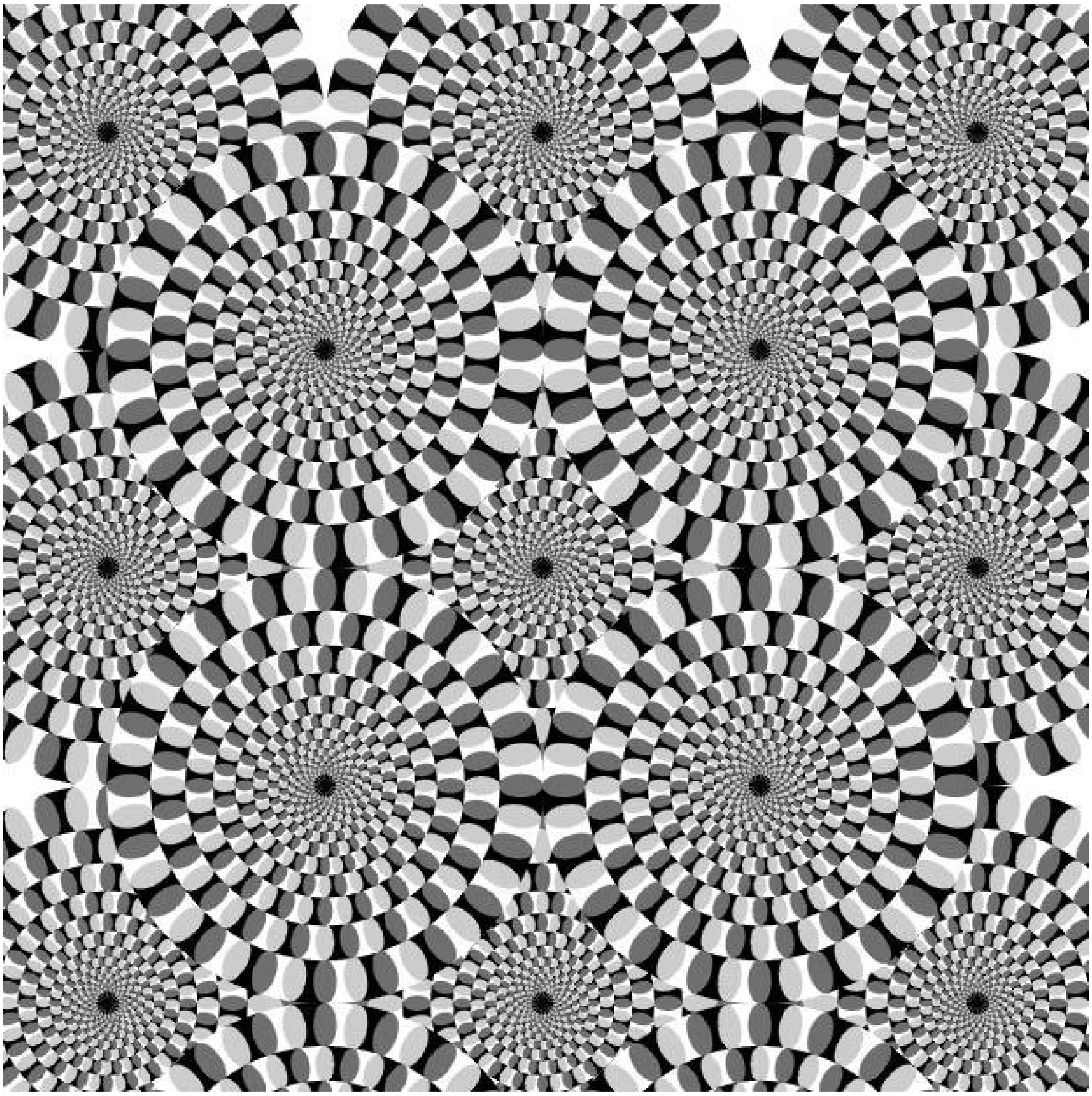
Issue 3, 2003

Paradox

The Magazine of the Melbourne University Mathematics and Statistics Society



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Paradox

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Words from the Editor…

Despite the fact that many of you will still be reeling from the deep maths and still chuckling from the hilarious jokes of the previous issue, the Paradox team have decided to bring out the third issue for the year of this prestigious periodical. You probably know that Paradox is the publication of MUMS, the Melbourne University Mathematics and Statistics Society. And you are probably wondering, as I did, why the acronym is MUMS and not MUMSS…but wonder no more! All will be revealed towards the bottom of this page.

As Paradox readers have come to expect, this issue contains maths problems to solve for cash prizes as well as the usual collection of maths jokes for you to laugh and/or groan at. Furthermore, this issue includes articles on calculating prodigies, the mind-boggling world of p-adic numbers, as well as a training guide for the maths olympics. What on earth is the maths olympics, you may well ask? Well, I'm glad you did…The maths olympics is the premier event on the MUMS calendar, a fun-filled competition where teams of five combine their mathematical prowess and athletic ability for eternal glory. I encourage you all to form your teams quickly and get a piece of the action which will be taking place on 19 September in Theatre A. Remember to e-mail any queries, mathematical jokes or articles to us at paradox@ms.unimelb.edu.au.

