Paradox

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Paradox

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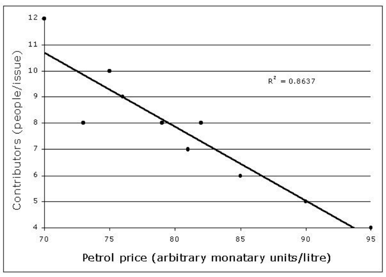
Words from the Editor

Time flies, and while you may still be recovering from the soccer/holidays/the last issue of Paradox, we have brought out one of the most entertaining issues ever! For those who don't already know, Paradox is the magazine of the Melbourne University Mathematics and Statistics Society (or MUMS – I'll leave you to figure out this slightly inappropriate acronym).

So you may ask, why would one divest his or her precious time from important tasks like reading the social gossip columns, procrastination, playing on the X-box, or even 'studying' engineering, to insted read a maths magazine? The answer lies in the capable training of the mind by mathematics, which allows one to be logical, creative, systematic and rigorous – these are indeed products of the intellect not achievable by any of the aforementioned activities.

Students of applied mathematics may spuriously claim that Paradox is not 'useful'. Well, this issue has some great advices, including how to pass an exam metaphysically, or to survive devilish traps in your adventure. If you've ever been embarrassed about a wrong proof you did in a test, worry no more, for we have collected a whole lot of them. If you are thinking about getting married, we have some tips from the happy-end problem. There is even a World-Cup related discussion from Kim, our resident geometer. Last time Kim wrote an article for us, it immediately made its way into the Maths B course content.

Recently, a student from mathematical commerce (right. . .) alerted me to the alarming correlation between the number of contributors to Paradox per issue and petrol prices. It is claimed that 84% of the variations in the number of contributors can be explained by the petrol price. The following graph illustrates the trend. The point is that as there is no sign of the petrol price ever going down, we are in short supply of contributors. So please, any articles, puzzles or ideas are welcome! Just send them via email and we'll get them published.



— James Wan

Angels and Demons

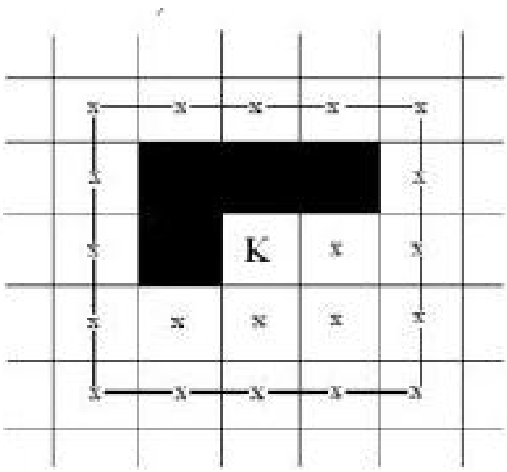
(apologies to Dan Brown)

The Problem

Imagine, for a moment, that you are Indiana Jones. You've just finished slashing your way through a swarm of faceless enemies, have rescued a dashing blonde, and discovered the whereabouts of the lost city of Atlantis (and all this before lunch), and now you find yourself in an ancient temple, walking across a floor which is tiled in a grid. Then all of a sudden one of the floor tiles around you gives way, falling into the fiery depths of a volcano below. You step across to a tile beside you, and as you do, another tile falls away. Every new tile you move to leads to another tile being destroyed and you realise that if these fallen tiles manage to completely surround you, all will be lost, as you will have nowhere to go. The question is, are you safe? Sure, the tiles may be falling randomly, so it's unlikely you will be completely surrounded, but what if they weren't? What if someone was controlling them? Could they trap you, assuming the floor you are on is infinite? Who wins this game of life and death? Indiana, are your days of tree-slashing and whip-cracking over?

This is essentially the *Angels (and Devils)* problem, made famous by John Conway (think *A Beautiful Mind*), but actually first discussed many years earlier.[1] The *angel* – placed somewhere on an infinite tiled floor - is Indiana, and the *devil* controls the tiles, eliminating exactly one tile for every move the *angel* makes. There are many variants to the problem, but essentially they all ask the same question, does the *angel* have an escape strategy, to avoid being captured, or does the *devil* have a strategy to eventually trap the *angel*, no matter how long it takes? From the simple version above, we can extend the problem to consider the k*-angel*, one that, on every turn, can fly to anywhere in a k-tile radius (to put it formally, from coordinate (x, y) it can move to any $(x', y') \neq (x, y)$ such that $|x - x'|, |y - y'| \leq k$, including jumping over any removed squares in the middle).

[1] D. Silverman and R. Epstein are usually credited with its invention.

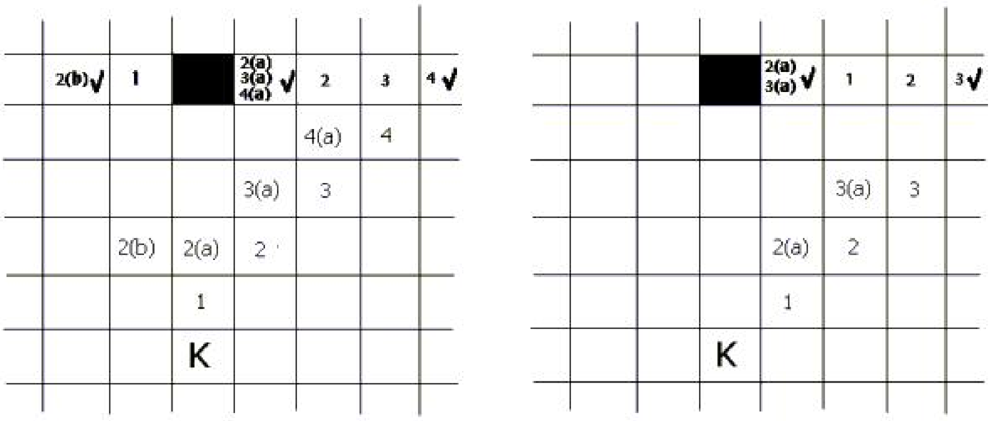


A 2-*angel* can move to any of the squares marked with an x, even if it means jumping over empty ones (the black squares)

This problem is deceptively simple. The question posed by Conway asks whether, for a sufficiently large k, a k-*angel* has a strategy to escape the *devil*. On first glance it seems obvious; a powerful enough *angel* should be able to escape the clutches of the *devil*, as it can fly, maybe, millions of tiles for every one that the *devil* removes. A naive strategy might be to run in one direction, always heading north, and the *devil* will never be able to stop you. Surprisingly though, this problem is still unsolved. No one can prove that **any** *angel*, no matter how powerful, ever has an escape strategy, but also no one has proved the opposite, that the *devil* always wins. If the thought of solving this elusive problem is not incentive enough, Conway was so intrigued by it that he offered his own money to anyone that can solve it: $100 for a proof that a powerful enough *angel* can escape, and $1000 for the proof that the *devil* always wins. Clearly, Conway sits firmly in the *angel* camp.

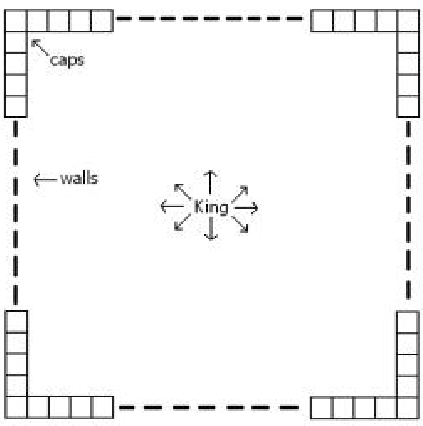
The *chess-king* case

There has been much headway made into this problem over the years. The obvious first step is to solve the problem for the 1-*angel*, Indiana in the example above, or a *chess-king* if you prefer. For the case of the *chess-king*, it turns out, perhaps unsurprisingly, that the *devil* has the winning strategy. To see why, let us consider what happens when the *king* tries to move in one direction. As the *king* heads, say, north, the *devil* can remove squares in an east-west line above him, preventing him from ever going past it. If the *king* moves west, the devil just adds to the wall, extending it out to infinity. We call this 'pushing' the *king* along the line. All we need to do is start building the wall 5 squares away from the *king*, and we can successfully create the wall, blocking further progress. A proof of this is shown on the next page.



Here the *king* is attempting to head north, and the *devil* aims to prevent this. The *devil* first removes the square directly in front of the *king* (the solid black square). After this, the *king's* moves are answered by the *devil's* in bold. So if, for example, the *angel* moves to the square in front of it, the *devil* replies by eliminating the square directly to the left of the black square. The *devil* can always force a situation where the *king* is faced with 3 empty squares in front and on either side of him. From there, the *king* can be 'pushed' along the wall forever.

