(y \not\leq x_L)$

Case 2: $(x \geq z_R)$ This means that (z_R, x, y) are *naughty* numbers for similar reasons to case 1. Can you see them?

If we call the day on which a number is created its *day number*, then in both cases the sum of the three day numbers has decreased, because a number can only be made up of sets of numbers that were previously made. This can't keep going forever because the smallest possible day sum is 3. Basically it's proof by induction on the day sum number: We prove that it is true for a day sum of 3 $(x = y = z = 0)$. Then we have shown that if it is true for all previous day sum numbers, then it is true for the next one. Now, look at the diagram at the beginning of this article. It makes a little more sense now doesn't it!

We can go on and prove some important things, like the fact that all numbers are related:

$$x \not\geq y \Rightarrow x \leq y$$

and that a number is between its two sets,

$$X_L < x < X_R$$

Addition is defined in a similar recursive way to $\leq$.

Rule 3 – Defining addition

$$x + y = ((X_L + y) \cup (Y_L + x), (Y_R + x) \cup (X_R + y))$$

What does this mean? It means that to get the left set of $x + y$, you take every element x_L of X_L and compute $x_L + y$, then do the same with x and y swapped. Then just take all the results of these additions and put them in the left set of $x + y$. For the right set of $x + y$ you just do exactly the same thing but with R's instead of L's.

For example,

$$0 + 0 = (\emptyset, \emptyset) = 0$$
$$0 + 1 = (\{0 + 0\}, \emptyset) = (\{0\}, \emptyset) = 1$$

The other two rules you need for addition are really to do with subtraction.

Rule 4 – Defining negative

$$-x = (-X_R, -X_L)$$

Rule 5 – Defining subtraction

$$x - y = x + (-y)$$

With these the surreals become a group under addition. However, there's one more subtlety we've so far forgotten about: we have to prove that $x + y$ and $-x$ are both numbers; i.e. that they satisfy Rule 1. For this I'll refer you to Knuth's book.

So far we've only created numbers that have a finite binary representation (and every number with a finite binary representation gets created eventually). To get the remaining (real) numbers we need to keep going until we get to $\aleph_0$ day (the limit as the day number tends to infinity)! On this day all the remaining real numbers were created. To see this we just have to realise that now X_L and X_R can be *infinite* sets. So any remaining real number r can just be made with a set of binary numbers below r in R_L that converge to r, and similarly for R_H. Now the rest of the diagram is beginning to make sense... We also get infinity!

$$\omega = (\{1, 2, 3...\}, \emptyset)$$

Something else intriguing is created on $\aleph_0$ day. Consider this number,

$$\epsilon = (\{0\}, \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}...\})$$

This is greater than 0 but smaller than every positive fraction (using the result that a number is between each of its two sets)! In fact we get this kind of number around every number that was created before $\aleph_0$ day. On the next day the rest of the reals get numbers like this close to them (i.e. that are closer to them than any other real number).

There is one important subtlety I've glossed over. The day sum argument doesn't work once we pass $\aleph_0$ day. We need a new way of handling induction. Basically, we always showed that if a theorem fails for x then it fails for some element x_L in X_L and then it fails for an element in one of x_L's sets and so on. This gives us a sequence of *ancestors* of x. All we need to do is realise that this sequence cannot be infinite. To see this we could, whenever we create a new number x, prove simultaneously that there is no infinite sequence of *ancestors* of x. NICE!

With the final rule, which defines multiplication, we get the full tapestry of the surreal numbers and their arithmetic. We can then prove all kinds of interesting things. Like,

$$\epsilon\omega = 1$$

Now, as promised,

$$\omega - 1 = (\{1, 2, 3, ...\}, \{\omega\})$$

and

$$\sqrt{\omega} = (\{1, 2, 3, ...\}, \{\frac{\omega}{1}, \frac{\omega}{2}, \frac{\omega}{3}, ...\})$$

— Ian Preston

For more information, here are a few very good books:

Donald Ervin Knuth, *Surreal Numbers:*
How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness
http://www-cs-faculty.stanford.edu/~knuth/sn.html

John Horton Conway, *On Numbers and Games*
David Wolfe and Elwyn R Berlekamp, *Mathematical Go: Chilling gets the last point*

In the beginning, everything was void, and J. H. W. H Conway began to create numbers. Conway said, 'Let there be two rules which bring forth all numbers large and small. This shall be the first rule: Every number corresponds to two sets of previously created numbers, such that no member of the left set is greater than or equal to any member of the right set. And the second rule shall be this: One number is less than or equal to another number if and only if no member of the first number's left set is greater than or equal to the second number, and no member of the second number's right set is less than or equal to the first number'. And Conway examined these two rules he had made, and behold! They were very good.