— Norman Do, Paradox Editor

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In the last issue of Paradox you raised the question
as to why the acronym "MUMS" as opposed to "MUMSS".
MUMS used to be the "Melbourne University
Mathematics Society".  (An aside, the equivalent
club at Sydney University has the really nice
acronym "SUMS".)  Due to budget cuts at the time,
the mathematics and statistics departments decided
to merge to cut costs.  As MUMS was always very
closely affiliated with the department we decided to
change our name and charter to include statistics as
well.  MUMSS seemed to be an unwieldy acronym and
after much deliberation we decided to leave off the
final S.

Lawrence Ip, Former President of MUMS
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. . . and some from the President

Welcome to the third Paradox for 2003. Paradox is the magazine of the Melbourne University Mathematics and Statistics Society (MUMS).

Next week we are running the major event of the year for MUMS — the University Maths Olympics. This is a tradition of the club which dates back many years. The Maths Olypmics is a relay race involving problem solving, guesswork, and physical endurance. Teams compete for book voucher prizes whilst spectators compete for edible goodies. Form teams amongst your friends and class mates, or just come along for the food. More information is available at the MUMS room (G06) or on the back cover of Paradox.

After the mid semester break, we have a number of activities planned. For the socialites amongst you, we have a BBQ and a trivia night. For the studious types, we have seminars and an honours information session, giving you a chance to find out what it is really like doing postgraduate study here at the University of Melbourne.

Hope you enjoy this Paradox and the activities that MUMS has in store for the rest of the semester.

— Joe Healy, President of MUMS

MUMS Dates Semester 2, 2003

Date	Event
19 September	University Maths Olympics
4 October	Welcome Back BBQ
15 October	Honours Information Session
24 October	Trivia Night

Rain People — The Story of Mental Calculators

Zerah Colburn was born in 1804 in Vermont, the son of a farmer. His parents had no extraordinary intellectual ability. However, while Zerah's father was working at a joiner's bench one day, he heard the child, who was playing in the woodchips, repeating multiplication tables to himself. Intrigued, he tested the child on the tables, which Zerah knew perfectly. He then speculatively asked what 13 multiplied by 97 was. Zerah replied instantly that the answer was 1261. He was, at that time, a month short of his sixth birthday, and had had but six weeks' schooling.

Knowing an opportunity when he saw one, Colburn's father began exhibiting him publicly for profit. Audience members would pose problems to the child, which he would answer extremely quickly. On one occasion, when he was aged eight, he calculated the value of 2^{48} (which is 281474976710656) in under 30 seconds. He also factorised the number $2^{32}+1$, which Fermat conjectured to be a prime, as 641×6700417. Curiously, this extraordinary computational ability left him as he received his formal education. In fact, in his adult life, he never demonstrated any of the genius that was expected of him as a result of his feats as a child.

History abounds with examples of such prodigious calculators as Zerah Colburn. Many, like Colburn, showed no special ability in higher mathematics; some, such as Carl Friedrich Gauss, were mathematicians of the most profound genius.[1] Computational ability, it seems, in no way guarantees mathematical ability (ie. the ability to solve mathematical problems that require creativity, as opposed to computational skill).[2] While mathematicians may be expected to have great computational facility, even at a young age, the most interesting cases are those in which the person involved has very little understanding of mathematics beyond the multiplication, division, addition, subtraction and extraction of roots at which they excel.

[1] Gauss was famous for the feat of summing the numbers from one to 100 in a few seconds at a young age, having quickly discovered the formula for the summation of an arithmetic progression. He also learnt to read, write and do arithmetic before he could speak — he apparently extrapolated from books he had seen!

[2] It is also noteworthy that mathematical ability does not imply computational ability. Ernst Kummer, who made very important contributions to the solution of Fermat's Last Theorem, was renowned for his poor arithmetic. He once turned to his students, in a lecture, for the solution to 7×9. One proposed 62; another ridiculed him, claiming that 69 was the answer. "Come, come, gentlemen, it must be one or the other," Kummer is recorded as saying.

In some cases, people of extraordinary computational ability may not only lack the genius that their skills suggest, but may also be incapable of grasping even the rudiments of higher mathematics. Henri Mondeux, a French shepherd, was discovered by a teacher at a nearby village. He was particularly skilled at calculating powers of numbers, for which he used the Binomial Theorem (without having received an education in mathematics). However, when the teacher attempted to further Mondeux's education in mathematics, he found that Mondeux was utterly uninterested in anything but calculation. Study of anything but his calculation made him physically sick.