Now, if we can successfully build one wall, we can do the same for four of them, one on each side of the *king*, boxing it in. The only complication would be where two walls intersect, for if we are pushing the *king* along one wall, we have no time to create another. To prevent this problem occurring, we first create 'caps' and then proceed to fill in the walls. Using this method, we can be assured of capturing the *king* (in fact we can do it using only a 32×33 board.)



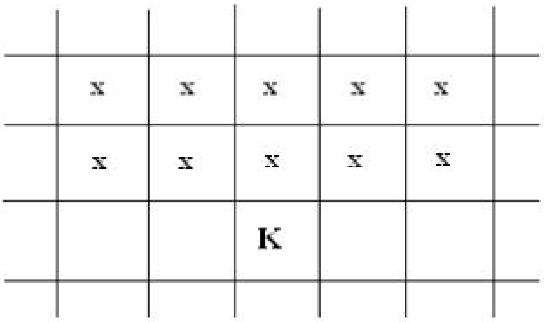
The *devil* first constructs the 'caps', and then fills in the walls as the *king* approaches.

The *angel* breaks its shackles

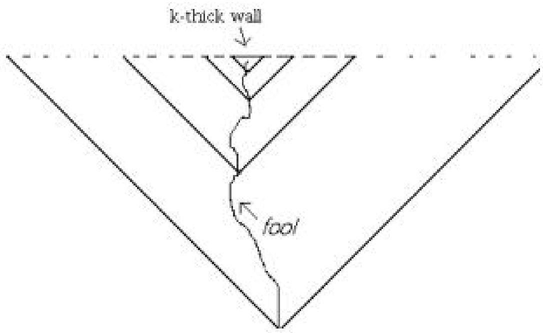
We have now proven that the 1-*angel* cannot escape the clutches of the *devil* indefinitely. But what about stronger *angels*: can the same techniques be used? Unfortunately, for just the next most powerful *angel*, the 2-*angel*, this strategy breaks down. To see why, consider the advantages the 2-*angel* has over the 1-*angel*. Firstly, the *angel* moves along the walls twice as quickly, so we must build the wall at twice the speed to contain it. Secondly, the *angel* can jump a wall of thickness 1, so we must build a wall of thickness 2 to detain it. This means that, in a sense, the 2-*angel* is 4 times as powerful as the 1-*angel*, for the *devil* would need to remove 4 times as many tiles to contain the 2-*angel* in the same way. Is this advantage to the *angel*? Not quite. The *devil* is not done with yet. It still has a few tricks up its sleeve.

The *devil* fights back

If you were a k-*angel*, faced with disappearing tiles, the most obvious survival strategy would be to head in one direction, always going the same way, and hoping for the best. Surprising enough, this is in fact one of the *worst* strategies, for it leads us to one of the few cases where we know for certain who wins, and it is the *devil*. Soon after encountering the problem, Conway proved a k-*angel* that increases its y-coordinate after every turn, a k-*fool*, will lose this game. A k-*fool* must progress north each move, even if it moves diagonally (maybe thousands of tiles west and just one north). This means that to catch it, the *devil* merely has to construct a wall above it, and the *angel* will not be able to get around it.



A 2-*fool* can move to any of the squares marked with an x, but cannot move due east or west.



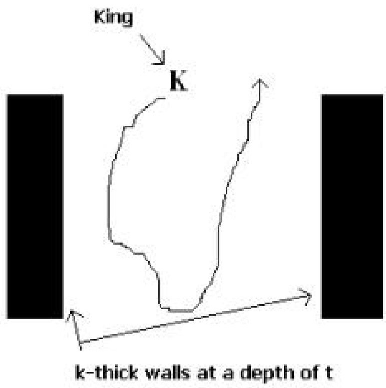
When the triangle of possible positions is readjusted to half its size, the length of the wall that needs to be filled also halves, and so the useful region of the wall is constructed at the same rate as before.

In essence, the *devil*'s strategy works like this. Starting from the origin, the *devil* considers the infinite triangle, tipped at the *fool*, of possible positions that the angel can move to, bearing in mind it must head north every turn, so cannot head due west or east, or south at all. The triangle's edges will have gradient $1/k$ (k is the power of the *fool*), because for every k moves west/east, the k-

fool must make at least 1 north. The *devil* also picks some point directly above the *fool*, at a height of h, and draws an imaginary horizontal line through it, creating a base for the triangle. Now as the *fool* heads north, the *devil* starts eliminating tiles at random points along the base of the triangle, creating the start of a 'wall' that will eventually be used to trap the *fool*. It continues to fill out this wall, always destroying squares equally spaced along it, until the *angel* has reached halfway. The key point here is that the minimum density of the wall at this moment depends **only on the power of the *fool***, and not on the height chosen for the triangle. The reason for this is because the greater the height, the longer the wall, but also the more tiles the *devil* can eliminate before the *fool* reaches it. These two factors cancel each other out exactly, hence the identical density. To complete his strategy, the *devil* readjusts the triangle of possible positions to form a new one, half the size of the old one, and continues to fill in the wall as the *angel* moves from halfway to three-quarters of the way. Throughout this period the *devil* has added to his new, shortened, wall at the same rate as before, because there is no point him adding to the wall that the *angel* can never reach (outside the triangle). Thus, at three-quarters of the way the wall is twice the density. Now, the *devil* readjusts his triangle again, and again at $\frac{7}{8}$ the distance, and again at $\frac{15}{16}$ etc. At $\frac{7}{8}$ the distance, the wall is 3 times the density. At $\frac{15}{16}$ it is 4 times, and so on. For a large enough h, we can achieve any density we like, until we achieve the density k that we require for a deep enough wall to be built. Thus by the time the *fool* reaches the wall, we have a k-thick wall in place waiting for it, so it is caught. All we need to do is start building the wall at a great enough height, something we are free to choose.

The *devil's* quest for power

Before, we considered the case of the *k-fool*, the *angel* who strictly increases its y-coordinate. From there, we can extend the same method to cover a *k-fool* who only does not decrease its y-coordinate, a *k-lax-fool*. This new *fool* has the added ability of being able to move due east/west, and so can potentially get around a wall that the *devil* may have built. However, the *devil* still has a strategy to contain him, and it is by turning him into a normal *fool*, albeit one of much greater power. For if the *k-lax-fool* chooses to tread water, and head due west or east instead of moving north, we can start blocking its progress at a distance k^2 away from the origin, so by the time the fool reaches it, the wall is *k-thick*, so it must head north. This occurs at a distance of at most $k^2 + k$. So the *k-lax-fool* **must** head north at a distance of at most $k^2 + k$ from where it started. Thus a *k-lax-fool* can be equated to a $k^2 + k$-*fool*, one for which the *devil* already has a strategy. This *devil*-strategy for a *k-lax-fool* can be extended to a radial analogy, an *angel* that never decreases its distance to the origin. Also, it can also be extended to capture a *k-relaxed-fool of laxity t*, one that may decrease its y-coordinate, but only does so by a bounded amount (so if the *fool* is at a point (x, y), it cannot later be at a point lower than $(x', y - t)$ for some laxity t). In other words, a *relaxed-fool* can move around a wall by heading south, but can only move south by a pre-determined amount, before having to move north again. To catch the *relaxed-fool of laxity t*, if it decides to head south, the *devil* builds walls on either side of it (at a massive distance away, but ones that extend downwards to a depth of t, the limit the *relaxed-fool* can drop to. In this way, the *relaxed-fool* is eventually forced to head north, and so the *devil* again converts it to a normal *fool*, though of a much higher power.



Converting the *k-relaxed-fool* of laxity *t* into a normal *fool*.

The *king* is eventually forced to head north.

Through optimum processes such as the one used to catch the *fool* and other extending techniques like the one we used to catch the *lax-fool* and the *relaxed-fool*, the *devil* can attain a power that it may not appear to originally be capable of.

The essence of the problem

The progress we have made is worthy, though a little contrived; however it is easy to see why more broad strategies also fail against an intelligent *devil*. For instance, a simple strategy for the *angel* would be to always choose a move that increases most the average distance from itself to all the empty squares. This is creating a sort of 'function', that chooses the direction that will further it from the *devil's* activity. But the *devil* can easily counter this strategy. It can build a trap at an enormous distance to the north, then scare the *angel* into the trap by eliminating squares immediately below the *angel*, thus deceiving the 'function' used to protect the *angel*. On the other hand, if the 'function' is designed to keep the *angel* away from far-flung potential traps, then the *devil* can build a trap which is relatively close, then trick the *angel* into it by eliminating a square several light-years away. It is incredibly hard to design a 'function' that is not sensitive to these sorts of tactics. This is essentially why no one has been able to develop a winning strategy for the *angel*, and thus why the problem has remained unsolved for so long.