Puzzle Hunt

The Puzzle Hunt is an annual event organised by us at MUMS that began in 2004. It's an involved fight to the death between dog-fighting teams battling it out on a battlefield comprising puzzles released each day, to a stunning backdrop of an intriguing story. Or, in other words, the Puzzle Hunt involves working with a team to solve sets of puzzles, culminating in the discovery of the location of an object, as revealed by the answers to all the puzzles channeled through a meta puzzle.

The 2005 Puzzle Hunt was held directly after the Easter break, starting on April 4^{th}, with five puzzles released each day for five days, with the meta puzzle released on the weekend. This year, 620 people, comprising 269 teams, competed, with a total of 888 correct answers from 11065 guesses.

While in 2004 the plot revolved around John Howard using a substance called mirror matter to create an army of zombie voters with which to win the election (accurately representing the events occurring at the time), this year was slightly more fanciful, involving a chase after some suspicious thieves from Sydney who stole a coin stabilising Melbourne, protecting it from detesselation. The coin was, at last, retrieved on April 13^{th}, after a spoiler hint was released for the seemingly uncrackable meta puzzle, by Team Room 187 (formerly 106), last year's winners, as well as this year's winners on points.

At this point, I should like to note the generous support from the Department of Mathematics and Statistics that enables us to run the Puzzle Hunt, providing us with the prizes.

While you are all waiting for the next Puzzle Hunt to come around, no doubt you'll be wanting to prepare yourself. Luckily, all the puzzles from the previous Puzzle Hunts are up on the website[3], with the hints and answers as well. There you can also pore over the answer logs, complete with threats and blandishments from the participants.

— Michael de Graaf

[3] `www.ms.unimelb.edu.au/~mums/puzzlehunt/`

Spot the Mathematician!

The pictures below come in pairs. Each consists of one mathematician and one 'normal' person. Try to work out which is which! Answers on page 23.

Novelist or mathematician?

Bohemian artist or mathematician?

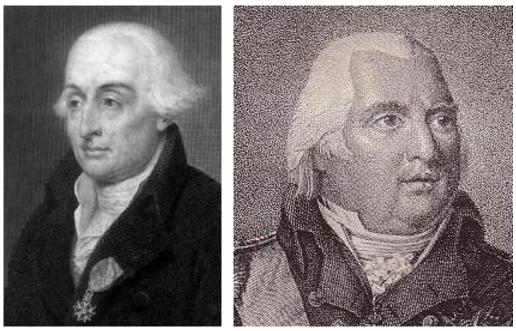
Murderer or mathematician?

Journalist or mathematician?

70s folk musician or mathematician?

Revolutionary or mathematician?

Russian writer or mathematician?



King or mathematician?

French marshall or mathematician?

Military strategist or mathematician?

Politician or mathematician?

Literary critic or mathematician?

— Concept due to Penny Wightwick

Solutions to Problems from Last Edition

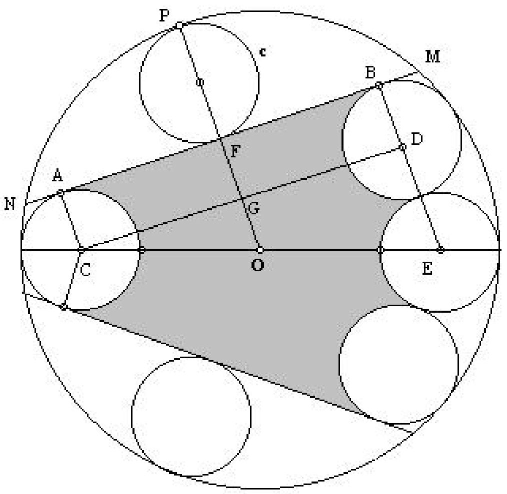
Problem 1. Show that a number is rational if and only if its decimal representation is eventually periodic.

Solution: (due to James Wan) If the decimal is eventually periodic, then the non-periodic part is obviously rational. Since the sum of rationals is rational, we just need to show the periodic part is rational. Single it out and multiply by a power of 10 so it is in the form $0.X$, where X is recurring and let p be the period. Then $(0.X) \times 10^p - (0.X)$ is an integer due to cancellation of decimals. Hence $0.X$ is rational, and so must be the periodic part, and thus the number we started with is rational.