This apparent lack of the expected degree of correlation between mathematical and computational ability has been the subject of research by psychologists. One of the most convincing explanations advanced for this phenomenon is that the ability of great calculators is a concrete ability, requiring the solution of a definite problem, usually by a known formula or method. Once mathematics advances beyond basic computation, however, it steadily becomes more abstract. Thus calculators may be incapable of the abstraction required for high-level mathematics, yet be able to do arithmetic extremely quickly.

This is supported by the example of Zerah Colburn. At the height of his powers, Colburn was completely unable to explain how he obtained his answers — they just happened. Of course, it was not magical — the processes by which he solved problems were simply subconscious. With practice, and with his unique mind, the processes of arithmetic became as intuitive as the processes of face recognition or speech are to other people. As Colburn's education advanced, he grew aware of the techniques that he had been using to perform calculations, revealing that they were not different from the normal techniques — he simply performed them extremely rapidly. This was, apparently, at around the time when his abilities were disappearing. His ability to abstract his techniques, to recognise the methods he was using, coincided with the loss of the incredible speed in calculation that he was known for.

Another interesting aspect of mental calculators is the distinction between 'auditory' and 'visual' calculators. Visual calculators mentally see the numbers involved in their calculations, while auditory calculators hear the numbers. Auditory calculators are known to mumble while they calculate, along with other tics and fidgets. Visual calculators often see their sums in a certain form — some would visualise a blackboard with the numbers written on them. An interesting exception to this rule was the case of a 'tactile' calculator, who was accustomed to calculating using cubarithms (counting symbols used by blind people to assist in computation). As he calculated,

> He moved his fingers feverishly over the lapel of his jacket and it was curious to watch him using these tactile images to obtain sensations corresponding to those he would have had in touching cubarithms.[3]

You may be interested in trying your abilities against some of the best calculators in recorded history. Test yourself against these feats — it's fun and free!

Truman Henry Safford, at age 10, calculated that

$$365365365365365365^2 = 133491850208566925016658299941583225$$

in less than a minute.

Shakuntala Devi found that

$$7686369774870 \times 2465099745779 = 18947668177995426462773730$$

in 28 seconds.

Thomas Fuller, an American slave in the 18th century, found that in his life of 70 years, 17 days and 12 hours, there had been 2210500800 seconds, in a minute and a half. When corrected by the man who set the question, Fuller pointed out that his examiner had forgotten about leap years.

Jedediah Buxton took a few months to calculate that

$$725935716098002055388532495854438851106^2$$

is equal to

$$527015363459557385673733542638591721213298966079307524904381389499251637423236.$$

Buxton was unable to read or write; this was done mentally. (Those readers who are on the ball will notice that there is a mistake: the 57th digit ought to be a 5, not a 4.)

— Nick Sheridan

References: *Smith, S. B., The Great Mental Calculators: The Psychology, Methods and Lives of Calculating Prodigies, Past and Present.*

Devi, S., Figuring: The Joy of Numbers.

[3] Smith, page 20.

A Training Guide for the Maths Olympics

The highlight of the MUMS calendar is the annual Maths Olympics. One of the best things you can do in the lead-up to this great event is to engage in the training that up until now has only been practised by one or two teams every year. I do not pretend to offer the final word when it comes to training methods for the Maths Olympics, but I will try to give a glimpse into the methods my team has used in the past seven years we have competed. What worked for us may not necessarily work for your team. Try them and see!

Firstly, a few words about the Maths Olympics itself. It's a maths relay with teams of five people. The team is split up into two groups on opposite sides of Theatre A with a runner relaying a question from the marker to the groups. Only one side may work on any one question and you can't work on the next question until you've either solved the previous question or given up on it. There's no penalty for wrong answers so you can guess as much as you want. Calculators aren't permitted for this contest — you're going to have to use old-fashioned pencil and paper!

Why should you participate in the Maths Olympics? Because it's a fun thing to do. In fact one of the rules explicitly says that "You shall have fun!" What other event at uni allows you to combine the joy of solving interesting problems, team work and physical activity all at the same time? You will have the chance to watch your lecturers embarrass themselves. The lecturer teams have often fared poorly (they're a bit too old!). It's probably the only chance you'll have to get physical with them — one year the then head of maths was taken out with bruised ribs. Listen out too for the witty comments by the compere.

Even if you don't feel like competing, come along and spectate. The start has always been a great sight — all runners simultaneously hustling for that first question. It can also be really funny seeing friends of yours being really pumped up after solving questions, or really flustered while attempting them! If that's not enough there's a spectator competition as well — questions already attempted by all the teams are put up on an overhead and spectators who solve them win chocolates for their efforts. For those who prefer to just sit back and watch, you'll find that the compere hurls sweets into the crowd at regular intervals.

Some tips

The following tips offer some ways to improve your performance and also enjoyment in the Maths Olympics (to a level even higher than what it already would be!).