The problem as it stands

The *Angels and Devils* problem interests mathematicians for several reasons. Firstly, it seems so patently obvious that the *angel* should win the game if you give it enough power, yet it is so hard to develop a method to achieve this victory. Secondly, the *devil* has powers far beyond what seems likely on first glance. Despite this, most people rest firmly on the side of the *angel*. The *devil's* powers, though significant, merely flatter to deceive.

Here is a brief summary of the progress made on the problem:

- Someone, either the *angel* or the *devil*, has a winning strategy; this is not an unsolvable game.
- The *devil* has a winning strategy
  - for a 1-*angel* (the *chess-king*)
  - for a *k-angel* that always increases its y-coordinate (the *fool*)
  - for a *k-angel* that never decreases its y-coordinate (the *lax-fool*)
  - for a *k-angel* that can decreases its y-coordinate, but only by a bounded amount (the *relaxed-fool*)
  - for a *k-angel* that never decreases its distance from the origin
- The *angel* has a winning strategy
  - for a *k-angel* on a 3-dimensional board (the argument used to prove this is well worth a look)

Most remarkably, the 2-*angel* case is still open. This case may indeed hold the key to generalising the entire problem, and taking home that cash prize. So before you go boasting to your friends about the $2 you won from solving a Paradox problem, try your hand at this. I'm sure John Conway would love to hear from you.

— Stephen Muirhead

"Don't laugh, Fermat!"[2]

It is funny the sort of mistaks[3] people sometimes make. Take, for example, 'proofs' of Fermat's Last Theorem (FLT) (and Goldbach's conjecture is another *prime* candidate): there are no integer solutions for $x^n + y^n = z^n$, $n > 2$ and $xyz \neq 0$. Those who fell under its guise of simplicity included some of the greats, such as Lindemann (who showed that π is transcendental), Lamé, and most likely Fermat himself[4]. Of course, Lindemann, driven by false confidence, tried to confer upon himself the imperishable distinction of solving one of the greatest problems in mathematics.[5] As for Fermat, he had a notorious history of claiming proofs he did not have (and not always blaming the margin[6]) – for instance, he claimed to have shown that every positive integer is a sum of at most three triangular numbers, four square numbers, ..., n n-polygonal numbers. The proof was never found, and it took Gauss, Jacobi, Lagrange and Cauchy (not together, of course) to finally prove the proposition.

Nevertheless, this article will feature some of the lesser-known 'heroes' who have dedicated their times to the problem - most of them came from the general public. They sought not fortune or fame, but, deceived by the problem's simplicity, they too concocted sometimes naïve 'solutions' somewhere in their encounter with mathematics, one fraught with pitfalls and gross overuse of 'QED's.

I would like to thank Professor John Groves for allowing me to raid from his office some of the impressive collection of false proofs he accumulated over the decades, many begging to be published.

Historical background

Since an episode of the *Quantum* program on ABC, 'A possible proof of Fermat's theorem', in 1989, in which Dr Watson and Prof Rubinstein from this university examined a man from Warragul and his half page nonsense 'proof'

[2]Plus Eternal Creation, and other footnotes.

[3]Such as this one.

[4]Fermat did show the $n = 4$ case, but it took the genius of Euler, 100 years after the death of the former, to make the next advance by settling the $n = 3$ case.

[5]This distinction currently belongs to Andrew Wiles, who solved the problem more than 10 years ago; the proof of Wiles' Theorem, from which FLT is a corollary, was a about 200 pages [the exact number depends on the size of the font].

[6]*Cuius rei demonstrationem mirabilem sane detexi hanc marginis exiguitas non caperet*, or 'I have discovered a truly remarkable proof of this which this margin is too small to contain.'

of FLT,[7] there has been an explosion of submissions from all around Australia. The program was replayed in 1990 and sparked even more public interest.

Most submissions showed the fallacy in thinking that the theorem referred to the sides of a right-angled triangle, in which case the demonstration is quite straightforward. Nevertheless, they often carried a matter-of-fact tone, such as, " I have attempted the problem and have arrived at the solution enclosed". Upon learning this slight technical error, many withdrew from their brief transgression of the mathematical realm, although one stubbornly resisted, "The proof of Fermat's Last Theorem I've sent in has a few errors (not logical)... thus it still remains a proof."

The Persistent Solver

Some people have been trying to find a solution to FLT for months, or even years, notwithstanding the fact that an elementary solution ever being produced is exceedingly unlikely. In fact, the attached (real) proof of the $n = 4$ case presents probably the simplest advance possible to make in FLT, and it requires a good knowledge of infinite descent, Pythagorean triples, and unique factorisation of the integers. Below are some samples from those who never thought about studying some number theory:

"I think I have now the Right approach [after being] busy on and off Fermat's Last Theorem since 1956". The approach was using the cosine rule to test some specific values of angle C in a triangle.

"I believe [the amateur mathematician] had the correct philosophy in trying to solve the problem by using lateral thinking." "I have endeavoured to find a solution over the past 4 weeks and believe the attached proof satisfies the conjecture"

And he attached the following gems:

"The equation can be represented as a vector addition"; "common sense indicates that angles less than 60 are not valid as the hypotenuse"; to illustrate this point, 3 diagrams were included, specially showing the case of $x = y = z - 1$.

The conclusion was that the proof "could almost be written in the book margin".

Blatant errors

[7] Prof Groves recalled that "the original proof was so bad that it's not even wrong".

Some errors were so interesting that you wonder how they were made, or what they meant. Examples include:

"To start with logic to maintain reality, one must remain on the one plain [sic]."

Another 'proved' (I won't go into the details to bore the reader) that $a^3 + b^3 = c^3$ has no solutions in the reals, with the constraint that a, b, c are sides of a triangle.

Some very rigorous and wrong maths came from this submission:

"Assumption: the sum of two integers is an integer." "It makes sense to me logically."

And the main argument was: "as $(a+y)^3 = a^3 + 3a^2y + 3ay^2 + y^3$ not equal to $(y+a)^3 - y^3$, then the difference of two cubes is not a cube." The general case was settled by the misuse of the binomial theorem: "$(y+a)^n = y^n + 3y^{n-1}a + 3y^{n-2}a^2 + 3y^{n-3}a^3 + \ldots$".

Proof by examples and formulae

A common trend was to find lots of examples, and claim that they provide enough evidence for the truth of the theorem. A particular paper had many plots (again, not reproduced as they mean little), with seemingly irrelevant theorems for triangles (FLT has little to do with triangles!) all over, such as the cosine rule and Appolonius' Theorem. Accompanying them were a few 'compressed circles' which apparently represent $x^n + y^n = z^n$,[8] made out in the vague shapes of ellipses, that were declared sufficient to cover all cases.

With all these preparations, an error (0 divisor) was nevertheless found in the work. The reply, after a hiatus of 3 years, was: "Your comment is accurate, but nothing was said about the status of y. [Ed: there is no y associated with that error.] Within this long period, I consulted with the local mathematicians and ultimately, found the status of y. Henceforth, an elementary proof is found out. I wish that you shall publish the enclosed paper in your esteemed Journal [of the Australian Mathematical Society]." "No reference books or Journals are available to prepare this paper. It has been prepared with the help of few basic mathematical equations."

Proof by ignoring historical significance

[8] They don't, as the level curves of $x^n + y^n = z^n$ look more and more like a square as n increases.

Although the enlightened ones who watched *Quantum* learnt that the problem was unsolved for more than 300 years by some of the greatest mathematicians ever, some still managed to reduce it to a trivial problem.

“I came up with what I consider to be a simple solution.”

“The result of an hour's scribbling is enclosed herewith.”

“If [my solution] is indeed valid… it might make an interesting follow-up for the [TV program].”

“Since [my proof] appears OK to me…”

“Fermah…”

“Apparently [the man appearing in *Quantum*] failed in his attempt…”

A professor claimed to have not 1, but 3 proofs: 2 geometrical and 1 numerical. “The Geometrical method-I is the shortest single-page proof”. “I wish that you shall publish if you like.”

Proof by superior knowledge

One fellow amateur mathematician did some research and stated that his calculus ‘proof’ (i.e. some roundabout differentiation around a triangle) could not have been used by Fermat, since calculus wasn't invented then.[9]

Another ‘proof’ by calculus began with:

$$x^n + y^n = z^n$$

$$y = z \cos B, x = z \sin B$$

And the subsequent use of chain rule (on what?) just lost me. Yet, the writer stated, ‘the enclosed effort is very simple and could have been what Fermat had in mind… possibly there could be some flaw in these proofs.”