If the number is rational, then it can be written as $\frac{p}{q}$ where p, q are integers. We carry out a long division, and obtain a remainder after each digit of the quotient calculated. But the remainder is $< q$, so it can only take values from 0 to $q-1$, and after at most $q+1$ answer digits, it must start repeating (the same remainder gives the same quotient). Hence the decimal will eventually repeat.

Both cases of the if and only if statement have been proven, so it must be true.

Problem 2. The little circles in the diagram below have radius 1. Show that the shaded area A is $16\sqrt{2} - 2\pi$.



Solution: (due to James Wan) Circle c is tangent to the big circle and the line NM; by symmetry c is centrally placed, hence PO bisects NM. As the circles centred C and D are equal and both tangent to NM and the big circle, $AN = BM$, so PO bisects AB.

As AC and $BD\perp$ to AB, and $AC = BD$, $ABCD$ is a rectangle, and $AB||CD$. Hence $BE\perp CD$ and $\triangle CDE$ is a right triangle. Also PO bisects CD; by similar triangles, $ED/GO = CD/CG$, so $GO = 1$. Hence the big circle has radius 4.

By Pythagoras, $CD^2 = CE^2 - DE^2$, so CD is $4\sqrt{2}$. Area A is just 2×([area of ACEB] – [area of small circle]). The areas subtracted is a circle because they happen to be a semicircle and 2 supplementary sectors. So $A = 2 \times [(1 + 3) \times 4\sqrt{2}/2 - \pi] = 16\sqrt{2} - 2\pi$.

(NB the answer given in the last edition, $8 - 2\pi$, was unfortunately incorrect.)

Problem 3. Find all subsets X of the reals such that for each 'stretch' S, where $S(x) = ax$, there exists a translation T, where $T(x) = x + t$, such that $S(X) = T(X)$.

Solution: (due to Angelo Di Pasquale) Let T be the set of all real numbers t such that $X + t = X$ (that is, all 'periods' of X). Now given $t \in T$, and $\alpha \in \mathbb{R}^+$, we know that $\alpha X = X + r$ for some $r \in \mathbb{R}$. Hence, we have

$$\begin{array}{rccc} & t \in T & & \\ \Rightarrow & X + t & = & X \\ \Rightarrow & \alpha(X + t) & = & \alpha X \\ \Rightarrow & \alpha X + \alpha t & = & \alpha X \\ \Rightarrow & X + r + \alpha t & = & X + r \\ \Rightarrow & X + \alpha t & = & X \\ \Rightarrow & \alpha t \in T & & \end{array}$$

So $t \in T \Rightarrow \alpha t \in T$ for all $\alpha \in \mathbb{R}^+$. Also, it is clear that $X + t = X \Rightarrow X - t = X$, so $t \in T \Rightarrow -t \in T$. Thus, either $T = \{0\}$ or $T = \mathbb{R}$.

If $T = \mathbb{R}$, then $x \in X \Rightarrow x + t \in X$ for all $t \in \mathbb{R}$, so $X = \emptyset$ or $X = \mathbb{R}$. These two sets clearly have the required property.

If $T = \{0\}$, then the only period of X is 0. Given $\alpha \in \mathbb{R}^+$, let us define $f(\alpha)$ so that $\alpha X = X + f(\alpha)$. Note that, for each such α, such a number $f(\alpha)$ exists by the definition of X. Furthermore, if two such numbers existed, we would have $\alpha X = X + f(\alpha) = X + f(\alpha)'$, in which case $X + (f(\alpha) - f(\alpha)') = X$. But then, as $T = \{0\}$ we must have $f(\alpha) - f(\alpha)' = 0$, and hence $f(\alpha) = f(\alpha)'$. Thus, the function $f : \mathbb{R}^+ \to \mathbb{R}$ is well-defined.