Enter early. You won't enjoy it as much if you're not competing! Due to the finite size of the lecture theatre, only about 28 teams can enter. In some years, almost 50 teams have submitted entries, only for many to be turned away.

Bring along lots of pens and paper. You'll need it! Look at the sample questions. Have a look at sample questions to get a feel for the type of questions posed. Questions tend to be similar in style to the ones in the Australian Mathematics Competition. As a result, it's not necessary to have done any maths at uni, in fact this may even be a handicap!

The runner can and should help. In years gone past, the runner was forbidden from helping with the question solving. However, due to blatant violations of this rule and the difficulty of enforcement, it was repealed. Keep working when the runner goes off, as the answer may have been wrong. One worthwhile tactic is to have the runner running back and forth guessing answers while the rest of the side tries to actually work out the answer.

Don't give up too easily! If you get stuck, read the question carefully, even if you have no idea, spend a minute as it may be easier than it first looks. Questions are usually designed to have a neat solution that doesn't require too much calculation. If you still have no idea, don't be afraid to abandon the question. One easy trap to fall into is to keep on investing more time in a question because it is psychologically difficult to give up on a question once you've spent a significant amount of time on it.

Guess. Always look for the opportunity to guess the answer. For example, consider the following question:

$$f(1) = 1,\ f(2n) = f(n) \text{ and } f(2n+1) = 1 - f(n). \text{ What is } f(98)?$$

At first you might think,

$$f(1) = 1, f(2) = f(1) = 1, f(3) = 1 - f(1) = 0,$$
$$f(4) = f(2) = 1, f(5) = 1 - f(2) = 0, f(6) = f(3) = 0.$$

Hmmm, no obvious pattern... Then you might say,

$$\begin{aligned} f(98) &= f(49) = 1 - f(24) = 1 - f(12) = 1 - f(6) \\ &= 1 - f(3) = 1 - (1 - f(1)) = 1 - (1 - 1) = 1. \end{aligned}$$

That wasn't too bad was it? However, another possibly quicker approach would be to notice that the rules imply that $f(n)$ must be either 1, or $1-1=0$. Thus a quick way to do this question would be to guess 0 and when that fails, try 1. Remember that there is no penalty for guessing, you can do it as many times as you like (provided the runner doesn't mind!). So let your legs do the thinking for you!

The team name. A vital part of the Maths Olympics is the ingenious team names that teams come up with every year. Some team names from the past have included, "No Real Solutions", "The Return of the Lemma", "Pythagoras was Wrong!", "We Don't Count", "Meds on Prozac", and "Gottim - Yes". Start thinking now! For those who really want to get into the spirit of things, team mascots, music, costumes and make-up are encouraged. Bring along a cheer squad too.

Lastly, don't take it too seriously. Have fun! Unless you're one of those people who can't stand coming 2nd, just sit back and enjoy the spectacle when the other half of your team is working on a problem. Yell and scream a bit! At the end, even if you receive no monetary reward, I guarantee you will have had one of your most entertaining hours at uni.

About the author: Lawrence Ip competed in the University Maths Olympics from 1991 through 1997, first competing when he was in year 11 at school. He captained the winning team in 1992, 1994, 1995 and 1997. Since retiring from competition he has spent the last 5 years pursuing a PhD in quantum computing at UC Berkeley. He often dreams of making a comeback but in moments of reason realises he is too old for this kind of thing. He writes:

> Although I was the one who wrote the training guide, the development of these training methods was not a solo effort. My main collaborator was Chaitanya Rao. At the time this guide was first published he was too shy to have his contributions recognised.

The above article was originally published in the second issue of Paradox in 1998. It is an excerpt from the longer training guide which appears at the official University Maths Olympics website

`http://www.ms.unimelb.edu.au/~mums/olympics/umo.html`

Maths Olympics Practice Questions

- Augustus de Morgan, a famous mathematician, used to boast that he was x years old in the year x^2. He died in 1874. How old was he when he died? (You may assume that he died on the same date as he did his boasting, which was not his birthday!)
- A newly developed fruit called a *yumidum* is a substance that is 99% water by mass. If one *yumidum*, weighing 500 grams, is allowed to stand overnight, the partially evaporated substance that remains in the morning is 98% water. What is the mass of the *yumidum* in grams in the morning?
- Which of the following statements are true? List all the true statements by their letter.