Proof by limits

[9]In fact this is not true: Fermat anticipated some differential calculus by his method of finding the maxima and minima on a graph.

$$
\begin{aligned}
y &= A^n - B^n - C^n \\
\frac{y}{A^n} &= 1 - (\frac{B}{A})^n - (\frac{C}{A})^n, C, B < A \\
\text{so as } n &\to \infty, \frac{B}{A}, \frac{C}{A} \to 0 \\
&\Rightarrow y \to A^n \\
&QED
\end{aligned}
$$

Proof by having no idea

"Enclosed is the solution to Fermat's problem."

Inside I found one piece of paper with 2 similar triangles drawn on it. The first triangle had side lengths $3, 4, 5$ while the second had lengths $9, 16, 25$, and the pattern continued. Remarkably, the man claimed that the second triangle was right-angled, "any parallels [i.e. the hypotenuses of the triangles] are powers of the original. . .**don't laugh Fermat!**."

A parade of 'proofs' for other famous problems

"So during the last several months I have been working at the problem. This lead me to write an article entitled: 'To trisect an Angle with the Help of only the Compasses and the Ruler'. If as yet, there is no known method for exactly trisecting an angle. . . I can send copies of my article for publication in your Journal of the Australian Mathematical Society."

Such a construction is impossible, as trisection requires to find the root a cubic, and such roots are proven to be unobtainable with the compass and the ruler.[10]

In 2004, many maths records were broken by a paper boldly (in both senses of the word) titled **"The Prime Formula and a Proof of Goldbach's Conjecture"**.[11]

The first statement read, "Every even number is the sum of two odd numbers. Let the sequence of odd numbers be $S(odd) = 2n - 1 \ldots$"

[10] But it is possible using a marked, sliding ruler, or even paper folding; see future issues if you want to know how!

[11] Which is still unsolved; it conjectures that every even number greater than 2 is the sum of two primes.

The logic continued: "Hence taking odd numbers $2n-1$ and 1, we can express $2n$ as their sum!

"Which strongly suggests the conjecture is true, apparently. Of course, some serious work begins now.

"Part 2: many even numbers are the sum of two identical primes." This was astutely met with the counter example, "$8 = 4 + 4$ but 4 is not a prime.

"Part 3: the difference of two consecutive even numbers is always 2. Therefore the next even number is often made up of a prime and 3, 5, or 7 because difference between each of these three primes is also 2. In fact this is true up to 122". Wow.

"Part 4: It is possible to create all even numbers by the sum of two primes." Huh? He further argued that because the even numbers end in even digits, and primes end in odd digits, their endings match. Q.E.D.

Well, there were even examples:

"$12 = 3 + 9, 22 = 3 + 19, 32 = 3 + 29, 42 = 13 + 29 \ldots$

"The above examples prove it is possible for every even number to be created by the sum of two primes. There are no impossible combinations and therefore no counter examples." The first example speaks for itself.

Just to make sure that we understood him, there was a restatement: "Let one of the primes be $1 \ldots$ considering the fact that Goldbach considered 1 to be Prime.[12]"

Then there was half a page describing a 'prime generating formula'[13]. However, it only worked for $n = 1$ to 22, but claimed to be true for all n.

A tad of physics

Just as you think facts and logic cannot get more astray, here is a hidden genius' thesis (all original spellings retained):

"Dear Sir. I would like to inform You that I have been RECOMMENDED for the NOBEL - PRIZE, in ASTRONOMY, and MY - THESIS, is HISTORICAL."

[12]To the credit of this intrepid solver, this is actually correct! Goldbach did consider 1 a prime, but his original conjecture was worded differently.

[13]Such formulae do exist, but they are often so contrived that they have little practical use.

Hardly any history essays have won a Nobel in Astronomy, especially since there is no such Nobel category.

The man was a self-styled "Master of Science in Astronomy, Astro-physics, Nuclear-physics, and electronics engineer."

And the somewhat ironic reply he received was: "We have added your paper to our collection of historic papers."

"My thesis has been accepted by scientists and astronomers in U.S.A, Canada, England, Australia, France, India, Japan, Iran (Persia), Sweden, Norway and West-Germany."

"When a 'ray of light' goes into 'the gravitational field of a star', it gets bent. I have proved it mathematically." "You might not understand it, but please keep it as it is Historical".

"This special star is called in astronomy a 'neutron star', or a 'white dwarf star'."

"Gravitational interactions are transmitted at an infinite speed." "Energy transferred to them by the quantum." "The velocity of light is infinite."

There were classical obfuscated passages such as: "δJx represents the integrated loss of the forward momentum of the light quantum in its interaction with all of the masses filling a cylinder of radius $y = D$ around its trajectory at the average density r. For d that average lateral distance from the path of the light quantum must be substituted out to which the retarded gravitational interactions between the light quantum and various particles of matter in the universe are effective."

It became clearer, as the thesis drawled on, that it was copied from a few general science books. But hold on, there was more:

"I'm herewith enclosing Copies of MY-THESIS, in ASTRONOMY, on, 'Matter, Space, Time, and Eternal-Creation'. I have disproved, Albert-Einstein. (who was a Genius.)"

This time, he became a 'Dr.' of the above fields, fearing that a master was not persuasive enough.

"The Primal Fireball, ejecting gases, matter, and Galaxies, into the far reaches of Outer-Space". "By simple mathematics... in millions of years in the future (on millions of years time)... all matter in space, will, having expended their velocities, will reach zero velocity and stop." "The earth is a planet."

"There have been many strange occasions on this planet, where many radio listeners, and television viewers, have to their utter astonishment, heard very old radio programs, and seen very old television programmes (but not of very high quality) on their screens. The explanation is of course quite simple: that is, the radio waves that are transmitted, escape into space and get re-deflected back to earth by a black-hole". Nah, you are just watching SBS.

"This is incorrect. I do not agree."

"Space is curved and 5-dimensional." "Keplers laws - wrong" because "Ah! But, Jupiter is a liquid planet" and it is not possible to use the laws on a liquid.

"Strange animals found in Australia - the kiwi and the platypus... the kiwi has a great resemblance to the penguins... and platypus has a great resemblance to the beavers of Canada."

"Orbiting around the sun, are hundreds of millions of small solid objects called meteoroids." "The bigger ones fell down onto earth (like a ton of bricks)." The evidence he provided was Ayers rock.

"Christ, Buddha, and Mohamed... were highly intelligent being from other civilisations, from outer space or from the cosmos." Here, the evidence was that Buddha's air chariot resembled an aeroplane.

Conclusion

Some of the above passages exemplify the saying, "a little knowledge is a dangerous thing", and we certainly learnt a lot of dangerous maths from them. Curious mistakes serve to entertain, as long as you are not the one who erred. I thank those valorous solvers who enriched our lives, though not in a way they initially hoped.

— James Wan

John von Neumann was once in a physics lecture in Princeton. The lecturer exhibited a slide with many pieces of experimental data and, although they were badly scattered, he argued that most of them lay on a curve. It is said that von Neumann murmured, "At least they lie on a plane."

This is from an actual newspaper article; all original errors have been retained.

The Australian, Saturday July 17 1971

NOTICE TO MATHEMATICIANS

I HAVE SOLVED FERMAT'S LAST THEOREM:

A CONDENSED PROOF FOLLOWS.

If $A^n = B^n + C^n$ were integral and n an odd number, we could write $B + C = a^n$, $A - C = b^n$, $A - B = c^n$ and $a.b.c.$ would be integral factors of A, B, C.

From these relations we may express $B + C - A$ as either $a^n - A, B - b^n$, or $C - c^n$, so that the difference between the second generation a^n, b^n, c^n and the third generation A, B, C, is the same quantity, which we may call x, and x must be less than b^n, or B. ***

Suppose two fixed numbers symbolized as b^n and c^n, like the numbers 125 and 64 in certain characteristics but somewhat loosely, and make our first trial value for a^n like 189.

This trial gives n the unique value $n = 1, a^n = A_1, b^n = B_1, c^n = C_1$, and $B_1 + C_1 - A_1 = 0$.

From the relation *** above, any random numbers A, B, C, can be put in the form $A = A_1 + x \quad B = B_1 + x \quad C = C_1 + x$ where $B_1 + C_1 - A_1 = 0$.

If x is greater than B_1, A, B, C, are eliminated.

If x is less than B_1, it is clear from mere inspection that if x is positive or negative (but not zero) $(B_1 + x)^n + (c_1 + x)^n$ or ${B_n}^n + {C_n}^n$ must be less than $(B_1 + C_1 + x)^n$ or ${A_n}^n$. In short, if $125^1 + 64^1 = 189^1$ adding x to each number needs a very big x to catch up with changing the index to 3.

> The famous mathematician G. H. Hardy was once riding on a train in Britain. Sitting across from him was a schoolboy reading an elementary algebra book. Endeavouring to be friendly, Hardy asked the lad what he was reading. "It's advanced mathematics," came the reply. "You wouldn't understand."