Now we look for properties of f. Note that

$$\begin{array}{rrcl} & (\alpha\beta)X & = & X + f(\alpha\beta) \\ \Rightarrow & \alpha(\beta X) & = & X + f(\alpha\beta) \\ \Rightarrow & \alpha(X + f(\beta)) & = & X + f(\alpha\beta) \\ \Rightarrow & \alpha X + \alpha f(\beta) & = & X + f(\alpha\beta) \\ \Rightarrow & X + f(\alpha) + \alpha f(\beta) & = & X + f(\alpha\beta) \\ \Rightarrow & f(\alpha) + \alpha f(\beta) & = & f(\alpha\beta) \end{array}$$

using the same reasoning as before for the last step. Now, this functional equation can be solved: using the symmetry of the right hand side, we have

$$f(\alpha) + \alpha f(\beta) = f(\alpha\beta) = f(\beta) + \beta f(\alpha)$$

Hence, fixing $\alpha = 2$, we can solve to find $f(\beta)$:

$$f(\beta) = -f(2)(1 - \beta)$$

So, if we let $k = -f(2)$, then we have:

$$f(\beta) = k(1 - \beta)$$

Now, let us redefine the origin of our coordinate system so that k become the new 0. Then, in the old coordinate system, we had

$$\alpha X = X + f(\alpha) = X + k(1 - \alpha)$$

In the new coordinate system, this translates to become

$$\begin{array}{rrcl} & \alpha(X - k) & = & (X - k) + k(1 - \alpha) \\ \Rightarrow & \alpha X & = & X \end{array}$$

Hence, for all $\alpha \in \mathbb{R}^+$, $\alpha X = X$. Thus, if any $x \in \mathbb{R}^+$, $x \in X$, then $\mathbb{R}^+ \subset X$. Similarly, if any $x \in \mathbb{R}^-$, $x \in X$, then $\mathbb{R}^- \subset X$. If we now translate this back into the original coordinate system, the possible sets X are $X = \emptyset, \mathbb{R}, [k, \infty), (k, \infty), (-\infty, k], (-\infty, k)$, where k may be any real number. These sets all clearly have the desired property, so we have found all solutions.

Paradox Problems

The following are some maths problems for which prize money is offered. The person who submits the best (clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number. Solutions may be emailed to `paradox@ms.unimelb.edu.au` or you can drop a hard copy into the MUMS pigeonhole near the Maths and Stats Office in the Richard Berry Building. Congratulations to James Wan who submitted the correct solution to Question 1 from the last edition of Paradox, and also to Sam Hart, who elegantly solved Question 2. They can come by the MUMS room to pick up their prizes whenever they feel like it.

1. ($5) A hole bored through the centre of a sphere has length 5 cm. What is the volume left in the sphere?
2. ($5) A man has to take exactly 1 tablet of type A and exactly 1 of type B every day, otherwise he dies. Each type of tablet has its own bottle; apart from this they are completely identical. One day the man accidentally took out 1 tablet of type A and 2 tablets of type B; he had no way of telling them apart. Without wastage of tablets, think of a way to save the man's life.
3. ($5) A polynomial of degree $n > 1$ with real coefficients has n distinct real roots. Show that the sum of the gradients of the normals to the graph of the polynomial at these roots is 0.

Solution to Mathematical Riddle: The boy bought a box in the shape of a cube with sidelength $5'$. The fishing rod will easily fit along the diagonal of the box, which is over $8'$ long.

Solutions to 'Spot the Mathematician' (reading pairs from left to right, top to bottom): On the left is author Barbara Kingsolver, on the right is Vaughan Jones of Jones polynomial fame; On the left is bohemian artist Amadeo Modigliani, on the right is mathematician Rufus Bowen; On the left is alleged murderer Alvin Scott, on the right is Fields Medallist Vladimir Drinfeld; On the left is mathematician Linda Keen, on the right is journalist Silvia Cattori; On the left is a 70s folk musician, on the right is mathematician Lenore Blum; On the left is Fields Medallist Enrico Bombieri, on the right is revolutionary Mikhail Kalinin; On the left is a youngish Leo Tolstoy, on the right is mathematician and anti-fascist activist Renato Caccioppoli; on the left is the celebrated mathematician Lagrange of multiplier fame, on the right is King Louis XVIII of France; on the left is mathematical celebrity Joseph Fourier, on the right is French marshall and King of Naples Murat; On the left AND right are Lazare Carnot, geometer and military strategist, described by Napoleon as the "Organizer of Victory"; On the left is Mario Soares, Portuguese politician, on the right is David Crighton, who only became interested in mathematics when one of his grammar-school masters said he would never be any good at it; On the left is mathematician Evelyn Nelson, on the right is literary critic P.K. Balakrishnan.

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- check out our website and subscribe to our mailing list at:

  `http://www.ms.unimelb.edu.au/~mums/mlist`