  A. Exactly one of these statements is false.

  B. Exactly two of these statements are false.

  C. Exactly three of these statements are false.

  D. Exactly four of these statements are false.

  E. Exactly five of these statements are false.
- How many four digit numbers are there such that, in their normal (base-10) representation, there is at least one repeated digit?
- Express the value of this sum in simplest form:

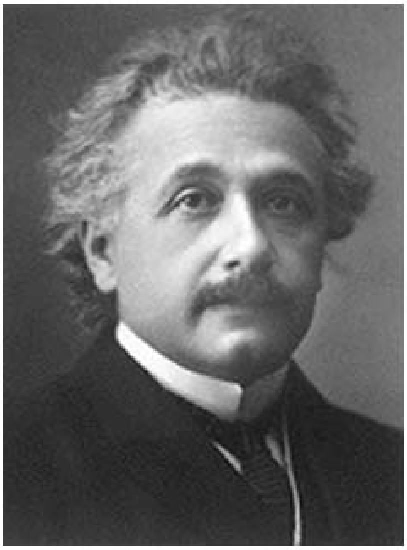
$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{1023}+\sqrt{1024}}.$$

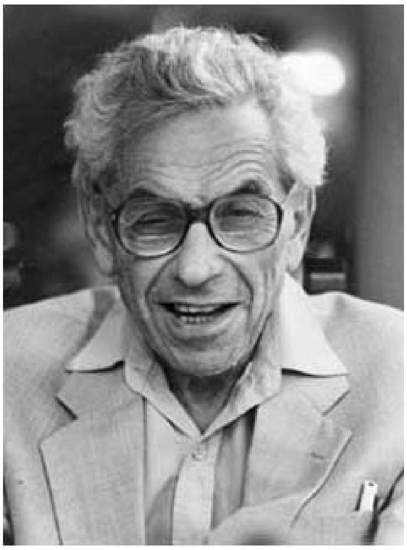
Know Your Mathematicians

See if you can match the following nine mathematicians to their quotes below and their pictures opposite. Answers appear on page 20.

Archimedes of Syracuse (287 BC - 212 BC)
René Descartes (1596 - 1650)
Albert Einstein (1879 - 1955)
Paul Erdös (1913 - 1996)
Leonhard Euler (1707 - 1783)
Pierre de Fermat (1601 - 1665)
Carl Friedrich Gauss (1777 - 1855)
Apu Nahasapeemapetilon
Sir Isaac Newton (1643 - 1727)

1. A mathematician is a machine for turning coffee into theorems.
2. Cogito ergo sum. (I think, therefore I am.)
3. In fact I can recite pi to 40 000 places. The last digit is one!
4. Put your hand on a hot stove for a minute, it seems like an hour. Sit with a pretty girl for an hour, and it seems like a minute. Now THAT'S relativity.
5. Give me a lever long enough and a fulcrum on which to place it, and I shall move the world.
6. I mean the word proof not in the sense of the lawyers, who set two half proofs equal to a whole one, but in the sense of a mathematician, where half proof equals 0, and it is demanded for proof that every doubt becomes impossible.
7. (upon losing the use of his right eye) Now I will have less distraction.
8. To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it.
9. If I have seen further, it is by standing upon the shoulders of giants.






MATT GROENING

Maths Jokes

A team of engineers were required to measure the height of a flag pole. They only had a measuring tape, and were getting quite frustrated trying to keep the tape along the pole. It kept falling down! A mathematician comes along, finds out their problem, and proceeds to remove the pole from the ground and measure it easily. When he leaves, one engineer says to the other: "Just like a mathematician! We need to know the height, and he gives us the length!"

At a conference, a mathematician proves a theorem. Someone in the audience interrupts him: "That proof must be wrong - I have a counterexample to your theorem." The speaker replies: "I don't care - I have another proof for it."

A biologist, a physicist and a mathematician were sitting in a street cafe watching the crowd. Across the street they saw a man and a woman entering a building. Ten minutes they reappeared together with a third person.
- "They have multiplied," said the biologist.
- "Oh no, an error in measurement," the physicist sighed.
- "If exactly one person enters the building now, it will be empty again," the mathematician concluded.

Problem: Expand $(a+b)^n$.

Solution:

$$(a+b)^n$$

$$(a \quad + \quad b)^n$$

$$(a \qquad + \qquad b)^n$$

$$(a \qquad\quad + \qquad\quad b)^n$$

etc.

Maths Quotes

I had a feeling once about Mathematics — that I saw it all. Depth beyond depth was revealed to me — the Byss and Abyss. But it was after dinner and I let it go.

Winston Churchill

Black holes are where God divided by zero.

Steven Wright

A mathematician is a blind man in a dark room looking for a black cat which isn't there.

Charles Darwin

Mathematicians are a species of Frenchmen: if you say something to them they translate it into their own language and presto! It is something entirely different.

Johann Wolfgang Goethe

If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?

David Hilbert

He uses statistics as a drunken man uses lamp posts — for support rather than illumination.

Andrew Lang

Bridges would be safer if only people who knew the proper definition of a real number were allowed to design them.

David Norman Mermin

The Exotic Realm of p-adic Numbers

Numbers come in many forms, shapes and sizes. We all use whole numbers, fractions, and real numbers every day, but many people never stop to ask why these types of numbers are special, or natural: why should these numbers be as they are? Some mathematicians, on the other hand, do ask what the hell is going on, and how did these numbers get here, and by the way where did I put my coffee, and oh can you remind me what my name is again?