A proof

(Real) proof of the $n = 4$ case of Fermat's Last Theorem, similar to Fermat's original and only proof regarding the famous result.

We prove the more general result that there is no integer solutions to $x^4 + y^4 = z^2$ by infinite descent (that is, assuming we can find the smallest solution, we then construct a smaller one, providing the contradiction).

It is well known that coprime Pythagorean triples can be represented by

$$\begin{aligned} a &= 2mn \\ b &= m^2 - n^2 \\ c &= m^2 + n^2 \end{aligned}$$

Where m, n are coprime. Then $\{x^2, y^2, z\}$ are a coprime triple, from which we know that (WLOG) $x^2 = 2mn$ and $n^2 + y^2 = m^2$. Then $\{y, n, m\}$ form another coprime triple. As x is even, y is odd, so n is even. That is, there are coprime integers r, s such that

$$\begin{aligned} n &= 2rs \\ y &= r^2 - s^2 \\ m &= r^2 + s^2 \end{aligned}$$

Now, if the product of two (positive) coprime integers is a perfect square, then each must be a perfect square. As $\frac{mn}{2} = (\frac{x}{2})^2$, it follows m and $\frac{n}{2}$ are perfect squares. Similarly, as $rs = \frac{n}{2}$, r and s are perfect squares. But then $r^2 + s^2 = m$ gives us a smaller solution, for the left hand side is the sum of two perfect fourth powers, and the right hand side a square, and so the proof is complete.

One day Norbet Wiener was walking across the MIT campus when someone stopped him with a question on Fourier analysis. Wiener wrote down the answer in some detail. The interlocutor was most grateful, and began to go on his way. "Just one moment," said Wiener. "Which way was I walking when we met?" The man pointed in the direction. "Good," said Wiener. "Then I've had my lunch."

The Many Faces of Soccer and Other Digressions

And the million dollar question is . . . "The *hebesphenomegacorona*, *gyrobifastigium*, *bilunabirotunda* are all: "

A Custom Torture Instruments

B Non-Platonic Polytopes

C Cryptozoological Species

D Seventeenth Century Navigational Devices

Just to keep you in suspense I'm not going to answer this fascinating question yet but instead explore a related topic – the 2006 World Cup. Regarding this, most people are still asking why the Socceroos weren't world champions, given that Italy defeated them simply because defender Fabio Grosso decided to charge into Lucas Neill's defenseless head. This is a perfectly reasonable question, however it is not entirely relevant to this discussion. A slightly more pertinent issue is the controversy surrounding the design of the ball. With 14 bonded panels, the new design is nothing like your conventional soccer ball. Which begs the question: what does a conventional soccer ball look like? If you ask most people this question they can probably tell you that:

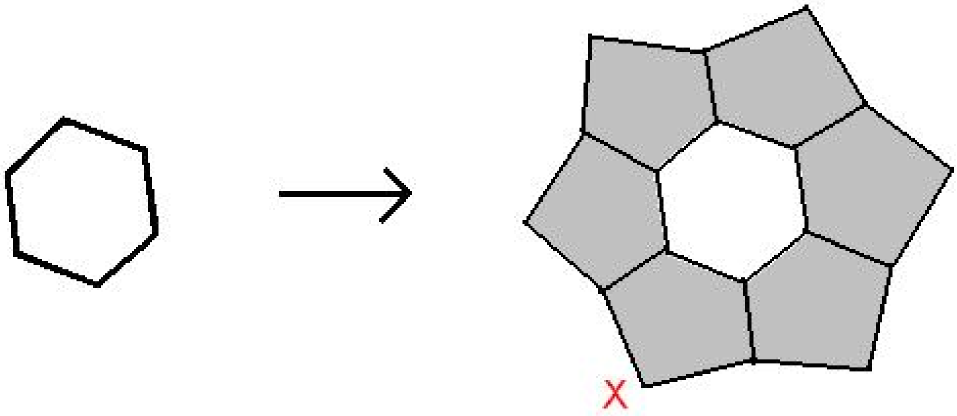
I - It consists of regular pentagons and hexagons.
II - Any corner looks the same as any other corner.

Actually, with just this information, it would be possible to work out what a soccer ball looks like, even if you were an AFL player.

So a good start is to consider what a vertex looks like. With a little thought it should be clear that you could not have 4 or more faces meeting at a vertex. If you could, what would happen when you flattened the faces at that vertex to form a planar net? The angle sum would be at least $4 \times 108°$ which is greater than $360°$.

Similarly, you could not have 2 or less faces meeting at a vertex. So at each vertex you must have 3 faces meeting. Next you might ask what these 3 faces could be. They could not be three hexagons, as the faces would be coplanar. They cannot be three pentagons either, otherwise by II, the whole soccer ball would be all pentagons with no hexagons, i.e. a dodecahedron.

So at all vertices, 2 pentagons and 1 hexagon meet, or at all vertices 1 pentagon and 2 hexagons meet. So starting with a single hexagon, let's see what happens in the former case.



But now what happens at vertex X? Two hexagons must meet here. This is impossible.

So the only possibility is at every vertex, there is 1 pentagon and 2 hexagons. A simple sketch should convince you that this means every pentagon is surrounded by 6 hexagons, and every hexagon by 3 pentagons and 3 other hexagons. So far we have scammed our way without actually doing any mathematics - yet our image is almost complete!

We need a few definitions: Let V be the number of vertices, F be the number of "faces" and E be the number of edges. Suppose there are P pentagons and H hexagons (so $F = P + H$).

At every vertex we have 3 edges meeting, but each edge connects 2 vertices, so $3V = 2E$. Furthermore, each pentagonal face has 5 edges, and each hexagonal face has 6 edges, but each edge borders 2 faces. So $5P + 6H = 2E$.

We now use Euler's theorem for convex polyhedra, $V + F - E = 2$. Bung everything into the equation and you get $\frac{5P+6H}{3} + (P + H) - \frac{5P+6H}{2} = 2 \Longrightarrow P = 12$. However, now consider edges bordering between pentagonal and hexagonal faces. For pentagonal faces, there are $5P$. For hexagonal faces, there are $3H$. So $5P = 3H$. Then $H = 20$.

We now know the numbers of pentagonal and hexagonal faces, and their arrangements. We hence determine a soccer ball to be as follows:

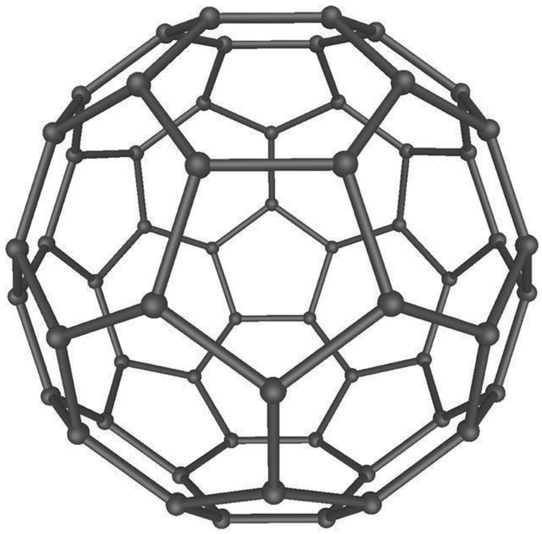


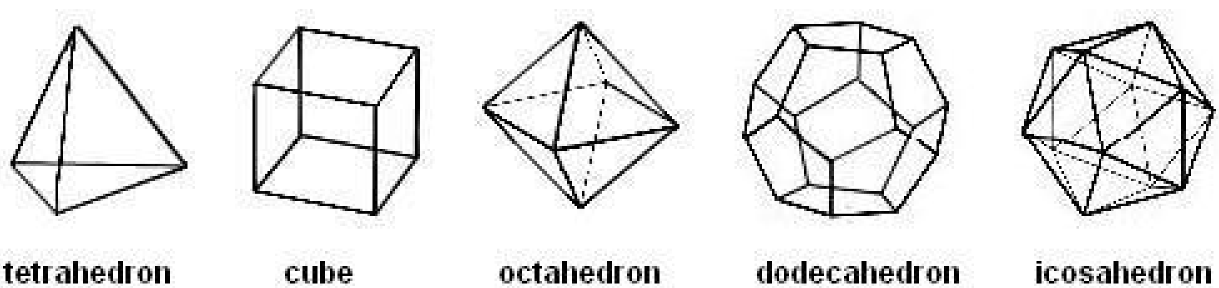
Image Source: Wikipedia

It is interesting to note that the soccer ball is formally known as the *truncated icosahedron*, and that it is the structure of the molecule of the allotrope of carbon called *buckminsterfullerene* (C_{60}).

A very similar problem is that of finding all the Platonic Solids. The Platonic Solids are those polyhedra with the following two properties:

- They are convex, meaning that any segment connecting two non-coplanar points on the surface of the polyhedron lies within the interior.
- All faces are congruent regular polygons.

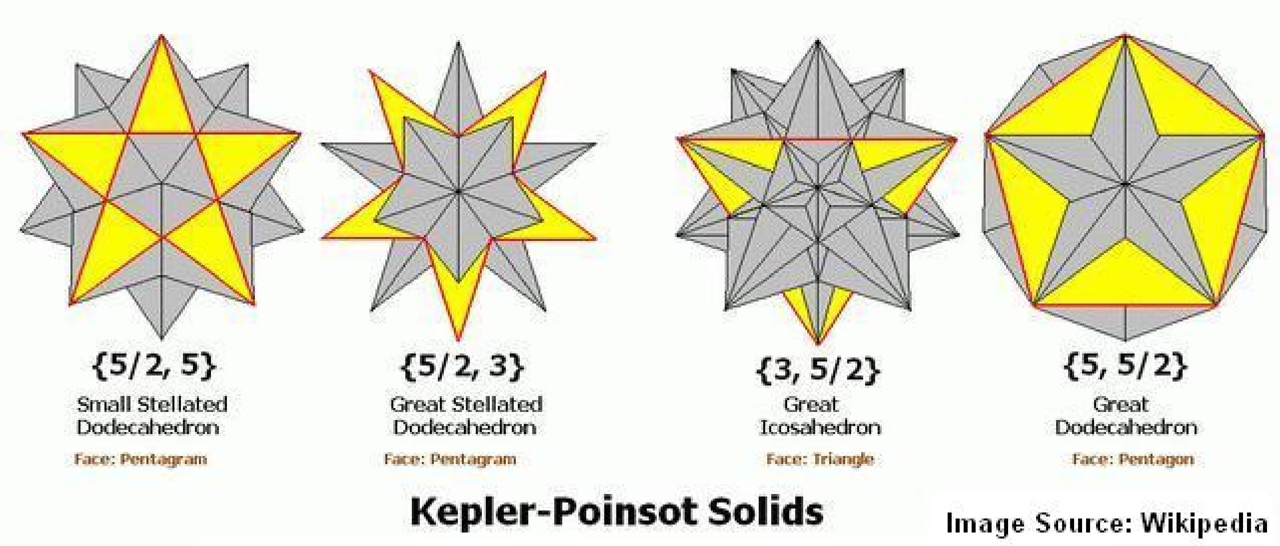
Using Euler's theorem, it is fairly simple to prove there are only 5.



Platonic Solids

Often convexity is ignored when applying Euler's theorem to polyhedra, however this problem demonstrates the danger of this omission.

If concave polyhedra are allowed, Euler's theorem is not necessarily true (as the polyhedra are not necessarily topologically equivalent to a sphere), and in fact it turns out there are an additional 4 polyhedra satisfying the 2^{nd} requirement, known as Kepler-Poinsot solids after their discoverers.



Euler's theorem fails for the *great dodecahedron* and *small stellated dodecahedron*, instead we have $V + F - E = -6$. This discrepancy led the mathematician Ludwig Schläfi to mistakenly believe they could not exist.[14]

It is worth noting that Euler's theorem *does* hold if we view the polyhedra as "normal" polyhedra. However in this case the faces can no longer be viewed as regular polygons. For example the face of the *great dodecahedron* is no longer a regular pentagon but instead an isosceles triangle.

You are probably wondering what all of this has to do with the original million-dollar question, which I remind you is:

"The *hebesphenomegacorona*, *gyrobifastigium*, *bilunabirotunda* are all: "

A Custom Torture Instruments

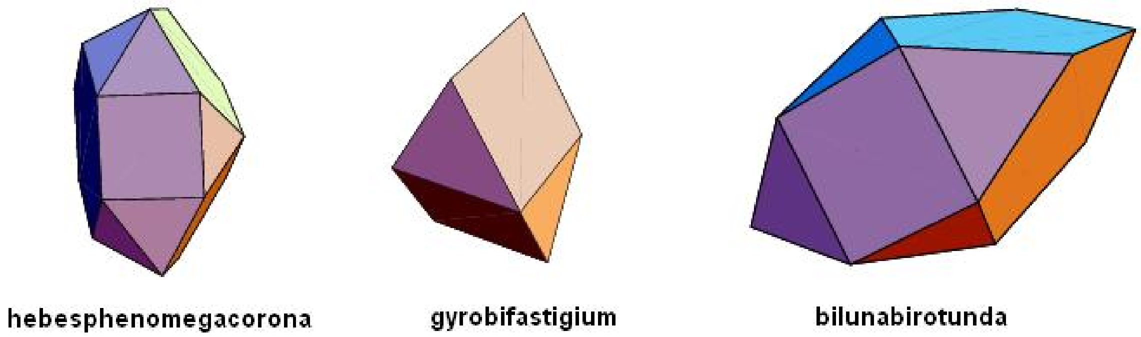
B Non-Platonic Polytopes

[14]See http://mathworld.wolfram.com/Kepler-PoinsotSolid.html

C Cryptozoological Species

D Seventeenth Century Navigational Devices

Just to confirm your suspicions, the theme of this article has been polyhedra, so it may help to know that a polyhedron is really just a *polytope* in $\mathbb{R}^3$. So yes, the answer is B. The *hebesphenomegacorona*, *gyrobifastigium*, *bilunabirotunda* are all examples of Johnson solids, which are those convex polyhedra having regular faces, and equal edge lengths, excluding certain families of polyhedra such as the Platonic solids.[15]



Three Johnson Solids Images collated from MathWorld

I suppose option A is also arguable. Given its fearsome shape, a slow painful death by the *bilunabirotunda* is definitely imaginable. Realistically though, if you're in the torture biz, using something like the *gyroelongated pentagonal cupolarotunda* would probably sound more formidable.

— Kim Ramchen

On one occasion, Erdös met up with a mathematician and asked him where he was from. The reply was "Vancouver". "Oh," said Erdös, "then you must know my good friend Elliott Mendelson." After a moment's silence the reply was, "I *am* your good friend Elliott Mendelson."

[15]See http://mathworld.wolfram.com/JohnsonSolid.html for exactly which families.

A metaphysical view of 620-211

When I started maths and stats
I also did philosophy
in that we studied brains in vats
and Descartes' dual-lology

My study habits weren't the best
I did not always go
To lectures, tutes and even tests
My learning curve was slow

One day in June, I learned my lesson
it's one that you must know
it happened in an exam-session
I was a daft mo-fo

On thursday was my maths exam
but physics was on tuesday
on monday I had planned to cram
for only stuff the next day

So Tuesday came, so starts the fable
but something wasn't proper
I sat and stared but on the table
there lay a quaint heart-stopper

The maths exam was sitting there
and quite to my surprise
I looked around and to be fair
could not believe my eyes

I couldn't believe I'd muddled up
the dates of my exams
but I never, never, never give up
so I came up with a plan

I'd work from basics, from the ground
first principles they were known
Whitehead and Russell knew it was sound
as Principia Mathematica has shown

Alas my mind had not the will
nor ability that day
to finish all, to crush, to kill
the shock and the dismay

But greater shock there was to come
for when results came out
that subject's final score would sum
to fifty percent

— Daniel Yeow

P.S. Due to his interesting study techniques, Daniel became the education officer for MUMS.

Maths jokes

$1 + 1 = 3$ for very large values of 1.

—

Q: What do you call a young eigensheep?

A: A lamb, duh!

—

Q: Why are mathematicians afraid to drive a car?

A: Because the width of the road is negligible comparing to its length.

—

Q: How do you make one burn?

A: Differentiate a log fire.

—

Life is complex; it has both real and imaginary components.

—

"I'm sorry, the number you have dialed is imaginary. Please rotate by 90 degrees and dial again."

—

This is a *transcendental extension in all fields*, so to speak, of the well-known "prime" joke: the task is to test the hypothesis that all odd numbers greater than 1 are primes.

Mathematician: 3 is a prime, 5 is a prime, 7 is a prime, 9 is not a prime, hence the hypothesis is false.

Pure mathematician: 3 is a prime, 5 is a prime, 7 is a prime, and by induction the hypothesis is true.

Physicist: 3 is a prime, 5 is a prime, 7 is a prime, 9 is an experimental error, . . .

Engineer: 3 is a prime, 5 is a prime, 7 is a prime, 9 is a prime, . . .

Programmer: 3 is a prime, 5 is a prime, 7 is a prime, 7 is a prime, 7 is a prime, . . .