It turns out that these hare-brained mathematicians have a point. While the "usual" numbers mentioned above do form part of the world of numbers, this world of numbers and number systems is immeasurably broader, full of amazing and strange lands. And one of the most exotic corners of this world is the realm known as *p-adic numbers* — a realm rarely visited by the average mathematician, much less the average person!

So, let us don our Thinking Caps and Number Theoretical Boots and head off to this undiscovered country! We shall attempt to observe this exotic species in its natural state. But beware, p-adic numbers are a highly twisted bunch!

Our journey starts at the familiar land of integers (whole numbers). We should all know what a whole number is! But as the great mathematician Dedekind once said, "God made the integers; all the rest is the work of man." We quickly move on to the nearby field of fractions, or rational numbers, which hopefully we should all know as well.

We can be content with our knowledge of fractions from primary school, but a pure mathematician might ask how we got there from the land of integers. The answer is, of course, you get rational numbers by *dividing* one integer by another! Starting from 3 and 5, you get $\frac{3}{5}$ by dividing 3 by 5. A pure mathematician might go further and actually *define* rational numbers *in terms* of integers — in fact as an *ordered pair* or integers — but let's not trouble ourselves with details. We need to get to our destination, after all! But you might note that, in order to gain a better understanding of a number system like the fractions, you should try to relate it to a simpler number system like the integers.

So, after a brisk traversal of the field of rational numbers, we move on and arrive at the kingdom of real numbers. Now we still might have a pretty good idea of what a real number is, from our intuition. A real number is a point on the real number line. Or, maybe slightly more accurately, a real number is one that can be written as a decimal, for instance

$$-1.2,\ 0.66666\cdots,\ 1.414213562373\cdots,\ 26.$$

Note that the decimal digits can terminate, or continue infinitely far, with or without repetition.

However, our pure mathematician friend (if he hasn't got lost yet and strayed over into the sphere of complex numbers) might want a bit more detail here. Yes, but how did we get the real numbers from the rational numbers? Well, there are a few ways to answer this, but one way might be as follows (our friend Dedekind had a different answer). We can think of real numbers as *numbers approximated by rational numbers.* So for instance

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421,\ 1.414213,\ldots$$

is a sequence of rational numbers which approximates the real number which we know as $1.414213562\cdots = \sqrt{2}$, while the boring sequence $221, 221, 221, 221, \ldots$ is a sequence of rational numbers approximating, you guessed it, 221! In this way we can approximate all rational numbers, but also we will add in extra numbers to get the entire real number line. Technically, these "approximating" sequences are called *Cauchy* sequences, but again let's not bother ourselves with details too much. This process is known as *completing* the rational numbers.

Having brought you this far on the journey, we must say *turn back!* We've actually gone too far, and need to go back to the field of rational numbers. So forget about the real numbers, go back to the fractions. And let's take a different route.

The clever mathematician (or maybe it's just too much exposure to the elements) might ask, "well, is there any *other* way to complete the rational numbers?" Because, while the main track through the field of rational numbers leads directly to the reals, there is another, less travelled road which, if you find it, leads to the exotic and surreal land of p-adic numbers.

How might we complete this task of completion? Remember that the real numbers are made by approximating them with rational numbers. But there is more than one way to approximate numbers! Here's a different way.

On a whim let's try to find numbers congruent to 221 modulo 7. So for instance we have

$$221 \equiv 4 \bmod 7.$$

Then, let's try for a larger modulus. Let's try modulo $7^2 = 49$, which (in a vague way) is a "refinement" of modulo 7.

$$221 \equiv 25 \bmod 7^2$$

And now let's keep going...

$$221 \equiv 221 \bmod 7^3, \qquad 221 \equiv 221 \bmod 7^4, \ldots$$

So, by refining our search by taking higher and higher modulos, we can obtain a sequence of rational numbers which "approximate" 221, in some fashion! The sequence is $4, 25, 221, 221, 221, 221, \ldots$.

Seems pretty ridiculous? Well, it only gets stranger. Let's see if we can get a sequence of numbers, using the same method, to approximate $\sqrt{2}$, which you might recall is *not* a rational number! Well, finding $\sqrt{2}$ is the same thing as finding a solution x to the equation $x^2 = 2$. So, again we'll investigate the problem modulo 7, 7^2, 7^3, and so on.

$$\begin{aligned} x^2 \equiv 2 \bmod 7 &\quad\Rightarrow\quad x \equiv 3, 4 \bmod 7 \\ x^2 \equiv 2 \bmod 7^2 &\quad\Rightarrow\quad x \equiv 10, 39 \bmod 7^2 \\ x^2 \equiv 2 \bmod 7^3 &\quad\Rightarrow\quad x \equiv 108, 235 \bmod 7^3 \\ x^2 \equiv 2 \bmod 7^4 &\quad\Rightarrow\quad x \equiv 2166, 235 \bmod 7^4 \end{aligned}$$