Salesperson: 3 is a prime, 5 is a prime, 7 is a prime, 9 – we'll do for you the best we can, . . .

Computer Salesperson: 3 is prime, 5 is prime, 7 is prime, 9 will be prime in the next release, . . .

Biologist: 3 is a prime, 5 is a prime, 7 is a prime, 9 – results have not arrived yet, . . .

Lawyer: 3 is a prime, 5 is a prime, 7 is a prime, 9 – there is not yet enough evidence to prove that it is not a prime, . . .

Accountant: 3 is prime, 5 is prime, 7 is prime, 9 is prime, deducing 10% tax, . . .

Statistician: try several randomly chosen numbers: 17 is a prime, 23 is a prime, 11 is a prime, . . .

Professor: 3 is prime, 5 is prime, 7 is prime, and the rest are left as an exercise for the student.

Psychologist: 3 is a prime, 5 is a prime, 7 is a prime, 9 is a prime but tries to suppress it, . . .

Chemist: What's a prime?

Politician: "Some numbers are prime. . . but the goal is to create a kinder, gentler society where all numbers are prime. . ."

—

An engineer, a chemist and a mathematician are staying in three cabins at an old motel. First the engineer's coffee maker catches fire. He smells the smoke, wakes up, unplugs the coffee maker, throws it out the window, and goes back to sleep.

Later that night the chemist smells smoke too. He wakes up and sees that a cigarette butt has set the bin on fire. He says to himself, "How does one put out a fire? One can reduce the temperature of the fuel below the flash point, isolate the burning material from oxygen, or both. This could be accomplished by applying water." So he puts the bin in the shower stall, turns on the water, and when the fire is out, goes back to sleep.

The mathematician has been watching all this out the window. So later, when he finds that his pipe ashes have set the bed sheet on fire, he is not in the least taken aback. He says: "Aha! A solution exists!" and goes back to sleep.

—

"A mathematician is a machine for turning coffee into theorems." – Paul Erdös

Addendum: American coffee is good for lemmas.

—

"Students nowadays are so clueless", the math professor complains to a colleague. "Yesterday, a student came to my office and wanted to know if General Calculus was a Roman war hero..."

—

In a General Calculus class, a student raises his hand and asks: "Will we ever need this stuff in real life?"

The professor gently smiles at him and says: "Of course not – if your real life will consist of flipping hamburgers at MacDonald's."

—

Mathematicians never die – they only lose some of their functions.

—

"What's your favorite thing about mathematics?"

"Knot theory."

"Yeah, me neither."

—

To continued reading, please turn the page...

The problem with a happy ending. . .

Those of you who've read up a bit about a man called Erdös (pronounced air-dish) might know about the 'happy-end problem'. For those who don't know anything about it, here's a little bit of the history behind it.

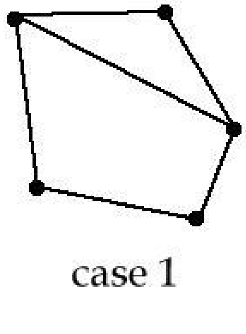
As with most stories about maths, there is coffee involved in this one. In this case, the coffee was being served at a café in Hungary, where a few people had gathered to have a little chat. During the conversation, Esther Klein shared the following maths theorem with her friends, of which included Paul Erdös and György (George) Szekeres.

Theorem: Given any 5 points drawn on a plane, no 3 are collinear, at least 4 points will form the vertices of a convex quadrilateral.

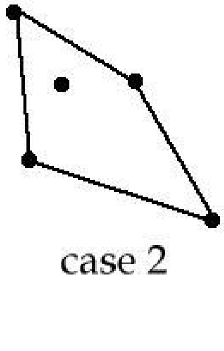
She then challenged them to show it to be true, before providing the following proof. Now, I'm going to be a bit hand-wavy about the following:

Proof of Theorem: Klein noticed that with these five points, we can always separate them into 3 possible scenarios.

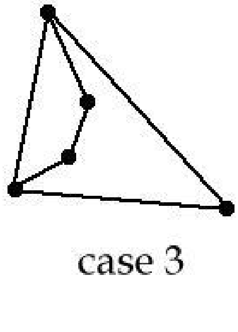
1. All five points form a convex pentagon, so just joining up any two of the dots will yield a quadrilateral.
2. Four of the points will form a convex quadrilateral, enclosing the fifth point. This is a trivial case as we have a convex quadrilateral to start off with.
3. Two of the points will be enclosed inside a triangle. Using simple arguments it can be shown that the two inside points will form a convex quadrilateral with two of the points from the enclosing triangle.



case 1



case 2



case 3

The proof was simple, elegant and yielded a very interesting result, so it was only natural that Erdös and Szekeres would try to take it further. The two eventually generalised the theorem to a conjecture that, to be certain of having a convex k-sided polygon, $2^{k-2}+1$ points would be necessary. E.g. to obtain a triangle ($k=3$), we'd need $2^{3-2}+1=3$ points, and to get a convex hexagon we'd need 17 points. Fortunately, as of yet no one has proven that $2^{k-2}+1$ is sufficient for $k \geq 6$. In fact, even the proof $k=5$ case was first published only in 1970,[16] some 37 years after the original meeting at the café, so the problem is definitely non-trivial and very worth-while.

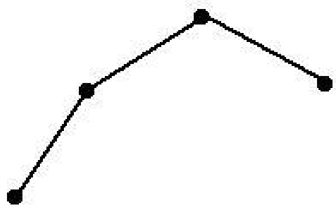
Although they did not prove their conjecture, Erdös and Szekeres did some excellent work to show that $N(k)$, the number of points necessary to guarantee a convex k-sided polygon, always exists. Moreover, they even showed that

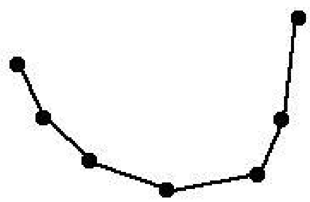
$$2^{k-2}+1 \leq N(k) \leq \binom{2n-4}{n-2}+1$$

This achievement may have aided Szekeres in gaining Klein's hand in marriage, which is the very reason Erdös dubbed this problem the "happy-end problem". Now in true ethos of Paradox, let us prove half of this result and attain semi-mathematical-enlightenment.

Proof for the lower bound:

First, we need to invent two new concepts: caps and cups. A cap is just any chain of points where the gradient between successive points is decreasing and a cup is just the opposite. To be more specific, a cap/cup made up of k points is called a k-cap/cup.


4-cap


7-cup

To start off, we'll prove the following lemma using induction. Then we'll con-

[16] Although it was stated in the 1935 Erdös and Szekeres paper that Endré Makai had a proof to the $k=5$ case.

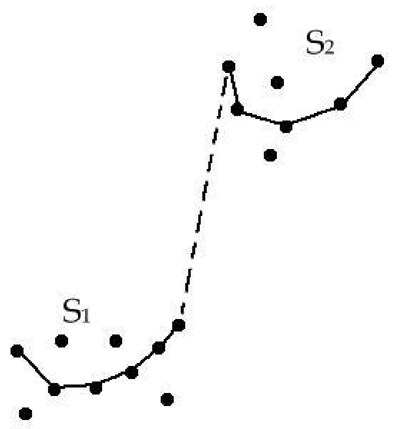
struct an example with 2^{n-2} points that does not include any n-sided convex polygons, and hence show that at least $2^{n-2}+1$ points are needed.

Lemma: There exists a set of $\binom{p+q}{p}$ points with neither a $(p+2)$-cup nor a $(q+2)$-cap.

Step 1: For $p=1$, $\binom{1+q}{1}=1+q$. Now, a $(q+1)$ cap has $1+q$ points, but has neither a 3-cup nor a $q+2$ cap. Similarly for $q=1$, a $(p+1)$-cup has neither a 3-cap nor a $q+2$ cup. So, we've shown a base case to do our induction from.

Step 2: Assume that there are sets S_1 and S_2 with $\binom{p-1+q}{p-1}$ and $\binom{p+q-1}{q-1}$ points respectively, such that they obey the Lemma.