Now, as a mathematician named Hensel found in the 19th century, it turns out that you get *exactly* two solutions for every modulus 7^n (can you prove it?), so we get two sequences of rational numbers (in fact just integers)

$$3, 10, 108, 2166, \ldots \text{ and } 4, 39, 235, 235, \ldots$$

which approximate the two numbers $\pm\sqrt{2}$ somehow! In fact, we say that these sequences *converge 7-adically* to $\pm\sqrt{2}$. So $\sqrt{2}$ is a 7-adic number, quite close to 108, though even closer to 2166!

If you can escape from the previous discussion with your brain intact, then you're well on the way to p-adic land! Because the p-adic numbers are just what you get when you complete the rational numbers, adding in all the necessary extra numbers, in this bizarre, insane, who-fried-my-brain kind of way. They are *numbers approximated by congruences modulo larger and larger powers of p*. Note that, as you might have guessed, the p here stands for a prime (we took $p = 7$ above).

Let's think a bit more about what we're saying by "approximating" here, because it has mind-bending implications. Normally, if we're given two numbers x, y and asked to find their "distance apart", we look at $|x - y|$. But this is no longer the case in the p-adic realm! Rather, we think of "distance apart" as *how many times $x - y$ is divisible by our prime p.* The more times divisible, the closer the numbers are. So, 1 and 1001 are quite close 2-adically, since 1000 is divisible by 2^3. The numbers 1 and 1000001 are even closer, since 1000000 is divisible by 2^6. But 1 and 0 are far apart 2-adically (in fact, any-adically), since 1 is not divisible by 2 (or any other prime p).

As one final glimpse into this surreal world, let's look at the series

$$1 + 2 + 4 + 8 + 16 + 32 + \cdots .$$

Consider the partial sums 2-adically, and compare them to -1.

$$\begin{aligned} 1 &\equiv -1 \bmod 2 \\ 1 + 2 = 3 &\equiv -1 \bmod 2^2 \\ 1 + 2 + 4 = 7 &\equiv -1 \bmod 2^3 \\ 1 + 2 + 4 + 8 = 15 &\equiv -1 \bmod 2^4 \end{aligned}$$

So we can see that these partial sums are getting closer and closer to -1. Therefore, in the limit we have the following astounding sum, which incidentally agrees with the formula you might have learnt for geometric series!

$$1 + 2 + 4 + 8 + 16 + 32 + \cdots = -1$$

Or, you could write this equation in "2-adic binary notation", in which case the left-hand side has an infinite expansion, but *before the decimal point*!!!

$$\cdots 1111111111111 = -1$$

You might ask, what's so special about p-adic numbers? Couldn't we have made up any dumb rules we wanted to complete the rational numbers and come up with a silly number system? Well, it turns out that the *only* way you can properly complete the rational numbers is to get either some p-adic numbers, or the reals! So they are quite important... and number theorists use them a lot as well. But unfortunately, our voyage is over, and we must return to the mundane land of integers... For some detail, see, for instance, *p-adic Numbers: An Introduction* by Gouvea.

— Daniel Mathews

Winners of Last Issue's Paradox Problems

The following people have won prizes from last issue's Paradox problems. Please drop by the MUMS room in the Richard Berry Building to receive your prize.

Problem 2 — Ian Preston ($5)

Problem 3 — Wendy Pan ($5)

Problem 5 — Vivienne Agus ($5)

Answers to "Know Your Mathematicians"

Quotes: 1. Erdös 2. Descartes 3. Apu 4. Einstein 5. Archimedes 6. Gauss 7. Euler 8. Fermat 9. Newton

Top row (left to right): Euler, Newton, Fermat
Middle row (left to right): Einstein, Archimedes, Erdös
Bottom row (left to right): Gauss, Apu, Descartes

Thanks

The Paradox team would like to thank Lawrence Ip, Daniel Mathews and Nicholas Sheridan, for contributing fantastic articles to this issue.

New York (CNN) — At John F. Kennedy International Airport today, a Caucasian male (later discovered to be a high school mathematics teacher) was arrested trying to board a flight while in possession of a compass, a protractor and a graphical calculator. According to law enforcement officials, he is believed to have ties to the Al-Gebra network. He will be charged with carrying weapons of math instruction.

Solutions to Last Issue's Paradox Problems

2. The plane is divided into regions by straight lines. Show that it is always possible to colour the regions with two colours so that adjacent regions are never the same colour.

   Solution: We will prove the result by induction on the number of lines. For zero lines, it is obvious how to colour the plane. Suppose that we can colour the plane divided into regions by k straight lines in two colours. Then, after adding a new line onto the plane, we can colour the plane divided into regions by $k+1$ straight lines by swapping the colours of the regions on one side of this line. It is easy to check that all of the adjacent regions away from this new line are of differing colours. Also, the regions which share a boundary created by this new line are of differing colours. Therefore, by induction, the result holds for any number of lines drawn in the plane.

3. *Solution:*

HOUSE 1	HOUSE 2	HOUSE 3	HOUSE 4	HOUSE 5
Yellow	Blue	Red	White	Green
America	Russia	England	Spain	Japan
Football	Table Tennis	Hockey	Basketball	Baseball
Water	Tea	Milk	Orange Juice	Coffee
Fox	Horse	Hamster	Dog	Monkey

5. *Solution:*

[1] 3	[2] 6		[3] 7	6	[4] 8		[5] 7	[6] 1
[7] 5	1	1	0		[8] 4	9	7	0
	4		[9] 8	[10] 5	8		8	
[11] 9	4	[12] 6		0		[13] 2	8	[14] 8
9		[15] 4	8		[16] 5	9		6
[17] 4	[18] 9	9		[19] 2		[20] 1	[21] 2	4
	0		[22] 9	0	[23] 7		0	
[24] 7	0	0	0		[25] 1	0	2	[26] 4
[27] 5	9		[28] 2	0	0		[29] 7	0

Paradox Problems

The following are some maths problems for which prize money is offered. The person who submits the best (i.e. clearest and most elegant) solution to each problem will be awarded the sum of money indicated beside the problem number. Solutions may be emailed to

`paradox@ms.unimelb.edu.au`

or you can drop a hard copy of your solution into the MUMS pigeonhole near the Maths and Stats Office in the Richard Berry Building.

1. *100 prisoners and a light bulb* ($10)

   There are 100 prisoners in solitary cells. There is a central living room with one light bulb; this light bulb is initially off. No prisoner can see the light bulb from his or her own cell. Every day, the warden picks a prisoner at random, each of them with equal probability, and that prisoner visits the living room. While there, the prisoner can toggle the bulb if he or she wishes. Also, the prisoner has the option of asserting that all 100 prisoners have been to the living room by that time. If this assertion is false, all 100 prisoners are shot. However, if it is indeed true, then all prisoners are set free. Thus, the assertion should only be made if the prisoner is 100% certain of its validity. The prisoners are allowed to get together one night in the courtyard, to discuss a plan. Can you devise a plan that they can use so that eventually they will all be set free?

2. *Tapeworm* ($5)

   Imagine a rubber tape one metre long. A worm starts at one end and travels along the tape at 1 centimetre per second. At the end of every second, the tape gets stretched so that it was one metre longer than before (the worm is carried along with the stretching). So the worm travels 1 centimetre, the tape gets stretched 1 whole metre, then the worm travels 1 centimetre farther on the stretched tape, the tape gets stretched again by another metre, and the worm travels 1 centimetre farther, etc...

   Does the worm ever reach the end of the tape?

3. *Birthday brothers* (\$5)

   One year in the 20th century, Alphonse noticed on his birthday that adding the four digits of the year of his birth gave his actual age. That same day, his brother Benedict — who shared Alphonse's birthday but was not the same age as him — also noticed this about his own birth year and age. That day, both were under 99. By how many years do their ages differ?

4. *An amazing problem* (\$5)

   A maze consists of an 8×8 grid, in each 1×1 cell of which is drawn an arrow pointing up, down, left or right. The top edge of the top right square is the exit from the maze. A token is placed on the bottom left square, and then is moved in a sequence of turns. On each turn, the token is moved one square in the direction of the arrow. Then the arrow in the square the token moved from is rotated $90°$ clockwise. If the arrow points off the board (and not through the exit), the token stays put and the arrow is rotated $90°$ clockwise. Prove that sooner or later the token will leave the maze.

5. *Sharing the Music* (\$5)

   Each of six friends is sharing the music by lending a CD of his or her favourite group to one of the other five. Can you determine which CD each of the six owns and who is currently borrowing it?

   (a) The six music sharers are Denise, the girl who owns the Cake CD, the boy who is borrowing the Múm CD, Helen, the teenager who owns the CD by Radiohead, and the one who is currently listening to another's Pulp CD.

   (b) Two of the boys are borrowing music owned by two of the girls; the boy who is listening to another boy's favourite has his Beatles CD.

   (c) Helen, who isn't the teen who owns the Múm CD, isn't the person who is borrowing the Cake music.

   (d) Norm is borrowing neither the Portishead nor the Radiohead CD.

   (e) Daphne is borrowing neither the Cake nor the Beatles CD.

   (f) Denise isn't the friend who is listening to Alan's favourite group, and Alan isn't the one listening to Shannon's loaned CD.

   (g) The teen who is borrowing the Radiohead CD doesn't own the Portishead CD.