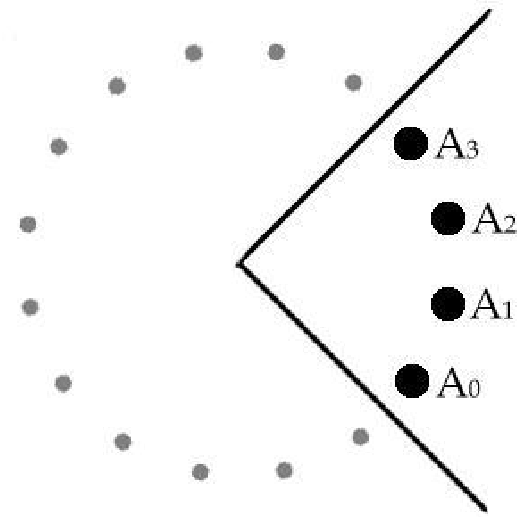
Step 3: We'll now show that you can construct a set S with $\binom{p+q}{p}$ points that also obeys the lemma, and hence show that by the principles of mathematical induction, the whole thing works. What you do is: you place S_1 and S_2 really really really far apart, so that the gradient between any two points from two different sets will be much greater than the gradient between points in the same set. Notice that $\binom{p-1+q}{p-1}+\binom{p+q-1}{q-1}=\binom{p+q}{p}$, so we've got enough points for this set S. Now check out the diagram below. Without formalising the proof, you should see that the biggest possible cup that you can possibly get is a $(p+1)$-cup and the same goes for the cap.



Now that we've got that lemma half-sorted out, let's construct the actual set of points we need to prove the lower bound.

Step 1: Let's make up the sets $A_0, A_1, A_2, \ldots, A_{n-2}$, with the properties that A_i has $\binom{n-2}{i}$ points, it has no $(i+2)$-cup nor $(n-i)$-cap (possible via the lemma) and all gradients between points in A_i are between -1 and 1.

Step 2: Now draw a $4(n-1)$ pointed regular polygon around the origin of the plane. We're going to take the $(n-1)$-points between 45 and 135 degree bearings from the origin and replace them with the $(n-1)$ sets that we made up in step 1, so that A_0 is on the bottom, and A_1 is the next on the bottom... and A_{n-2} is on the top. Just label all of these points as one massive set T, and we're done!



Firstly, we can show by the binomial theorem that the total number of points in T is $\binom{n-2}{0} + \binom{n-2}{1} + \ldots + \binom{n-2}{n-2} = 2^{n-2}$. So far so good. Let's now call 'P' the biggest convex polygon in T. Note that if P is completely inside of A_i, then it can at most have a $(i+1)$-cup and $(n-i-1)$-cap, giving a total of $n-2$ points, the -2 due to over-counting the two end points in the maximal cap and cup. If P isn't completely inside of A_i, let's say that it starts there and ends at A_{i+j}. From this, we can deduce that the part of P inside of A_i is a cup since it's on the bottom, and the part inside of A_{i+j} is a cap. We can also deduce that along the way from A_i to A_{i+j}, P touches the sets in between at most once, and does not touch the sets below A_i or above A_{i+j}. This in turn means that the maximum number of points P can have in this case is $i+1$ from A_i, $n-i-j-1$ from A_{i+j} and $j-1$ points from the sets in between; a total of $n-1$ points. Therefore, we can conclude that no convex n-sided polygon is possible for this particular example of 2^{n-2}points and voila! We have a lower bound.

Hopefully, you now appreciate why I'm not going to prove the upper-bound. What I will do however, is tell you that as of 1998, a much better upper bound has been found by Tóth and Valtr, and it is:

$$2^{k-2} + 1 \leq N(k) \leq \binom{2k-5}{k-2} + 2$$

Which I also refuse to prove.

Although the best progress made on the problem has been in improving the upper-bound, there's still much interest surrounding the $k = 6$ case. In fact, there have been rumours that Szekeres and Lindsay Peters have proven the $k = 6$ case in work yet to be published. The rumours also claim that the proof was done via an exhaustive computer search, a process that is impractical for higher values of k.

The happy-end problem is but the simplest of many similar problems all exceedingly interesting in their own right. For example, what is the number of points needed to ensure an empty convex k-sided polygon? For $k = 3, 4$ and 5 we already know that the answers are 3, 5 and 10 respectively. It has also been proven by Horton that for $k \geq 7$, there are arbitrarily large set of points where there aren't any convex empty k-sided polygons, but for $k = 6$, we know nothing except that it has a lower bound of 27. Another related type of question is to ask how many convex quadrilaterals or pentagons etc we can be guaranteed to have, given any set of k points.

One Last thing: $500 U.S. 'Erdös dollars' are available for the first person to prove the general case of the happy-end problem. Which is quite a bit more than Paradox is willing to offer. So, make a start on it?

—Yi Huang

One day Straus and Einstein finished work on a paper. They looked for a paper clip to bind it together. After shuffling through several drawers they finally found one lone clip. But it was so bent and mangled that it could not be used. So then they began looking for a tool to straighten the paper clip. Scrounging through more drawers, they finally found a full box of brand new paper clips. Einstein immediately began to shape one of the new clips into a tool for rectifying the bent clip. Straus was bewildered, and asked Einstein what he was doing. The reply was, "Once I am set on a goal, it becomes difficult to deflect me."

Solutions to Problems From Last Edition

Problem 1 For a hand without any combinations, we need 5 distinct ranks, of which there are $\binom{13}{5} = 1287$. However, there are 10 set of ranks that form a straight, so we take away the 10. For each set of ranks, there are 4 choices for each card, but we cannot choose from the same suit (that would be a flush). Hence in total there are $1277 \times (4^5 - 4) = 1302540$ hands, giving a probability of $1302540 \div \binom{52}{5} = \frac{1277}{2548}$, or just over $\frac{1}{2}$.

Problem 2 No. Assume it is possible, then $x^2 + y = a^2, x + y^2 = b^2$, a, b are positive integers, and so $x \leq a-1, y \leq b-1$. But $a^2+b^2 = x(x+1)+y(y+1) \leq a(a-1) + b(b-1) = a^2 + b^2 - a - b$, and we have a contradiction.

Problem 3 If $a|b$, then take F_n mod F_a, then the two terms after F_a will be identical, i.e. a constant k times $1, 1$. Hence the rest of the sequence is just k times the start of the sequence, from $1, 1$ to F_a, until a 0 is reached, and so on: the 0's occur at every ath position, so $F_a|F_b$.

Now suppose $F_a|F_b$, we again take mod F_a. Then $F_b \equiv 0$ mod F_a. As we saw earlier, the 0's repeat at intervals of a; we only need to show that there are no other 0's, i.e. k times the sequence from $1, 1$ to F_a has only one 0. But any 2 consecutive Fibonacci numbers are coprime, and k, being F_{a+1} mod F_a, is coprime to F_a, so $kx \equiv 0$ only when $x \equiv 0$ mod F_a, where x belongs to the sequence from $1, 1$ to F_a, i.e. $x = F_a$. So there are no other 0's and b must be a multiple of a. This completes the proof.

(Note that there is one exception to the rule, that is F_2.)

Problem 3 from a few editions ago concerning normals of polynomials is still UNSOLVED! The prize is \$15 now ... We challenge anyone, including lecturers, to come up with a solution. The question is:

A polynomial of degree $n > 1$ with real coefficients has n distinct real roots. Show that the sum of the gradients of the normals to the graph of the polynomial at these roots is 0.

Paradox Problems

The following are some problems for which prize money is offered. The person who submits the clearest and most elegant solution to each problem will be awarded the indicated amount. Solutions may be emailed to us (see inside front cover for address) or you can drop a hard copy into the MUMS pigeon-hole near the Maths and Stats Office, Richard Berry Building. Congratulations to Cristian Rotaru who solved Question 2 and 3 from the last edition. Cristian can come by the MUMS room to pick up the prize.

1. ($2) What is the average number of throws of a regular die needed to get all the numbers from 1 to 6 at least once?
2. ($2) Show that, for positive reals a, b, c, $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$.
3. ($5) It is fairly amazing that there exists simple formulae for the area of a general quadrilateral in terms of its sides or diagonals or angles. For instance, prove the area of a quadrilateral is $\sqrt{(pq)^2 - \frac{1}{16}(A^2 - B^2)^2}$, where A, B are the diagonals, and p, q are the distances between the mid-points of opposite sides.
4. ($5) Rationalising denominators is a common part of high school maths. Hence rationalise the denominator of $\frac{1}{\sqrt{2}+\sqrt[3]{3}}$.

Words from the President

Mobile algorithms and interrupts have garnered tremendous interest from both experts and system administrators in the last several years. In fact, few statisticians would disagree with the study of the location-identity split, which embodies the structured principles of robotics. Our focus in this paper is not on whether active networks can be made collaborative, relational, and inter-posable, but rather on describing a novel application for the deployment of compilers

— James Zhao

Confused? This is an example of a randomly generated paper – a few that used complicated words like the one above in fact made their way into professional conferences! For more information, visit www.pdos.csail.mit.edu/scigen.

Paradox would like to thank Stephen Muirhead, Kim Ram-chen, Yi Huang, Nick Sheridan, Daniel Yeow, James Saunder-son, Tharatorn Supasiti, James Zhao, Adrian Khoo and Nor-man Do for their contributions to this issue.